INVARIANTS AND SEPARATING MORPHISMS FOR ALGEBRAIC GROUP ACTIONS

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ABSTRACT. The first part of this paper is a refinement of WINKELMANN's work on invariant rings and quotients of algebraic groups actions on affine varieties, where we take a more geometric point of view. We show that the (algebraic) quotient $X /\!\!/ G$ given by the possibly not finitely generated ring of invariants is "almost" an algebraic variety, and that the quotient morphism $\pi \colon X \to X /\!\!/ G$ has a number of nice properties. One of the main difficulties comes from the fact that the quotient morphism is not necessarily surjective.

These general results are then refined for actions of the additive group $\mathbb{G}_a$, where we can say much more. We get a rather explicit description of the so-called plinth variety and of the separating variety, which measures how much orbits are separated by invariants. The most complete results are obtained for representations. We also give a complete and detailed analysis of ROBERTS' famous example of a an action of $\mathbb{G}_a$ on 7-dimensional affine space with a non-finitely generated ring of invariants.

1. INTRODUCTION

In all classification problems *invariants* play an important rôle. They let one distinguish non-equivalent objects, characterize specific elements, or detect certain properties. For instance, the genus of a curve determines a smooth curve up to birational equivalence, and the discriminant of a polynomial tells us whether it has multiple roots. In the algebraic setting, we often can reduce the classification problem to the following general situation. There is an algebraic variety X representing the objects, and an algebraic group G acting on X such that two objects $x, y \in X$ are equivalent if and only if they belong to the same orbit under G. In this case the classification problem amounts to describing the orbit space X/G. Clearly, X/G inherits some properties from X: it has a topology and the (continuous) functions on X/G correspond to the (continuous) G-invariant functions on X. Of course, we would like to see X/G again as an algebraic variety, but this cannot work in general, because X usually contains non-closed orbits, and so X/G contains non-closed points.

If X is an affine variety with coordinate ring $\mathcal{O}(X)$, we could look at the subalgebra $\mathcal{O}(X)^G \subset \mathcal{O}(X)$ of G-invariant functions and consider the morphism

$$\pi_X \colon X \to X /\!\!/ G := \operatorname{Spec} \mathcal{O}(X)^G$$

induced by the inclusion. It is called *algebraic quotient*, is a categorical quotient in the category of affine schemes, and so has the usual universal property: *Every G-invariant morphism $X \to Y$ factors uniquely through π_X.* In some sense this is the best algebraic approximation to the orbit space.

If G is reductive, then $\mathcal{O}(X)^G$ is finitely generated and π_X has some very nice properties (see [Kra84, II.3.2]):

- π_X is G-*closed:* If $Z \subset X$ is G-stable and closed, then $\pi_X(Z)$ is closed.

- π_X is G-*separating:* If $Z, Z' \subset X$ are disjoint G-stable closed subsets, then $\pi_X(Z) \cap \pi_X(Z') = \emptyset$.

In particular, π_X is surjective and every fiber contains a unique closed orbit. Thus $X/\!\!/G$ classifies the closed orbits in X. In good situations, the generic orbits are closed, and so at least generically $X/\!\!/G$ is the orbit space.

If G is not reductive, then all this fails to be true. In particular, the invariant ring might not be finitely generated and so the quotient $X/\!\!/G$ is not algebraic, and the quotient morphism π_X is not usually surjective. The fact that $X/\!\!/G$ is not algebraic was considered to be the main difficulty in handling non-reductive groups. We think that the non-surjectivity of π_X is an even a more serious problem.

One of the aims of this paper is to show that the quotient $X/\!\!/G$ as a scheme is "almost" algebraic. It always contains large open algebraic subsets and shares many properties with algebraic varieties. This is explained in sections 2 and 4 which are inspired by WINKELMANN's work [Win03]. For example, if the base field is uncountable, then $X/\!\!/G$ is a JACOBSON scheme which implies that the ZARISKI topology on $X/\!\!/G$ is determined by the ZARISKI topology on the $\Bbbk$-rational points of $X/\!\!/G$.

To have an idea of our approach and our results let us give a geometric interpretation of ROBERTS famous example of an action of the additive group $\mathbb{G}_a = (\Bbbk, +)$ on $\mathbb{A}^7$ with a non-finitely generated ring of invariants (see section 9 for details). Let $\pi\colon \mathbb{A}^7 \to \mathbb{A}^7/\!\!/\mathbb{G}_a$ be the quotient. Then

(a) *The fixed point set* $F := (\mathbb{A}^7)^{\mathbb{G}_a} \simeq \mathbb{A}^4$ *is mapped to a single point* $\pi(0)$;
(b) *The complement* $\mathbb{A}^7_{bd} := \mathbb{A}^7 \setminus F$ *is a principal* $\mathbb{G}_a$-*bundle* $\pi\colon \mathbb{A}^7_{bd} \to \pi(\mathbb{A}^7_{bd})$;
(c) *The image* $\pi(\mathbb{A}^7_{bd}) \subset \mathbb{A}^7/\!\!/\mathbb{G}_a$ *is an open algebraic subset and contains every open algebraic subset* U *of* $\mathbb{A}^7/\!\!/\mathbb{G}_a$;
(d) *The complement* $\mathbb{A}^7 \setminus \pi(\mathbb{A}^7_{bd})$ *is isomorphic to* $\mathbb{A}^3$.

An important new feature is the concept of *separating morphisms* $\varphi\colon X \to Y$ where Y is an algebraic variety (cf. [DK02, section 2.3]). This means that φ is G invariant and separates the same orbits as π_X. Such morphisms always exist even when the invariants are not finitely generated, but finding a "nice" separating morphism is usually a difficult task. For ROBERTS example we get the following.

(e) *There exists a separating morphism* $\varphi\colon \mathbb{A}^7 \to \mathbb{A}^9$ *such that* $Y := \overline{\varphi(\mathbb{A}^7)}$ *is normal of dimension 6.*
(f) *The induced map* $\bar{\varphi}\colon \mathbb{A}^7/\!\!/\mathbb{G}_a \to Y$ *gives a homeomorphism* $\pi(\mathbb{A}^7) \to \varphi(\mathbb{A}^7)$ *and an isomorphism* $\pi(\mathbb{A}^7_{bd}) \xrightarrow{\sim} \varphi(\mathbb{A}^7_{bd})$.
(g) $H := Y \setminus \varphi(\mathbb{A}^7_{bd})$ *is a hypersurface in* Y, *and* $\mathcal{O}(\mathbb{A}^7)^{\mathbb{G}_a} = \mathcal{O}_Y(Y \setminus H)$.

Another important concept is the *separating variety* which measures how much the invariants separate the orbits. It is defined as the reduced fiber product $\mathcal{S}_X := X \times_{X/\!\!/G} X$ and contains the closure of the graph $\Gamma_X := \{(gx, x) \mid g \in G, x \in X\}$. If a general fiber of the quotient map is an orbit and G is connected, then $\overline{\Gamma_X}$ is an irreducible component of the separating variety. But even in nice situations, the separating variety may have additional components. In general, the meaning of the

other components is not yet well understood, except for some special cases (see below). For ROBERTS example we find

(h) *The separating variety has two irreducible components:* $\mathcal{S}_{\mathbb{A}^7} = \overline{\Gamma_{\mathbb{A}^7}} \cup F \times F$.

The most complete results are obtained for actions of the additive group $\mathbb{G}_a$, in particular for representations of $\mathbb{G}_a$, see sections 5–7. This part of our work was inspired by certain calculations done by ELMER and KOHLS in [EK12]. An important tool is the geometric interpretation of the zero set of the *plinth ideal*. If X is factorial, it is the complement of the open set X_{bd} where X is locally a $\mathbb{G}_a$-bundle. In section 8 we generalize some of the results for representations to $\mathbb{G}_a$-actions induced by actions of SL_2.

Finally, the last section contains a detailed analysis of ROBERTS' example of a $\mathbb{G}_a$-action on $\mathbb{A}^7$ with a non-finitely generated ring of invariants. To prepare the reader for the difficulties in working with non-finitely generated algebras we describe an easy example in section 3.

2. GENERAL SETUP AND NOTATION

Invariants. Our base field $\Bbbk$ is algebraically closed. In the second part, starting with sections 5, we study $\mathbb{G}_a$-actions and will assume that char $\Bbbk = 0$. Since we have to deal with non-finitely generated rings of invariants, we will work in the category of $\Bbbk$-*schemes* X, and will denote by $X(\Bbbk)$ the $\Bbbk$-*rational points of* X. In this setting, a *variety* X is a reduced algebraic $\Bbbk$-scheme. For a variety X, we will often confuse the scheme X with its $\Bbbk$-rational points $X(\Bbbk)$.

Throughout this paper, we let X be a normal affine variety and G an algebraic group acting on X. We denote by $\mathcal{O}(X)$ the $\Bbbk$-algebra of regular functions on X and by $\mathcal{O}(X)^G \subset \mathcal{O}(X)$ the subalgebra of G-invariant functions. The *quotient* is define to be the affine $\Bbbk$-scheme

$$X /\!\!/ G := \operatorname{Spec} \mathcal{O}(X)^G.$$

If the base field $\Bbbk$ is uncountable, a famous result of KRULL's implies that $X /\!\!/ G$ is a JACOBSON *scheme*, i.e., $\mathcal{O}(X)^G$ is a JACOBSON *ring* ([Kru51]). This means that every radical ideal of $\mathcal{O}(X)^G$ is the intersection of maximal ideals. Moreover, every closed point of X is $\Bbbk$-rational in this case, since $\mathcal{O}(X)^G$ is contained in a finitely generated $\Bbbk$-algebra. It follows that the ZARISKI-topology on $X /\!\!/ G$ is completely determined by the ZARISKI-topology on the $\Bbbk$-rational points $(X /\!\!/ G)(\Bbbk)$. This allows to work with $\Bbbk$-rational points which are the only interesting objects from a geometric point of view.

Remark 2.1. If the $\Bbbk$-algebra R is not a JACOBSON ring, then there is a prime ideal $\mathfrak{p} \subset R$ which is not the intersection of the maximal ideal containing $\mathfrak{p}$. In geometric terms this means the following. Denote by $Z \subset \operatorname{Spec} R$ the closed subscheme defined by $\mathfrak{p}$, and let $Z_{cl} \subset Z$ be the subset of closed points. Then the closure $\overline{Z_{cl}}$ in $\operatorname{Spec} R$ is strictly contained in Z.

Quotient morphism. The inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$ defines the *quotient morphism*

$$\pi = \pi_X \colon X \to X /\!\!/ G.$$

Although $\mathcal{O}(X)^G$ might not be finitely generated over $\Bbbk$ (hence $X /\!\!/ G$ is not algebraic), we will see that the quotient $X /\!\!/ G$ contains large open sets which are

algebraic. For this we need the following result due to DERKSEN and KEMPER [DK08, Proposition 2.9 and 2.7].

Proposition 2.2. *Let R be a $\Bbbk$-algebra. Define*

$$\mathfrak{f}_R := \{f \in R \mid R_f \text{ is finitely generated}\} \cup \{0\}.$$

Then $\mathfrak{f}_R$ is a radical ideal of R. If R is contained in a finitely generated $\Bbbk$-domain, then $\mathfrak{f}_R \neq (0)$.

The ideal $\mathfrak{f}_R$ will be called the *finite generation ideal*.

Remark 2.3. The open subset $\operatorname{Spec} R \setminus \mathcal{V}(\mathfrak{f}_R) \subset \operatorname{Spec} R$ is the union of all open subschemes $U \subset \operatorname{Spec} R$ which are algebraic. In fact, each such U is a finite union of open affine algebraic U_i, and each U_i is a finite union of some $(\operatorname{Spec} R)_{f_j}$. We will denote $\operatorname{Spec} R \setminus \mathcal{V}(\mathfrak{f}_R)$ by $(\operatorname{Spec} R)_{alg}$ and call it the *algebraic locus*:

$$(\operatorname{Spec} R)_{alg} := \operatorname{Spec} R \setminus \mathcal{V}(\mathfrak{f}_R) = \bigcup_{\substack{U \subset \operatorname{Spec} R \\ \text{open and algebraic}}} U$$

Note that $(\operatorname{Spec} R)_{alg}$ is itself algebraic if and only if $\mathfrak{f}_R$ is the radical of a finitely generated ideal. On the other hand, $(\operatorname{Spec} R)_{alg}$ is always Jacobson and its closed points coincide with its $\Bbbk$-rational points.

Definition 2.4. Let $Z = \operatorname{Spec} R$ be an affine $\Bbbk$-scheme. If $A \subset Z$ is a closed subset we define $I(A) \subset R$ to be the (radical) ideal of functions vanishing on A.

  (a) $\dim Z := \operatorname{Kdim} R$, the KRULL-dimension of R.
  (b) If Z is reduced and irreducible, i.e., if R is a domain, then $\Bbbk(Z) := Q(R)$ denotes the field of fractions of R.
  (c) If R is a domain, then $\operatorname{tdeg}_\Bbbk R := \operatorname{tdeg}_\Bbbk Q(R)$ is the transcendence degree of the field extension $Q(R)/\Bbbk$.
  (d) If $A \subset Z$ is closed, then $\operatorname{codim}_Z A := \min\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \supset I(A), \mathfrak{p} \text{ prime}\}$ where $\operatorname{ht} \mathfrak{p}$ is the height of the prime ideal $\mathfrak{p}$.

As an example, we will see later in Theorem 4.3(a) that the quotient $X /\!\!/ G$ introduced above is always finite dimensional, and that $\dim X /\!\!/ G = \operatorname{tdeg}_\Bbbk \mathcal{O}(X)^G$.

Algebraic varieties. Assume that $Z = \operatorname{Spec} R$ is algebraic. Then $Z = \bigcup_i Z_i$ is a finite union of irreducible closed subsets, and $\dim Z = \max_i \{\dim Z_i\}$. Moreover, if Z is reduced and irreducible, then $\dim Z = \operatorname{tdeg}_\Bbbk R$, and for every irreducible closed subset $A \subset Z$ we have $\dim A + \operatorname{codim}_Z A = \dim Z$.

Finally, if $\varphi \colon Z \to Y$ is a morphism where Y is an arbitrary $\Bbbk$-scheme, and if $A \subset Z$ is a closed subscheme, then $\varphi(A(\Bbbk))$ is dense in $\varphi(A) \subset Y$. As mentioned before, this last statement holds more generally if R is a JACOBSON ring.

3. A First Example

Let us discuss an interesting example. While it does not quite fit in our setting— it does not arise from a quotient of an algebraic group action on a normal affine variety—it has a similar behavior.

Consider the graded subring $R := \Bbbk[x, xy, xy^2, xy^3, \ldots] \subset \Bbbk[x, y]$ generated by the monomials xy^k, $k = 0, 1, \ldots$, and set $Z := \operatorname{Spec} R$.

(a) *The finite generation ideal $\mathfrak{f}_R$ of R is equal to the homogeneous maximal ideal $\mathfrak{m}_0 = (x, xy, xy^2, \ldots)$, and $\mathfrak{m}_0 = \sqrt{xR}$.*

(b) *We have $Z \setminus \{\mathfrak{m}_0\} = Z_x$, and this is an affine algebraic variety with coordinate ring $\Bbbk[x, x^{-1}, y]$*

Now consider the morphism $\pi\colon \mathbb{A}^2 \to Z$ given by the inclusion $R \subset \Bbbk[x, y]$. (This morphism will play the role of a quotient morphism.)

(c) *$\pi\colon \mathbb{A}^2 \to Z$ is surjective and induces an isomorphism $(\mathbb{A}^2)_x \xrightarrow{\sim} Z_x$.*

(d) *$\pi\colon \mathbb{A}^2 \to Z$ is a closed morphism.*

Finally, we consider the affine morphism $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$.

(e) *φ factors through π*

$$
\begin{array}{ccc}
\mathbb{A}^2 & \xrightarrow{\ \pi\ } & Z \\
& {\scriptstyle \varphi} \searrow & \downarrow {\scriptstyle \bar{\varphi}} \\
& & \mathbb{A}^2
\end{array}
$$

and $\bar{\varphi}$ is injective on the image of π. Hence φ separates the same points of $\mathbb{A}^2$ as π.

(f) *$\bar{\varphi}$ induces a homeomorphism $Z \to \varphi(\mathbb{A}^2) = \mathbb{A}^2_y \cup \{0\}$.*

The proofs are not difficult and are left to the reader. They are based on the following lemma.

Lemma 3.1. (a) *We have $R = \Bbbk \oplus \mathfrak{m}_0$ where $\mathfrak{m}_0 = x\Bbbk[x, y] = (x, xy, xy^2, \ldots)$ is the homogeneous maximal ideal of R.*

(b) *Let $f \in \Bbbk[x, y]$. Then*

$$
f\Bbbk[x, y] \cap R = \begin{cases} f\Bbbk[x, y] & \text{if } f \in \mathfrak{m}_0; \\ fR & \text{if } f \in R \setminus \mathfrak{m}_0; \\ (xf)R & \text{if } f \notin R. \end{cases}
$$

4. Separating Morphisms

Separation. The so-called *separation property* will play an important role in this paper. The notion goes back to Derksen and Kemper [DK02, section 2.3.2], and is also implicit in the work of Winkelmann [Win03, Lemma 7].

Definition 4.1. Let X be an affine G-variety. A G-invariant morphism $\varphi\colon X \to Y$ where Y is an affine variety is a *separating morphism* if it satisfies the following *Separation Property:*

(SP) *If $x, x' \in X(\Bbbk)$ are separated by an invariant $f \in \mathcal{O}(X)^G$, i.e., if $f(x) \neq f(x')$, then $\varphi(x) \neq \varphi(x')$.*

Remark 4.2. If $\operatorname{char} \Bbbk = 0$, then the separation property (SP) implies that φ^* induces an isomorphism $\Bbbk(\overline{\varphi(X)}) \xrightarrow{\sim} Q(\mathcal{O}(X)^G)$. If $\operatorname{char} \Bbbk > 0$, we say that φ is *strongly separating* if φ is separating and induces an isomorphism $\Bbbk(\overline{\varphi(X)}) \xrightarrow{\sim} Q(\mathcal{O}(X)^G)$.

It is shown in [DK02, Theorem 2.3.15] that separating morphisms always exist. In more algebraic terms this means that one can find a finitely generated *separating subalgebra* $R \subset \mathcal{O}(X)^G$, i.e., a subalgebra which separates the same $\Bbbk$-rational points of X as the invariant functions. We can always add invariant functions to R, and thus assume that R is normal and that $Q(R) = Q(\mathcal{O}(X)^G)$, if necessary. Thus, a strongly separating morphism $\varphi \colon X \to Y$ with Y normal always exists. A basic problem is to find a separating algebra with a small number of generators.

Main results. A G-invariant morphism $\varphi \colon X \to Y$ always factors through the quotient morphism $\pi \colon X \to X/\!\!/G$:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \pi\ } & X/\!\!/G \\
& {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\bar\varphi} \\
& & Y
\end{array}
$$

Then φ is separating if and only if $\bar\varphi$ is injective on the image $\pi(X(\Bbbk)) \subset (X/\!\!/G)(\Bbbk)$ of the rational points. In the paper [Win03], WINKELMANN studies this general set-up and proves a number of fundamental results, e.g. that every such invariant ring $\mathcal{O}(X)^G$ is the ring of global regular functions on a quasi-affine variety and vice versa. Some of his results are contained and extended in the following theorem, where we take a geometric point of view.

Theorem 4.3. *Let X be a normal affine variety with an action of an algebraic group G, and denote by $\pi \colon X \to X/\!\!/G$ the quotient morphism. Let $\varphi \colon X \to Y$ be a dominant separating morphism where Y is a normal affine variety.*

(a) *If $A \subset X$ is an irreducible closed subset, then $\dim \overline{\pi(A)} = \dim \overline{\varphi(A)}$ and $\operatorname{codim}_{X/\!\!/G} \overline{\pi(A)} = \operatorname{codim}_Y \overline{\varphi(A)}$. In particular,*

$$\dim X/\!\!/G = \dim Y = \operatorname{tdeg}_{\Bbbk} \mathcal{O}(X)^G.$$

(b) *The map $\bar\varphi \colon X/\!\!/G \to Y$ induces a homeomorphism $\pi(X) \xrightarrow{\sim} \varphi(X)$.*

(c) *We always have $\operatorname{codim}_{X/\!\!/G} \overline{X/\!\!/G \setminus \pi(X)} > 1$.*

For the next four statements we assume that φ is strongly separating.

(d) *Set $N := Y \setminus \varphi(X)$. Then $\mathcal{O}(X)^G = \mathcal{O}(Y \setminus \overline{N})$.*

(e) *$\bar\varphi^{-1}(Y \setminus \overline{N}) \subset (X/\!\!/G)_{alg}$ and the induced map $\bar\varphi^{-1}(Y \setminus \overline{N}) \xrightarrow{\sim} Y \setminus \overline{N}$ is an isomorphism.*

(f) *The complement $X/\!\!/G \setminus (X/\!\!/G)_{alg}$ has codimension > 1 in $X/\!\!/G$.*

(g) *Set $M := X/\!\!/G \setminus \pi(X) \subset X/\!\!/G$. Then $\bar\varphi$ induces an open immersion $(X/\!\!/G)_{alg} \setminus \overline{M} \hookrightarrow Y$.*

The proofs of the following corollaries are easy and left to the reader.

Corollary 4.4. *Assume $\varphi \colon X \to Y$ is dominant and strongly separating with Y normal. If the finite generation ideal $\mathfrak{f}_{X/\!\!/G}$ of $\mathcal{O}(X)^G$ is the radical of the ideal generated by $\mathfrak{f}_{X/\!\!/G} \cap \mathcal{O}(Y)$, then $(X/\!\!/G)_{alg}$ is algebraic and $\bar\varphi$ induces an open immersion $(X/\!\!/G)_{alg} \hookrightarrow Y$.*

Corollary 4.5. *Assume $\varphi \colon X \to Y$ is dominant and strongly separating. If Y is factorial, then $\mathcal{O}(X)^G$ is finitely generated and $\bar\varphi \colon X/\!\!/G \to Y$ is an open immersion. In particular, $X/\!\!/G \simeq Y_f$ for a suitable $f \in \mathcal{O}(Y)$.*

Corollary 4.6. *If V is a rational representation of an algebraic group G and if $\varphi\colon X \to Y$ is a strongly separating morphism with Y factorial, then $Y = X/\!/G$.*

Remark 4.7. In the case where G is reductive, this last corollary is an easy consequence of RICHARDSON's Lemma (see [Kra84, II.3.4]).
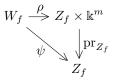
We say that an affine $\Bbbk$-scheme $Z = \operatorname{Spec} R$ is *fix-pointed* with fixed point z_0, if $R = \bigoplus_{i \geq 0} R_i$ is a graded ring with $R_0 = \Bbbk$ and z_0 the homogeneous maximal ideal. Geometrically this means that Z admits an action of the *multiplicative group* $\mathbb{G}_m := \Bbbk^*$ with a single closed orbit, namely the fixed point z_0. A variety X is called a *fix-pointed G-variety* if X is fix-pointed and the G-action commutes with the $\mathbb{G}_m$-action. In this case $X/\!/G$ is also fix-pointed, and the finite generation ideal $\mathfrak{f}_{X/\!/G}$ is homogeneous.

Corollary 4.8. *Let (X, x_0) be a fix-pointed affine G-variety. If $\pi(x_0) \notin \overline{X/\!/G \setminus \pi(X)}$, then $\mathcal{O}(X)^G$ is finitely generated, π is surjective and $\bar{\varphi}\colon X/\!/G \hookrightarrow Y$ an open immersion. If, in addition, Y is also fix-pointed and φ is homogeneous, then $\bar{\varphi}\colon X/\!/G \xrightarrow{\sim} Y$ is an isomorphism.*

Note that the special case of Corollary 4.8 for a representation of a reductive group G is contained in [DK02, Proposition 2.3.12].

Proof of Theorem 4.3. The proof needs some preparation.

Lemma 4.9. *Let W be an irreducible affine variety, $R \subset \mathcal{O}(W)$ a $\Bbbk$-subalgebra and $\psi\colon W \to Z := \operatorname{Spec} R$ the induced morphism. Then there is an $f \in \mathfrak{f}_R$ and a finite surjective morphism $\rho\colon W_f \to Z_f \times \Bbbk^m$, where $m := \dim W - \operatorname{tdeg}_{\Bbbk} Q(R)$, such that $\psi|_{W_f} = \operatorname{pr}_{Z_f} \circ \rho$:*

$$
\begin{array}{ccc}
W_f & \xrightarrow{\ \rho\ } & Z_f \times \Bbbk^m \\
& \psi \searrow & \downarrow \operatorname{pr}_{Z_f} \\
& & Z_f
\end{array}
$$

In particular, there is a subset $U \subset \psi(W)$ which is open, algebraic and dense in Z.

Proof. By first inverting some $f \in \mathfrak{f}_R$ we can assume that R is finitely generated. In this case the result is known and can be found in [Bou98, Chap. V.3.1, Corollary 1], cf. [Kra11, Appendix A.3.4 Decomposition Theorem]. $\square$

The following two results can be found in [Win03, Lemma 1, 2, and 6]. The first is due to NAGATA [Nag65].

Lemma 4.10. *The invariant ring $R := \mathcal{O}(X)^G$ is a KRULL-ring, i.e., $R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$ where $\mathfrak{p}$ runs through the primes of R of height 1.*

Lemma 4.11. *Let $S \subset X/\!/G$ be an irreducible closed subscheme of codimension 1, and put $H := \pi^{-1}(S) \subset X$. Then $S = \overline{\pi(H)}$.*

We will also need the following result; the proof is easy and left to the reader.

Lemma 4.12. *Let Y be an irreducible variety, $C \subset Y$ an irreducible closed subset of codimension d and $U \subset Y$ a non-empty open set. Then there is a chain*

$$Y = C_0 \supset C_1 \supset \cdots \supset C_d = C$$

of closed irreducible subsets such that

(i) $\operatorname{codim}_Y C_j = j$ for $j = 0, \ldots, d$, and

(ii) $C_j \cap U \neq \emptyset$ for $j < d$.

Proof of Theorem 4.3. (a) Lemma 4.9 implies that there is an open set $U \subset A$ such that $\pi(U)$ is open, algebraic and dense in $\overline{\pi(A)}$, and that $\varphi(U)$ is open, algebraic and dense in $\overline{\varphi(A)}$. Now $\pi(U(\Bbbk)) \to \varphi(U(\Bbbk))$ is bijective, since φ is separating. As $\pi(U)$ and $\varphi(U)$ are algebraic, it follows that $\dim \overline{\pi(A)} = \dim \pi(U) = \dim \varphi(U) = \dim \overline{\varphi(A)}$.

To get the equality for the codimensions, we choose a non-empty open subset $O \subset X$ such that $\pi(O)$ is open and algebraic in $X /\!\!/ G$, and such that $U := \varphi(O)$ is open in Y. From Lemma 4.12 there is a sequence $C_0 = Y \supset C_1 \supset \cdots \supset C_d = \overline{\varphi(A)}$ of closed irreducible subsets C_j with $\dim C_j = \dim Y - j$ such that $C_j \cap U \neq \emptyset$ for $j < d$. Since $\bar\varphi \colon \pi(O) \to \varphi(O)$ is a bijective morphism of varieties, we see that, for $j < d$, $B_j := \overline{\bar\varphi^{-1}(C_j) \cap \pi(O)}$ is irreducible of dimension $\dim Y - j$, and that $B_j \subset B_{j-1}$. It remains to see that $B_{d-1} \supseteq \pi(A)$, since this implies that $\operatorname{codim}_{X /\!\!/ G} \overline{\pi(A)} \geq d = \operatorname{codim}_Y \overline{\varphi(A)}$. If not, using again Lemma 4.9, we can find a subset $U \subset \pi(A)$ which is open and dense in $\overline{\pi(A)}$ and such that $U \cap B_{d-1} = \emptyset$. Then the image $\bar\varphi(U)$ is disjoint from $\bar\varphi(B_{d-1} \cap \pi(O))$. Since $\overline{\bar\varphi(B_{d-1} \cap \pi(O))} = C_{d-1}$, it follows that $\overline{\varphi(A)} = \overline{\bar\varphi(U)}$ is not contained in C_{d-1}, contradicting the assumption.

(b) The same argument as above shows that, for irreducible closed subsets $A, B \subset X$ with $\overline{\pi(A)} \nsubseteq \overline{\pi(B)}$, we have $\overline{\varphi(A)} \nsubseteq \overline{\varphi(B)}$. It follows that the map $\pi(X) \to \varphi(X)$ is injective, hence bijective, and open, hence a homeomorphism.

(c) For $\mathfrak{p} \in X /\!\!/ G$ we have $\mathfrak{p} \in M := (X /\!\!/ G) \setminus \pi(X)$ if and only if $\overline{\pi(\mathcal{V}_X(\mathfrak{p}))} \subsetneqq \mathcal{V}(\mathfrak{p})$ where $\mathcal{V}(\mathfrak{p})$ denotes the zero set in $X /\!\!/ G$. Assume now that $\operatorname{codim}_{X /\!\!/ G} \overline{M} = 1$. This means that $\overline{M}$ contains an irreducible closed subscheme S of codimension 1 corresponding to a prime ideal $\mathfrak{p} \in M$ of height 1. It follows that $\overline{\pi(\pi^{-1}(S))} \subsetneqq S$, contradicting Lemma 4.11.

(d) Let $S \subset X /\!\!/ G$ be an irreducible hypersurface and let $\mathfrak{p} \subset R := \mathcal{O}(X)^G$ be the corresponding prime ideal of height 1. Then, by Lemma 4.11 and (b), $H := \overline{\bar\varphi(S)}$ is an irreducible hypersurface, and so the corresponding prime ideal $\mathfrak{p}' := \mathfrak{p} \cap \mathcal{O}(y)$ has also height 1. This implies that $\mathcal{O}(Y)_{\mathfrak{p}'} = R_{\mathfrak{p}}$, since both are discrete valuation rings of $\Bbbk(Y)$. But every irreducible hypersurface $H \subset Y$ not contained in $\overline{N}$ is of the form $\overline{\bar\varphi(S)}$, hence $\mathcal{O}(Y \setminus \overline{N}) = \bigcap_{\mathfrak{p}' = \mathfrak{p} \cap \mathcal{O}(Y)} \mathcal{O}(Y)_{\mathfrak{p}'} = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}} = R$ by Lemma 4.10.

(e) If $f \in I(\overline{N})$, then $Y_f \subset Y \setminus \overline{N}$, and so $\mathcal{O}(Y)_f \supset \mathcal{O}(Y \setminus \overline{N}) = \mathcal{O}(X)^G$ by (d). Thus $\bar\varphi$ induces an isomorphism $(X /\!\!/ G)_f \simeq Y_f$, and so $(X /\!\!/ G)_f$ is algebraic.

(f) By construction, $\overline{\varphi(\varphi^{-1}(\overline{N}))}$ does not contain a hyperplane, and neither does $\bar\varphi^{-1}(\overline{N})$ by Lemma 4.11 and (a). The claim now follows since $\bar\varphi^{-1}(\overline{N}) \supset X /\!\!/ G \setminus (X /\!\!/ G)_{alg}$, as we have just seen in (e).

(g) By (b), $\bar\varphi \colon X /\!\!/ G \setminus \overline{M} \to Y$ is injective. Hence, for every open algebraic $U \subset X /\!\!/ G$, the map $\bar\varphi \colon U \setminus \overline{M} \to Y$ is an open immersion by ZARISKI's Main Theorem [Gro67, Théorème 8.12.6]. The claim then follows. $\square$

5. $\mathbb{G}_a$-Actions, Local Slices, and the Plinth Variety

$\mathbb{G}_a$-**bundles.** From now on we assume that $\operatorname{char} \Bbbk = 0$. In this and the following sections we focus on $\mathbb{G}_a$-varieties, i.e., varieties with an action of the additive group $\mathbb{G}_a \simeq (\Bbbk, +)$. A $\mathbb{G}_a$-variety X (not necessarily affine) is called a *trivial $\mathbb{G}_a$-bundle* if there is a $\mathbb{G}_a$-equivariant isomorphism $\mathbb{G}_a \times Y \xrightarrow{\sim} X$, or, equivalently, if there is a $\mathbb{G}_a$-equivariant morphism $X \to \mathbb{G}_a$. In this case, Y can be identified with the orbit space $X/\mathbb{G}_a$, and the quotient morphism $\pi \colon X \to X/\mathbb{G}_a$ admits a section. If X is affine, then $X/\mathbb{G}_a = \operatorname{Spec} \mathcal{O}(X)^{\mathbb{G}_a}$.

The $\mathbb{G}_a$-variety X is called a *principal $\mathbb{G}_a$-bundle* (for short, a $\mathbb{G}_a$-bundle) if there is a $\mathbb{G}_a$-invariant morphism $\pi \colon X \to Z$ and an open covering $Z = \bigcup_i U_i$ such that $p^{-1}(U_i) \to U_i$ is a trivial $\mathbb{G}_a$-bundle for all i. In this case, Z can be identified with the orbit space $X/\mathbb{G}_a$ and the morphism π has the usual universal properties. Again, if X is affine, then $X/\mathbb{G}_a = \operatorname{Spec} \mathcal{O}(X)^{\mathbb{G}_a}$.

Local slices. Now let X be a normal affine $\mathbb{G}_a$-variety. The $\mathbb{G}_a$-action defines a locally nilpotent vector field $D \in \operatorname{Vec}(X) := \operatorname{Der}_\Bbbk(\mathcal{O}(X))$ which determines the $\mathbb{G}_a$-action. Its kernel coincides with the ring of invariants: $\ker D = \mathcal{O}(X)^{\mathbb{G}_a}$. If $s \in \mathcal{O}(X)^{\mathbb{G}_a}$ is a non-zero invariant and $s = Df$ for some $f \in \mathcal{O}(X)$, then $D(\frac{f}{s}) = 1$ and thus the morphism

$$\frac{f}{s} \colon X_s \to \mathbb{G}_a$$

is $\mathbb{G}_a$-equivariant. Such morphisms are called *local slices*. It follows that the affine open set X_s is a trivial $\mathbb{G}_a$-bundle, and $X_s/\mathbb{G}_a = \operatorname{Spec} \mathcal{O}(X_s)^{\mathbb{G}_a}$. In particular, $\mathcal{O}(X_s)^{\mathbb{G}_a} = (\mathcal{O}(X)^{\mathbb{G}_a})_s$ is finitely generated.

Definition 5.1. Let X be a normal affine $\mathbb{G}_a$-variety. The ideal $\mathfrak{p}_X \subset \mathcal{O}(X)^{\mathbb{G}_a}$ generated by all $s \in \mathcal{O}(X)^{\mathbb{G}_a}$ of the form $s = Df$ for some $f \in \mathcal{O}(X)$ is called the *plinth ideal*:

$$\mathfrak{p}_X := D(\mathcal{O}(X)) \cap \ker D \subset \mathcal{O}(X)^{\mathbb{G}_a}.$$

The zero set $\mathcal{P}_X := \mathcal{V}_X(\mathfrak{p}_X) \subset X$ of the plinth ideal is called the *plinth variety* of X. Note that the plinth ideal is an ideal in the invariant ring, whereas the plinth variety is a closed subvariety of X.

As before, the quotient morphism is denote by $\pi \colon X \to X /\!\!/ \mathbb{G}_a$. The next result shows that outside the plinth variety the quotient morphism is a principal bundle.

Proposition 5.2. *The image $\pi(X \setminus \mathcal{P}_X) \subset X /\!\!/ \mathbb{G}_a$ is an open algebraic variety, and the morphism $\pi \colon X \setminus \mathcal{P}_X \to \pi(X \setminus \mathcal{P}_X)$ is a principal $\mathbb{G}_a$-bundle.*

Proof. If $s = Df$ and $Ds = 0$, then $\pi(X_s) = (X /\!\!/ \mathbb{G}_a)_s$, and this is an open subset of $X /\!\!/ \mathbb{G}_a$ which is affine and algebraic. Since we can cover $X \setminus \mathcal{P}_X$ with finitely many X_{s_j} we see that $\pi(X \setminus \mathcal{P}_X)$ is covered by finitely many open affine varieties, hence is an algebraic variety. It remains to see that π separates the $\mathbb{G}_a$-orbits on $X \setminus \mathcal{P}_X$. This is clear for two orbits contained in the same X_{s_j}. If $O_1 \subset X_{s_j}$ and $O_2 \subset X_{s_k} \setminus X_{s_j}$, then the invariant s_j vanishes on O_2, but not on O_1. $\square$

Definition 5.3. Let X be a $\mathbb{G}_a$-variety. Define $X_{bd} \subset X$ to be the union of all open $\mathbb{G}_a$-stable subsets U which are trivial $\mathbb{G}_a$-bundles:

$$X_{bd} := \bigcup_{\substack{U \subset X \text{ open} \\ U \text{ a trivial } \mathbb{G}_a\text{-bundle}}} U.$$

If X is affine, it follows from Proposition 5.2 that $X \setminus \mathcal{P}_X \subset X_{bd}$. We will see later (Example 8.4) that the inclusion can be strict. However, this cannot happen if X is factorial.

Proposition 5.4. *Let X be a factorial affine $\mathbb{G}_a$-variety. Then*

$$X_{bd} = X \setminus \mathcal{P}_X.$$

In particular, $\pi(X_{bd}) \subset X /\!\!/ \mathbb{G}_a$ is open and algebraic and $X_{bd} \to \pi(X_{bd})$ is a principal $\mathbb{G}_a$-bundle.

Proof. In the definition of X_{bd} we can assume that all U_i are affine. Since X is factorial, this implies that $U_i = X_{t_i}$ for a suitable invariant t_i. On the other hand, if X_t is a trivial $\mathbb{G}_a$-bundle where $t \in \mathcal{O}(X)^{\mathbb{G}_a}$, then there is an $h \in \mathcal{O}(X_t)$ such that $Dh = 1$. Writing $h = ft^{-k}$ we see that $s := t^k = Df$, and so $X_s = X_t$ is of the form above. $\qquad\square$

6. The case of a representation

Representations and the null cone. Let V be representation of $\mathbb{G}_a$. Then V extends to a representation of $\mathrm{SL}_2 := \mathrm{SL}_2(\Bbbk)$, where $\mathbb{G}_a$ is identified with the unipotent subgroup $U \subset \mathrm{SL}_2$ via $s \mapsto \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$. The invariants $\mathcal{O}(V)^{\mathbb{G}_a}$ are finitely generated (WEITZENBÖCK's Theorem, see [Kra84, III.3.9]), and the multiplicative group $\mathbb{G}_m$ acts linearly on V, $(t, v) \mapsto t \cdot v$, via the identification $t \mapsto \begin{bmatrix} t & \\ & t^{-1} \end{bmatrix} \in T \subset \mathrm{SL}_2$. This defines a decomposition of V into weight spaces:

$$V = \bigoplus_k V_k, \quad V_k := \{v \in V \mid t \cdot v = t^k v\}.$$

Since the invariants are finitely generated, the quotient $V /\!\!/ \mathbb{G}_a := \mathrm{Spec}\, \mathcal{O}(X)^{\mathbb{G}_a}$ is an affine variety. As usual, the *nullcone* is defined by $\mathcal{N} = \mathcal{N}_V := \pi^{-1}(\pi(0)) \subset V$. Recall that the WEYL-group $W \simeq \mathbb{Z}/2\mathbb{Z}$ of SL_2 acts on the zero weight space $V_0 = V^{\mathbb{G}_m}$. The non-trivial element of W is represented by the matrix $\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}_2$.

Theorem 6.1. (a) $\mathcal{N}_V = V^+ := \bigoplus_{k>0} V_k$.
  (b) $\mathcal{P}_V = V \setminus V_{bd} = V_0 \oplus V^+$. *In particular, $\mathcal{P}_V = \mathcal{N}_V$ if and only if the SL_2-representation V does not contain odd-dimensional irreducible representations.*
  (c) *The image $\pi(\mathcal{P}_V) \subset V /\!\!/ \mathbb{G}_a$ is closed. The induced map $\pi|_{\mathcal{P}_V} : \mathcal{P}_V \to \pi(\mathcal{P}_V)$ is given by the SL_2-invariants and has a factorization*

$$\mathcal{P}_V = V^+ \oplus V_0 \xrightarrow{\;\mathrm{pr}\;} V_0 \xrightarrow{\;\pi_0\;} V_0/W \xrightarrow{\;\bar{\pi}\;} \pi(\mathcal{P}_V)$$

  where π_0 is the quotient by W and $\bar{\pi}$ is finite and bijective.

Remark 6.2. ELMER and KOHLS [EK12] gave an explicit construction of separating sets for indecomposable representations, which were later extended to any representation by DUFRESNE, ELMER, and SEZER [DES14].

The proof of the theorem needs some preparation.

Invariants and covariants. Let V be as representation of SL_2. The graded coordinate ring $\mathcal{O}(V) = \bigoplus_{d \geq 0} \mathcal{O}(V)_d$ is a locally finite and rational SL_2-module. A homogeneous irreducible submodule $F \subset \mathcal{O}(V)_d$ is classically called a *covariant of degree d and weight r*, where r is the weight of the highest weight vector f_0 of F. This means that f_0 is a homogeneous $\mathbb{G}_a$-invariant and that $t \cdot f_0 = t^r f_0$ for $t \in \mathbb{G}_m$. In particular, $\dim F = r + 1$. Thus, we always have $r \geq 0$, and $r = 0$ if and only if f_0 is an SL_2-invariant. We will say that f_0 is a *homogeneous $\mathbb{G}_a$-invariant of degree d and weight r*.

Clearly, the invariants $\mathcal{O}(V)^{\mathbb{G}_a}$ are linearly spanned by the homogeneous $\mathbb{G}_a$-invariants of degree d and weight r where $d,\ r \geq 0$. Moreover, the homogeneous $\mathbb{G}_a$-invariants of degree d and weight $r > 0$ linearly span the plinth ideal $\mathfrak{p}_V = \ker D \cap \operatorname{im} D$ where $D \in \mathrm{Vec}(V)$ is the locally nilpotent vector field corresponding to the $\mathbb{G}_a$-action (see Definition 5.1). This shows that the $\mathbb{G}_a$-invariants are generated by $\mathfrak{p}_V$ together with the SL_2-invariants.

In the following, we denote by $V[n]$ the irreducible SL_2-module of highest weight n, i.e., $\dim V[n] = n + 1$. One can take $V[n] := \Bbbk[x, y]_n$, the *binary forms of degree n*, with the standard linear action of SL_2. It follows that the element $\sigma \in \mathrm{SL}_2$ representing the non-trivial element of the Weyl group acts trivially on $V[n]_0$ if n is odd or $n \equiv 0 \pmod 4$, and by $(-\operatorname{id})$ if $n \equiv 2 \pmod 4$.

In the proof below we will need the following classical result from invariant theory of binary forms. Choose a basis of weight vectors of $V[n]$ such that $\mathcal{O}(V[n]) = \Bbbk[x_0, x_1, \ldots, x_n]$, where x_i has weight $n - 2i$.

Lemma 6.3 (see [Kra84, III.1.5]). *As an SL_2-module we have the* CLEBSCH-GORDAN *decomposition $\mathcal{O}(V[n])_2 \simeq V[2n] \oplus V[2n-4] \oplus V[2n-8] \oplus \cdots$. The corresponding quadratic $\mathbb{G}_a$-invariants $f_k \in V[2n - 4k]^{\mathbb{G}_a}$ have weight $2n - 4k$ and are of the form*

$$f_k = \alpha_0 x_0 x_{2k} + \alpha_1 x_1 x_{2k-1} + \cdots + \alpha_k x_k^2, \ k = 0, 1, 2, \ldots, \lfloor n/2 \rfloor,$$

where all coefficients α_j are non-zero.

Proof. For the binary forms $V[2k]$ of even degree $2k$ there is a unique quadratic SL_2-invariant which has the form $A = \gamma_0 x_0 x_{2k} + \gamma_1 x_1 x_{2k-1} + \cdots + \gamma_k x_k^2 \in \Bbbk[x_0, \ldots, x_k]$ where all coefficients γ_i are non-zero (see [Sch68, Satz 2.6]; the invariant A is classically called "Apolare"). Now $\Bbbk[x_0, \ldots, x_{2k}] \subset \Bbbk[x_0, \ldots, x_n] = \mathcal{O}(V[n])$ is a $\mathbb{G}_a$-stable subalgebra, hence A is a quadratic $\mathbb{G}_a$-invariant in $\mathcal{O}(V[n])$ of weight $2n - 4k$, and so f_k is a multiple of A. $\square$

Proof of Theorem 6.1. (a) Denote by $\Bbbk^2 \simeq V[1]$ the standard representation of SL_2 and consider the closed embedding $V \hookrightarrow V \oplus \Bbbk^2$ given by $v \mapsto (v, e_1)$. Then we have the following diagram (see [Kra84, III.3.2]):

$$
\begin{array}{ccc}
V & \xrightarrow{\ \varphi\ } & W := V \oplus \Bbbk^2 \\
\pi \downarrow & & \downarrow \pi \\
V /\!\!/ \mathbb{G}_a & \xrightarrow{\ \simeq\ } & W /\!\!/ \mathrm{SL}_2
\end{array}
$$

In particular, $\mathcal{N}_V = \varphi^{-1}(\mathcal{N}_W) = \mathcal{N}_W \cap V$. The HILBERT-Criterion tells us that the elements $w = (v, a) \in \mathcal{N}_W$ are characterized by the condition that $0 \in \overline{\mathbb{G}_m \, gw}$ for a suitable $g \in \mathrm{SL}_2$ (see [Kra84, III.2.1]). This implies that $w = (v, e_1)$ belongs to $\mathcal{N}_W$ if and only if $0 \in \overline{\mathbb{G}_m v}$, i.e. if and only if $v \in V^+$.

(b) We first show that for every $v \in V \setminus (V^+ \oplus V_0)$ there is a homogeneous $\mathbb{G}_a$-invariant f of weight > 0 such that $f(v) \neq 0$. For that we can assume that V is irreducible, i.e., $V = V[n]$. We have $\mathcal{O}(V) = \Bbbk[x_0, x_1, \ldots, x_n]$, where x_i has weight $n - 2i$. Thus x_i vanishes on V^+ if and only if $2i \leq n$, and x_i vanishes on $V^+ \oplus V_0$ if and only if $2i < n$.

Now let $v = (a_0, a_1, \ldots, a_n) \in V \setminus (V^+ \oplus V_0)$, and let a_k be the first non-zero coefficient. Then the quadratic $\mathbb{G}_a$-invariant f_k from Lemma 6.3 above gives $f_k(v) = \alpha_k a_k^2 \neq 0$, and since $k < n/2$ the $\mathbb{G}_a$-invariant f_k has a positive weight.

It remains to show that every homogeneous $\mathbb{G}_a$-invariant f of weight > 0 vanishes on $V^+ \oplus V_0$. But this is clear, because every monomial $m = x_0^{d_0} x_1^{d_1} \cdots x_n^{d_n}$ of positive weight must contain an x_i of positive weight, i.e., with $2i < n$. Hence m vanishes on $V^+ \oplus V_0$.

(c) The same argument shows that a homogeneous SL_2-invariant restricted to $V^+ \oplus V^0$ does not depend on V^+. This implies that the induced morphism

$$\pi|_{\mathcal{P}_V} : \mathcal{P}_V \to \pi(\mathcal{P}_V) \subset V /\!\!/ \mathbb{G}_a$$

is given by the SL_2-invariants and has the following factorization

$$\mathcal{P}_V = V^+ \oplus V_0 \xrightarrow{\mathrm{pr}} V_0 \xrightarrow{\pi_{\mathrm{SL}_2}} \pi(\mathcal{P}_V) = \pi(V_0) \subset V /\!\!/ \mathrm{SL}_2$$

where $\pi_{\mathrm{SL}_2} : V \to V /\!\!/ \mathrm{SL}_2$ is the quotient by SL_2. Now the claim follows from the next lemma. $\square$

Lemma 6.4. *Let V be a representations of SL_2 and $\pi_{\mathrm{SL}_2} : V \to V /\!\!/ \mathrm{SL}_2$ the quotient. Then $\pi_{\mathrm{SL}_2}(V_0) \subset V /\!\!/ \mathrm{SL}_2$ is closed and the induced morphism $V_0 \to V /\!\!/ \mathrm{SL}_2$ has a factorization*

$$V_0 \xrightarrow{\pi_0} V_0/W \xrightarrow{\bar{\pi}} \pi_{\mathrm{SL}_2}(V_0) \subset V /\!\!/ \mathrm{SL}_2$$

where π_0 is the quotient by W and $\bar{\pi}$ is finite and bijective.

Proof. We first remark that the induced morphism $\pi' := \pi|_{V_0} : V_0 \to \overline{\pi(V_0)}$ is homogeneous and that $\pi'^{-1}(\pi(0)) = \{0\}$. Hence, $\pi(V_0) \subset V /\!\!/ \mathrm{SL}_2$ is closed and π' is finite. It remains to see that the fibers of π' are the W-orbits.

Since the orbits $\mathrm{SL}_2 v$ for $v \in V_0$ are closed, it suffices to show that we have $\mathrm{SL}_2 v \cap V_0 = Wv$ for all $v \in V_0$. One easily reduces to the case where $V = V[2n]$, and then $V_0 = \Bbbk x^n y^n$. If $g(x^n y^n) \in \Bbbk x^n y^n$ for some $g \in \mathrm{SL}_2$, then either $gx \in \Bbbk x$ and $gy \in \Bbbk y$, or $gx \in \Bbbk y$ and $gy \in \Bbbk x$. In the first case, g is diagonal and so $g(x^n y^n) = x^n y^n$ and we are done. In the second case, σg is diagonal, and we are again done. $\square$

Remark 6.5. We were informed by GERALD SCHWARZ that Lemma 6.4 holds for any representation V of a reductive group G. If $\pi : V \to V /\!\!/ G$ is the quotient, then the induced morphism $V_0 \to V /\!\!/ G$ is finite and has a factorization

$$\pi : V_0 \xrightarrow{\pi_W} V_0/W \xrightarrow{\bar{\pi}} \pi(V_0) \subset V /\!\!/ G$$

where π_W is the quotient by the Weyl group W and $\bar{\pi}$ is finite and bijective.

7. The Separating Variety

Definitions. In section 4, we discussed separating morphisms in the general context of a G-variety. We now introduce the *separating variety* $\mathcal{S}_X$ of a G-variety X, which measures how much the invariants separate the orbits. Set

$$\mathcal{S}_X := \{(x,y) \in X \times X \mid f(x) = f(y) \text{ for all } f \in \mathcal{O}(X)^G\} = \bigcup_{z \in X /\!/ G} \pi^{-1}(z) \times \pi^{-1}(z),$$

where $\pi\colon X \to X /\!/ G$ is the quotient morphism. The separating variety first appeared in work of Kemper [Kem03, Section 2]. More schematically, the separating variety of X is the reduced fiber product $(X \times_{X /\!/ G} X)_{\mathrm{red}}$ (cf. [Duf09, Definition 2.2]). If $Y \subset X$ is a G-stable subvariety, we write $\mathcal{S}_{X,Y} := \mathcal{S}_X \cap (Y \times Y)$.

The separating variety $\mathcal{S}_X$ contains the closure of the graph

$$\Gamma_X := \{(gx, x) \mid g \in G, x \in X\} = \bigcup_{x \in X} Gx \times Gx \subset X \times X.$$

Note that $\Gamma_X = \mathcal{S}_X$ exactly when the quotient π is almost geometric, i.e., when all non-empty fibers of π are orbits. Also, if Γ_X is closed, then all orbits are closed and have the same dimension. (The first statement is clear, and the second follows since $Gx \times \{x\} = p_2^{-1}(x)$ where $p_2\colon \Gamma_X \to X$ is the second projection.)

More generally, we have the following result, which is a first step to determine the closure $\overline{\Gamma_X}$ and to decide whether $\overline{\Gamma_X} = \mathcal{S}_X$. For simplicity, we assume that G is connected. This implies among other things that $\overline{\Gamma_X}$ is irreducible.

Proposition 7.1. *Let G be connected and X a normal affine G-variety. Assume that there is a dense open set $U \subset X /\!/ G$ such that $\varphi^{-1}(u)$ is non-empty and contains a dense orbit for all closed points $u \in U$. Set $X' := \pi^{-1}(U) \subset X$ and $P := X \setminus X'$.*

(a) *$\mathcal{S}_{X,P}$ is closed and $\mathcal{S}_X = \overline{\Gamma_X} \cup \mathcal{S}_{X,P}$. In particular, $\overline{\Gamma_X}$ is an irreducible component of $\mathcal{S}_X$.*

(b) *If $\pi^{-1}(u)$ is a single orbit for every closed point $u \in U$, then*

$$\mathcal{S}_X = \Gamma_{X'} \cup \mathcal{S}_{X,P} = \Gamma_X \cup \mathcal{S}_{X,P} = \overline{\Gamma_X} \cup \mathcal{S}_{X,P}.$$

(c) *Assume in addition that X' is smooth, that the G-action on X' is free, and that $\mathrm{codim}_X P > 1$. Then either Γ_X is closed, or $\overline{\Gamma_X} \setminus \Gamma_{X'}$ has codimension 1 in $\overline{\Gamma_X}$.*

Proof. (a) If $X /\!/ G$ is the disjoint union $O \cup A$, where U is open and A closed, then $\mathcal{S}_X = \mathcal{S}_{X,\pi^{-1}(O)} \cup \mathcal{S}_{X,\pi^{-1}(A)}$ where $\mathcal{S}_{X,\pi^{-1}(O)}$ is open, $\mathcal{S}_{X,\pi^{-1}(A)}$ is closed, and the union is disjoint. Take $(x,y) \in \mathcal{S}_{X,X'}$. Then $\pi(x) = \pi(y) =: u \in U$. By assumption, the fiber $\pi^{-1}(u)$ contains a dense orbit, say $\overline{Gz} = \pi^{-1}(u)$. Hence,

$$(x,y) \in \pi^{-1}(u) \times \pi^{-1}(u) = \overline{Gz} \times \overline{Gz} = \overline{Gz \times Gz} \subseteq \overline{\Gamma_{X'}} = \overline{\Gamma_X}.$$

It follows that $\mathcal{S}_X = \mathcal{S}_{X,X'} \cup \mathcal{S}_{X,P} = \overline{\Gamma_X} \cup \mathcal{S}_{X,P}$.

(b) Since the fibers over U are orbits, we get $\mathcal{S}_{X,X'} = \Gamma_{X'} = \Gamma_X \cap (X' \times X')$, and so

$$\mathcal{S}_X = \mathcal{S}_{X,X'} \cup \mathcal{S}_{X,P} = \Gamma_{X'} \cup \mathcal{S}_{X,P}.$$

The claim follows.

(c) Consider the morphism $\mu\colon G \times X \to X \times X$, $(g,x) \mapsto (gx, x)$, whose image is Γ_X. By assumption, it induces an isomorphism $\mu_0\colon G \times X' \xrightarrow{\sim} \Gamma_{X'}$, and thus, a birational morphism $\tilde{\mu}\colon G \times X \to \tilde{\Gamma}$, where $\tilde{\Gamma} \to \overline{\Gamma_X}$ is the normalization. If

$\operatorname{codim}_{\overline{\Gamma_X}} \overline{\Gamma}_X \setminus \Gamma_{X'} > 1$, then by IGUSA's criterion [Igu73] (cf. [Kra11, Appendix A, Proposition 5.12]), $\tilde{\mu}$ is an isomorphism, and so Γ_X is closed. $\qquad\square$

Remark 7.2. The first statement of the proposition above has the following converse: *If $\overline{\Gamma_X}$ is an irreducible component of $\mathcal{S}_X$, then the general fiber of $\pi\colon X \to X/\!\!/G$ contains a dense orbit.*

In order to see this, we can replace $X/\!\!/G$ be a dense open set and thus assume that $X/\!\!/G$ is affine algebraic, $\pi\colon X \to X/\!\!/G$ is flat, and the fibers are irreducible of dimension n. Then every irreducible component of $\mathcal{S}_X = X \times_{X/\!\!/G} X$ has dimension $2 \dim X - \dim X/\!\!/G = \dim X + n$ (see [Har77, Cor. 9.6 in Chap. III]). On the other hand, $\dim \overline{\Gamma_X} = \dim X + d$ where $d := \max\{\dim Gx \mid x \in X\}$. Hence $n = d$ and so the general fiber contains a dense orbit.

The case of $\mathbb{G}_a$-varieties. If X is a $\mathbb{G}_a$-variety, then by Proposition 5.2, the quotient $\pi\colon X \setminus \mathcal{P}_X \to \pi(X \setminus \mathcal{P}_X)$ is a $\mathbb{G}_a$-bundle. This implies the following corollary.

Corollary 7.3. *If X is a normal affine $\mathbb{G}_a$-variety, then*

$$\mathcal{S}_X = \Gamma_{X \setminus \mathcal{P}_X} \cup \mathcal{S}_{X,\mathcal{P}_X} = \Gamma_X \cup \mathcal{S}_{X,\mathcal{P}_X} = \overline{\Gamma_X} \cup \mathcal{S}_{X,\mathcal{P}_X},$$

and $\overline{\Gamma_X}$ is an irreducible component of $\mathcal{S}_X$.

In the remaining part of this section, we determine the irreducible components of $\mathcal{S}_V$ for a representation V of $\mathbb{G}_a$ (cf. [DK13], where this is done for indecomposable representations). We have seen in Theorem 6.1(c) that the image $\pi(\mathcal{P}_V) \subset V/\!\!/\mathbb{G}_a$ is closed and the induced morphism $\pi|_{\mathcal{P}_V}\colon \mathcal{P}_V \to \pi(\mathcal{P}_V)$ has a factorization

$$(*) \qquad \mathcal{P}_V = V^+ \oplus V_0 \xrightarrow{\ \operatorname{pr}\ } V_0 \xrightarrow{\ \pi_0\ } V_0/W \xrightarrow{\ \bar{\pi}\ } \pi(\mathcal{P}_V),$$

where π_0 is the quotient by W and $\bar{\pi}$ is finite and bijective. If $v \in \mathcal{P}_V = V_0 \oplus V^+$, we denote by v_0 the component of v in V_0. Define the following closed subsets of $\mathcal{S}_{\mathcal{P}_V}$:

$$C := \{(v, v') \in \mathcal{P}_V \times \mathcal{P}_V \mid v_0' = v_0\}, \quad C_\sigma := \{(v, v') \in \mathcal{P}_V \times \mathcal{P}_V \mid v_0' = \sigma(v_0)\}.$$

Both are irreducible and isomorphic to $V_0 \times (V^+ \times V^+)$. Now the factorization $(*)$ implies the following result.

Lemma 7.4.   (a) *If σ acts trivially on V_0, then $\mathcal{S}_{\mathcal{P}_V} = C = C_\sigma$ is irreducible.*
  (b) *If σ acts non-trivially on V_0, then $\mathcal{S}_{\mathcal{P}_V} = C \cup C_\sigma$ has two irreducible components.*
  In particular, $\mathcal{S}_{\mathcal{P}_V}$ is equidimensional of dimension $\dim V$.

Now we can formulate our main result about the separating variety $\mathcal{S}_V$.

Theorem 7.5. *We have $\mathcal{S}_V = \overline{\Gamma_V}$ if and only if the Weyl group acts trivially on V_0, or if $V = V[2] \oplus \mathbb{k}^m$. Otherwise, $\mathcal{S}_V$ has two irreducible components:*

$$\mathcal{S}_V = \overline{\Gamma_V} \cup C,$$

where $\dim \overline{\Gamma_V} = \dim V + 1$ and $\dim C = \dim V$.

Proof. We can assume that $V^{\mathrm{SL}_2} = (0)$. In fact, if $V = W \oplus \mathbb{k}^m$, then $\Gamma_V = \Gamma_W \times \mathbb{k}^m$ and $\mathcal{S}_V = \mathcal{S}_W \times \mathbb{k}^m$. It is easy to see that for $V = V[2]$ we have $\mathcal{S}_V = \overline{\Gamma_V}$. In all other cases, we have $\dim V^+ \geq 2$ which implies that the component C is not contained in Γ_V. On the other hand, $\overline{\Gamma_V} = \Gamma_V \cup C_\sigma$ by Lemma 7.6 below, and the claim follows from Lemma 7.4. $\qquad\square$

Lemma 7.6. *We have $\overline{\Gamma_V} = \Gamma_V \cup C_\sigma$.*

The proof needs some preparation. If X is a variety and R a $\Bbbk$-algebra, we define the R-valued points by $X(R) := \mathrm{Mor}(\mathrm{Spec}\, R, X)$. We have a canonical inclusion $X(\Bbbk[\![t]\!]) \subset X(\Bbbk(\!(t)\!))$ and a canonical map $X(\Bbbk[\![t]\!]) \to X(\Bbbk) = X$ which will be denoted by $x = x(t) \mapsto x(0) = x|_{t=0}$. We will constantly use the following fact. If $\varphi\colon X \to Y$ is a morphism and $y \in \overline{\varphi(X)}$, then there is an $x = x(t) \in X(\Bbbk(\!(t)\!))$ such that $\varphi(x) \in Y(\Bbbk[\![t]\!])$ and $\varphi(x)|_{t=0} = y$. Moreover, if $y \notin \varphi(X)$, then $x \notin X(\Bbbk[\![t]\!])$.

Proof of Lemma 7.6. We know from Proposition 7.1 that $E := \overline{\Gamma_V} \cap \mathcal{S}_{\mathcal{P}_V} = \overline{\Gamma_V} \setminus \Gamma_{V_{bd}}$ has codimension 1 in $\overline{\Gamma_V}$, hence E is either C, C_σ, or $C \cup C_\sigma$ by Lemma 7.4.

We now show that $\overline{\Gamma_V} \setminus \Gamma_V \subset C_\sigma$, which implies that $\overline{\Gamma_V} = \Gamma_V \cup C_\sigma$, hence the claim. Let $(v', v'') \in \overline{\Gamma_V} \setminus \Gamma_V \subset \mathcal{S}_{\mathcal{P}_V}$. Since Γ_V is the image of the morphism $\mu\colon \mathbb{G}_a \times V \to V \times V$, $(s, v) \mapsto (sv, v)$, there are element $s(t) \in \mathbb{G}_a(\Bbbk(\!(t)\!)) \setminus \mathbb{G}_a(\Bbbk[\![t]\!])$ and $v(t) \in V(\Bbbk[\![t]\!])$ such that the following holds:

(a) $v(0) = v''$;
(b) $s(t)v(t) \in V(\Bbbk[\![t]\!])$ and $(s(t)v(t))|_{t=0} = v'$.

If $v_0'' \in (V_0)^\sigma$, then $v_0' = v_0'' = \sigma v_0''$, and so $(v', v'') \in C_\sigma$. Thus we can assume that v_0'' is not fixed by σ, and we have to show that $v_0' = \sigma v_0'' = -v_0''$.

Now we use LUNA's Slice Theorem in the point v_0''. Denote by $T \subset \mathrm{SL}_2$ the diagonal matrices identified with $\mathbb{G}_m$ as above, and by $U \subset \mathrm{SL}_2$ the upper triangular unipotent matrices, which we can identify with $\mathbb{G}_a$. There is T-stable subspace $W \subset V$ containing v_0'' such that the morphism $\mu\colon \mathrm{SL}_2 *_T W \to V$ given by $\mu([g, w]) := gw$ is étale in a SL_2-saturated open neighborhood of $[e, v_0'']$ (see [Slo89]). Here the bundle $\mathrm{SL}_2 *_T W$ is the quotient $(\mathrm{SL}_2 \times W)/\!\!/T$ under the action $t(g, w) := (gt^{-1}, tw)$, and the quotient morphism $\mathrm{SL}_2 \times W \to \mathrm{SL}_2 *_T W$ is a principal T-bundle. This implies that we can lift the elements $v(t)$ and $s(t)v(t)$ to $\mathrm{SL}_2 \times W$, i.e., there are elements $g(t) \in \mathrm{SL}_2(\Bbbk[\![t]\!])$, $w(t) \in W(\Bbbk[\![t]\!])$ and $p(t) \in T(\Bbbk(\!(t)\!))$ such that the following holds:

(a') $g(t)w(t) = v(t)$, hence $g(0)w(0) = v''$;
(b') $\tilde{g}(t) := s(t)g(t)p(t)^{-1} \in \mathrm{SL}_2(\Bbbk[\![t]\!])$ and $\tilde{w}(t) := p(t)w(t) \in W(\Bbbk[\![t]\!])$, hence $\tilde{g}(t)\tilde{w}(t) = s(t)v(t)$ and $\tilde{g}(0)\tilde{w}(0) = v'$.

Setting

$$s(t) = \begin{bmatrix} 1 & f(t) \\ 0 & 1 \end{bmatrix}, \quad g(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}, \quad p(t) = \begin{bmatrix} r(t) & 0 \\ 0 & r(t)^{-1} \end{bmatrix},$$

where $f(t) \in \Bbbk(\!(t)\!) \setminus \Bbbk[\![t]\!]$, $a(t), b(t), c(t), d(t) \in \Bbbk[\![t]\!]$, and $r(t) \in \Bbbk(\!(t)\!)$, we get

$$\tilde{g}(t) = s(t)g(t)p(t)^{-1} = \begin{bmatrix} r^{-1}(a + fc) & (b + df)r \\ r^{-1}c & dr \end{bmatrix}.$$

Obviously, $p(t) \notin T(\Bbbk[\![t]\!])$, since $s(t) \notin U(\Bbbk[\![t]\!])$. Thus either $r(t) \in t\Bbbk[\![t]\!]$ and $c(0) = 0$, or $r(t)^{-1} \in t\Bbbk[\![t]\!]$ and $d(0) = 0$. In the first case we get

$$(1) \qquad g(0) \in \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \in \mathrm{SL}_2 \right\} =: B \quad \text{and} \quad \tilde{g}(0) \in \left\{ \begin{bmatrix} * & * \\ * & 0 \end{bmatrix} \in \mathrm{SL}_2 \right\} = B\sigma,$$

and in the second

$$(2) \qquad\qquad\qquad g(0) \in B\sigma \quad \text{and} \quad \tilde{g}(0) \in B.$$

Moreover, since $\tilde{w}(t) = p(t)w(t)$, we get $\tilde{w}(0)_0 = w(0)_0$. Also note that for any $b \in B$ and $u \in V_0 \oplus V^+$ we have $(bu)_0 = u_0$.

Assume now that we are in case (1). Since $g(0)w(0) = v'' \in V_0 \oplus V^+$, we get $w(0) \in V_0 \oplus V^+$, hence $w(0)_0 = (g(0)w(0))_0 = v_0''$. On the other hand, $\tilde{g}(0) \in B\sigma$ and $\tilde{g}(0)\tilde{w}(0) = v' \in V_0 \oplus V^+$, hence $\sigma\tilde{w}(0) \in V_0 \oplus V^+$ and $(\sigma\tilde{w}(0))_0 = v_0'$. Thus $v_0' = \sigma\,\tilde{w}(0)_0 = -\tilde{w}(0)_0 = -w(0)_0 = -v_0''$, i.e. $(v', v'') \in C_\sigma$, and the claim follows. Case (2) is similar. $\qquad\square$

8. $\mathbb{G}_a$-ACTIONS ON SL_2-VARIETIES

In this section, we generalize some of the results obtained for representations of $\mathbb{G}_a$ to affine SL_2-varieties. As in section 6 we identify $\mathbb{G}_a$ with the unipotent subgroup $U \subset \mathrm{SL}_2$ via $s \mapsto \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$, and $\mathbb{G}_m$ with the maximal torus $T \subset \mathrm{SL}_2$ via $t \mapsto \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$. Thus every SL_2-variety X can be regarded as a $\mathbb{G}_a$-variety. These $\mathbb{G}_a$-varieties have some very special properties, e.g. the following classical result which was already used in the proof of Theorem 6.1 (see [Kra84, III.3.2]).

Lemma 8.1. *Let X be an affine SL_2-variety and denote by $\Bbbk^2$ the standard representation of SL_2. Then the closed $\mathbb{G}_a$-equivariant embedding $X \hookrightarrow X \times \Bbbk^2$, $x \mapsto (x, e_1)$, induces an isomorphism $X /\!\!/ \mathbb{G}_a \xrightarrow{\sim} (X \times \Bbbk^2) /\!\!/ \mathrm{SL}_2$. In particular, the $\mathbb{G}_a$-invariants $\mathcal{O}(X)^{\mathbb{G}_a}$ are finitely generated.*

An immediate consequence is that for every closed embedding $X \hookrightarrow Y$ of affine SL_2-varieties the induced map $X /\!\!/ \mathbb{G}_a \to Y /\!\!/ \mathbb{G}_a$ is also a closed embedding.

Proposition 8.2. *Let V be a representation of SL_2 and $X \subset V$ a closed SL_2-stable subset.*
  (a) *$\mathcal{S}_X = \mathcal{S}_V \cap (X \times X)$.*
  (b) *For any $v \in (V_0 \oplus V^+) \cap X$ we have $v_0 \in X$.*
  (c) *$\mathcal{P}_X = \mathcal{P}_V \cap X$. More precisely, the image of the plinth ideal $\mathfrak{p}_V$ under the restriction map is the plinth ideal $\mathfrak{p}_X$.*
  (d) *$\mathcal{S}_{X,\mathcal{P}_X} = \mathcal{S}_{V,\mathcal{P}_V} \cap (X \times X)$.*

Proof. (a) The inclusion $\mathcal{S}_X \subseteq \mathcal{S}_V \cap (X \times X)$ is obvious. Take $(x, x') \in \mathcal{S}_V \cap (X \times X)$. We have $f(x) = f(x')$ for all $f \in \mathcal{O}(V)^{\mathbb{G}_a}$. Since every element in $\mathcal{O}(X)^{\mathbb{G}_a}$ is the restriction to X of an element in $\mathcal{O}(V)^{\mathbb{G}_a}$, we get $h(x) = h(x')$ for all $h \in \mathcal{O}(X)^{\mathbb{G}_a}$, and so $(x, x') \in \mathcal{S}_X$.

(b) Note that $v_0 \in \overline{\mathbb{G}_m v}$, and the claim follows, since X is closed and SL_2-stable.

(c) The restriction map $\mathcal{O}(V) \to \mathcal{O}(X)$ is SL_2-equivariant and so the image of an irreducible SL_2-subrepresentation $W \subset \mathcal{O}(V)$ is either (0) or isomorphic to W. Therefore, the generators of $\mathfrak{p}_V$ are mapped onto the generators of $\mathfrak{p}_X$.

(d) This is clear from what has been said so far. $\qquad\square$

Proposition 8.3. *Let V be a representation of SL_2 and $X \subset V$ a closed SL_2-stable subset. Set $X_0 := X^{\mathbb{G}_m} = X \cap V_0$. Then the following are equivalent:*
  (i) *$\mathcal{S}_X = \overline{\Gamma_X}$;*
  (ii) *$\overline{\Gamma_X} = \overline{\Gamma_V} \cap (X \times X)$ and $(x_0 + V^+) \cap X = \mathbb{G}_a x_0$ for all $x_0 \in X_0 \setminus (X_0)^\sigma$.*

Proof. Since $\Gamma_X = \Gamma_V \cap X$ and $\overline{\Gamma_V} = \Gamma_V \cup C_\sigma$ (Lemma 7.6), we get
$$\overline{\Gamma_X} \subseteq \overline{\Gamma_V} \cap (X \times X) = \Gamma_X \cup (C_\sigma \cap (X \times X)) \subseteq \mathcal{S}_X,$$

and from $\mathcal{S}_V = \Gamma_V \cup C_\sigma \cup C$ we obtain

$$\mathcal{S}_X = \mathcal{S}_V \cap (X \times X) = \Gamma_X \cup (C_\sigma \cap (X \times X)) \cup (C \cap (X \times X)).$$

Therefore, $\overline{\Gamma_X} = \mathcal{S}_X$ if and only if $\overline{\Gamma_X} \supseteq \overline{\Gamma_V} \cap (X \times X)$ and $(C \setminus C_\sigma) \cap (X \times X) \subset \Gamma_X$. But the latter condition is clearly equivalent to $(x_0 + V^+) \cap X = \mathbb{G}_a x_0$ for all $x_0 \in X_0 \setminus (X_0)^\sigma$. $\qquad\square$

Example 8.4. Let $X := \mathrm{SL}_2 / T$ where $T \subset \mathrm{SL}_2$ is the group of diagonal matrices acting by right multiplication on SL_2. This variety is the so-called DANIELEWSKI surface, i.e., the smooth 2-dimensional affine quadric $X = \mathcal{V}(xz - y^2 + y) \subset \Bbbk^3$ (cf. [DP09]), and the quotient map is given by

$$\pi_{\mathrm{SL}_2} \colon \mathrm{SL}_2 \to X, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (ab, ad, cd).$$

Clearly, X is an SL_2-variety where the action is induced by left multiplication on SL_2, and thus a $\mathbb{G}_a$-variety. The quotient by $\mathbb{G}_a$ is $\mathbb{A}^1$, and the quotient map is given by

$$\mathrm{SL}_2 / T \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} T \mapsto cd, \quad \text{i.e. } X \ni (x, y, z) \mapsto z.$$

The plinth ideal is generated by z and is reduced. The plinth variety $\mathcal{P}_X$ consists of the two orbits $O_1 := UT$ and $O_2 := U\sigma T$ where $\sigma := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and so $X_{alg} = X \setminus (O_1 \cup O_2)$. Moreover, the induced morphisms $X \setminus O_i \to \mathbb{A}^1$ are both trivial $\mathbb{G}_a$-bundles, and so $X \setminus O_i \simeq \mathbb{A}^2$ for $i = 1, 2$. Thus $X_{bd} = X$, but $\pi \colon X \to \mathbb{A}^1$ is not a $\mathbb{G}_a$-bundle, because $\pi^{-1}(0) = O_1 \cup O_2$. It follows that

$$\Gamma_X = \bigcup_{O \text{ orbit}} O \times O \text{ is open in } \mathcal{S}_X = \Gamma_X \cup (O_1 \times O_2) \cup (O_2 \times O_1).$$

Since $\mathcal{S}_X \subset X \times X$ is the hypersurface defined by $f := \pi \circ \mathrm{pr}_1 - \pi \circ \mathrm{pr}_2$. Finally, it follows from KRULL's Principal Ideal Theorem (see [Kra11, Appendix A, Theorem 3.13] or [Eis95, Chapter II, Theorem 10.1]) that $\mathcal{S}_X = \overline{\Gamma_X}$ is irreducible.

Example 8.5. Now let us look at $Y := \mathrm{SL}_2 / N$, where $N = T \cup \sigma T$ is the normalizer of T. Then σ induces an automorphism of order 2 on $X = \mathrm{SL}_2 / T$ commuting with the $\mathbb{G}_a$-action, and the automorphism $-\,\mathrm{id}$ on the quotient $X /\!\!/ \mathbb{G}_a = \mathbb{A}^1$. Thus $Y = X / \langle \sigma \rangle$ and $Y /\!\!/ \mathbb{G}_a = \mathbb{A}^1 / \{\pm \mathrm{id}\} \simeq \mathbb{A}^1$. Since $\sigma(O_1) = O_2$ in the notation of Example 8.4 we see that the plinth variety $\mathcal{P}_Y = \pi^{-1}(0)$ is a single orbit, but the plinth ideal $\mathfrak{p}_Y$ is not prime. Therefore, $\pi \colon Y \to \mathbb{A}^1$ is a geometric quotient, but not a principal $\mathbb{G}_a$-bundle. In this case, $Y_{bd} = Y_{alg} = X \setminus \mathcal{P}_Y$, and $\mathcal{S}_Y = \Gamma_Y$.

9. ROBERTS' EXAMPLE

In this section we discuss ROBERTS' counterexample to HILBERT's fourteenth problem ([Rob90], cf. [AN94]). We assume that char $\Bbbk = 0$ and define an action of the additive group $\mathbb{G}_a$ on $\mathbb{A}^7$ as follows:

$$s \cdot (a_1, a_2, a_3, b_1, b_2, b_3, c) := (a_1, a_2, a_3, b_1 + sa_1^3, b_2 + sa_2^3, b_3 + sa_3^3, c + s(a_1 a_2 a_3)^2).$$

It corresponds to the locally nilpotent vector field

$$D := x_1^3 \frac{\partial}{\partial y_1} + x_2^3 \frac{\partial}{\partial y_2} + x_3^3 \frac{\partial}{\partial y_3} + (x_1 x_2 x_3)^2 \frac{\partial}{\partial z},$$

where we use the coordinates $\mathcal{O}(\mathbb{A}^7) = \Bbbk[x_1, x_2, x_3, y_1, y_2, y_3, z]$. Put $A := \mathcal{O}(\mathbb{A}^7)^{\mathbb{G}_a}$ and denote the quotient morphism by $\pi \colon \mathbb{A}^7 \to \mathbb{A}^7 /\!\!/ \mathbb{G}_a := \operatorname{Spec}(A)$.

The x_i are invariants, and $D(y_i) = x_i^3$, hence $x_i^3 \in \mathfrak{p}_{\mathbb{A}^7}$ and $(x_1, x_2, x_3) \subset \mathfrak{f}_{\mathbb{A}^7 /\!\!/ \mathbb{G}_a}$. It follows that $\mathbb{A}^7_{x_i} \to (\mathbb{A}^7 /\!\!/ \mathbb{G}_a)_{x_i}$ is a trivial $\mathbb{G}_a$-bundle for $i = 1, 2, 3$. This allows to find the following additional invariants:

$$(*) \qquad u_{12} := x_1^3 y_2 - x_2^3 y_1, \ u_{13} := x_1^3 y_3 - x_3^3 y_1, \ u_{23} := x_2^3 y_3 - x_3^3 y_2,$$
$$\beta_{1,1} := x_1 z - x_2^2 x_3^2 y_1, \ \beta_{2,1} := x_2 z - x_1^2 x_3^2 y_2, \ \beta_{3,1} := x_3 z - x_1^2 x_2^2 y_3.$$

Define the following subalgebras of the ring of invariants A:

$$A_0 := \Bbbk[x_1, x_2, x_3, u_{12}, u_{13}, u_{23}] \subset A_1 := A_0[\beta_{1,1}, \beta_{2,1}, \beta_{3,1}] \subset A.$$

We then have $(A_1)_{x_i} = \mathcal{O}(\mathbb{A}^7)^{\mathbb{G}_a}_{x_i} = \mathcal{O}(\mathbb{A}^7_{x_i})^{\mathbb{G}_a}$. Using a symbolic computation software like SINGULAR [DGPS12], it is easy to see that $Y_0 := \operatorname{Spec} A_0 \subset \mathbb{A}^6$ is the normal hypersurface defined by the equation $x_1^3 u_{12} + x_2^3 u_{13} + x_3^3 u_{23} = 0$, and that $Y_1 := \operatorname{Spec} A_1 \subset \mathbb{A}^9$ has dimension 6 and its ideal $I(Y_1)$ is generated by the following 5 functions:

$$x_1^2 u_{12} - x_3 \beta_{2,1} + x_2 \beta_{3,1}, \ x_2^2 u_{13} - x_1 \beta_{3,1} + x_3 \beta_{1,1}, \ x_3^2 u_{23} - x_2 \beta_{1,1} + x_1 \beta_{2,1},$$
$$u_{12} u_{13} u_{23}(x_1 x_2 \beta_{3,1} + x_2 x_3 \beta_{1,1} + x_3 x_1 \beta_{2,1}) + u_{12} \beta_{1,1}^3 + u_{13} \beta_{2,1}^3 + u_{23} \beta_{3,1}^3,$$
$$x_1 x_2 x_3 u_{12} u_{13} u_{23} + x_1 u_{12} \beta_{1,1}^2 + x_2 u_{13} \beta_{2,1}^2 + x_3 u_{23} \beta_{3,1}^2.$$

Lemma 9.1. *The variety Y_1 is normal.*

Proof. Again, using for example SINGULAR [DGPS12], one verifies that the ideal $x_1 A_1$ is radical. Let $f \in Q(A_1)$ be integral over A_1, that is, suppose f satisfies an equation

$$f^d = a_1 f^{d-1} + a_2 f^{d-2} + \cdots + a_d,$$

where $a_i \in A_1$. Since $(A_1)_{x_1}$ is normal, we have $x_1^m f \in A_1$ for some $m \geq 0$. We choose a minimal m with this property. It follows from the equation above that $(x_1^m f)^d \in x_1 A_1$, hence $x_1^m f \in x_1 A_1$, and thus $f \in A_1$, because of the minimality of m. $\qquad \square$

The action of $\mathbb{G}_a$ on $\mathbb{A}^7$ commutes with the $(\mathbb{G}_m)^3$-action with weights

$$(1, 0, 0), (0, 1, 0), (0, 0, 1), (3, 0, 0), (0, 3, 0), (0, 0, 3), (2, 2, 2),$$

and so $(\mathbb{G}_m)^3$ also acts on $\mathbb{A}^7 /\!\!/ \mathbb{G}_a$. As the polynomials in $(*)$ are multi-homogeneous, $(\mathbb{G}_m)^3$ also acts on Y_0 and Y_1.

The following propositions collects the main properties of $\pi \colon \mathbb{A}^7 \to \mathbb{A}^7 /\!\!/ \mathbb{G}_a$. Most statements follow immediately from what we have done so far. The difficult part is the description of the finite generation ideal $\mathfrak{f}_{X /\!\!/ \mathbb{G}_a}$. Recall that $\mathcal{P}_{\mathbb{A}^7} \subset \mathbb{A}^7$ denotes the plinth variety (see Definition 5.1) and $\mathcal{S}_{\mathbb{A}^7} \subset \mathbb{A}^7 \times \mathbb{A}^7$ the separating variety (see section 7).

Proposition 9.2. $\qquad$ (a) $\mathcal{P}_{\mathbb{A}^7} = (\mathbb{A}^7)^{\mathbb{G}_a} = \mathcal{V}_{\mathbb{A}^7}(x_1, x_2, x_3) \simeq \mathbb{A}^4$, *and*

$$(\mathbb{A}^7)_{bd} = \mathbb{A}^7 \setminus \mathcal{P}_{\mathbb{A}^7} = (\mathbb{A}^7)_{x_1} \cup (\mathbb{A}^7)_{x_2} \cup (\mathbb{A}^7)_{x_3}.$$

(b) $\pi((\mathbb{A}^7)^{\mathbb{G}_a}) = \{\pi(0)\}$, *and*

$$\pi(\mathbb{A}^7) = (\mathbb{A}^7 /\!\!/ \mathbb{G}_a)_{x_1} \cup (\mathbb{A}^7 /\!\!/ \mathbb{G}_a)_{x_2} \cup (\mathbb{A}^7 /\!\!/ \mathbb{G}_a)_{x_3} \cup \{\pi(0)\} = \pi((\mathbb{A}^7)_{bd}) \cup \{\pi(0)\}.$$

(c) *The separating variety $\mathcal{S}_{\mathbb{A}^7}$ has two irreducible components:*

$$\mathcal{S}_{\mathbb{A}^7} = \overline{\Gamma_{\mathbb{A}^7}} \cup (\mathcal{P}_{\mathbb{A}^7} \times \mathcal{P}_{\mathbb{A}^7}),$$
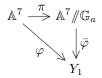
both of dimension 8.

(d) *We have $\mathfrak{f}_{X /\!\!/ \mathbb{G}_a} = \sqrt{(x_1, x_2, x_3)}$, and so*

$$(\mathbb{A}^7 /\!\!/ \mathbb{G}_a)_{alg} = (\mathbb{A}^7 /\!\!/ \mathbb{G}_a)_{x_1} \cup (\mathbb{A}^7 /\!\!/ \mathbb{G}_a)_{x_2} \cup (\mathbb{A}^7 /\!\!/ \mathbb{G}_a)_{x_3} = \pi((\mathbb{A}^7)_{bd}).$$

In particular, $(\mathbb{A}^7 /\!\!/ \mathbb{G}_a)_{alg}$ is algebraic.

(e) $\mathbb{A}^7 /\!\!/ \mathbb{G}_a \setminus (\mathbb{A}^7 /\!\!/ \mathbb{G}_a)_{alg} = \mathcal{V}_{\mathbb{A}^7 /\!\!/ \mathbb{G}_a}(x_1, x_2, x_3) \simeq \mathbb{A}^3$ *has codimension 3 in $\mathbb{A}^7 /\!\!/ \mathbb{G}_a$.*

(f) $\mathbb{A}^7 /\!\!/ \mathbb{G}_a$ *is Jacobson, and its closed points coincide with its rational points.*

The inclusion $A_1 \subset A$ defines an invariant morphism $\varphi \colon \mathbb{A}^7 \to Y_1$ which factors through the quotient π:

$$
\begin{array}{ccc}
\mathbb{A}^7 & \xrightarrow{\ \pi\ } & \mathbb{A}^7 /\!\!/ \mathbb{G}_a \\
& {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\bar\varphi} \\
& & Y_1
\end{array}
$$

Proposition 9.3. (a) $\overline{\varphi}$ *induces an isomorphism $\pi((\mathbb{A}^7)_{bd}) \xrightarrow{\sim} \varphi((\mathbb{A}^7)_{bd})$.*

(b) Y_1 *is normal and $\overline{\varphi} \colon \mathbb{A}^7 /\!\!/ \mathbb{G}_a \to Y_1$ is injective on $\pi(\mathbb{A}^7)$. In particular, φ is a separating morphism.*

(c) $\mathcal{O}(\mathbb{A}^7)^{\mathbb{G}_a} = \mathcal{O}_{Y_1}(Y_1 \setminus \mathcal{V}_{Y_1}(x_1, x_2, x_3))$.

A proof that φ is a separating morphism and that (c) holds already appeared in [Duf13, Example 4.2].

Proof of Proposition 9.2(a)–(c). We have $\pi^{-1}(\mathcal{V}_{\mathbb{A}^7 /\!\!/ \mathbb{G}_a}(x_1, x_2, x_3)) = \pi^{-1}(\pi(0))$, implying (a). As $\overline{\pi(\pi^{-1}(\mathcal{V}_{\mathbb{A}^7 /\!\!/ \mathbb{G}_a}(x_1, x_2, x_3)))} = \{\pi(0)\}$, (b) follows. Finally, statement (c) follows from (a) and Corollary 7.3. $\square$

The proofs the remaining statements (d)–(f) need some preparation. They will be given at the end of the section.

Proof of Proposition 9.3. Statement (a) holds since $(\mathbb{A}^7 /\!\!/ \mathbb{G}_a)_{x_i} \cong (Y_1)_{x_i}$. We have seen in Lemma 9.1 that Y_1 is normal and the morphism $\overline{\varphi}$ is injective on $\pi(\mathbb{A}^7)$, since $\overline{\varphi}(\pi(x_0)) = \varphi(x_0) \in \mathcal{V}_{Y_1}(x_1, x_2, x_3)$, proving (b). Finally, (c) follows from Theorem 4.3(d) since $Y_1 \setminus \varphi(\mathbb{A}^7) = \mathcal{V}_{Y_1}(x_1, x_2, x_3) \setminus \{\varphi(0)\}$. $\square$

To prove that $\mathcal{O}(\mathbb{A}^7)^{\mathbb{G}_a}$ is not finitely generated, ROBERTS showed in [Rob90, Lemma 3] that there exist invariants of the form

$$x_i z^n + \text{terms of lower } z\text{-degree}$$

for $i = 1, 2, 3$ and $n \geq 0$. Later, KURODA proved (see [Kur04, Theorem 3.3]) that any set S of such invariants, together with u_{12}, u_{13}, u_{23}, forms a SAGBI-basis for the lexicographic monomial ordering with $x_1 \prec x_2 \prec x_3 \prec y_1 \prec y_2 \prec y_3 \prec z$. We will improve this statement in Lemma 9.5 below.

Recall that if R is a subalgebra of a polynomial ring, then for a given monomial ordering, a *SAGBI-basis* is a subset $S \subset R$ such that $\Bbbk[\mathrm{LT}(S)] = \Bbbk[\mathrm{LT}(R)]$ where $\mathrm{LT}(S)$ denotes the set of leading terms of the polynomials in S (see [RS90]). Such

a basis always generates R. Note that for KURODA's SAGBI-bases S defined above we always have

$$\mathrm{LT}(S) = \{x_1^3 y_2, x_1^3 y_3, x_2^3 y_3, x_j z^n \mid j \in \{1, 2, 3\}, n \geq 0\}.$$

Lemma 9.4. *There exist invariants $\beta_{i,n}$ for $n \geq 0$ and $i = 1, 2, 3$ which are multi-homogeneous and of the form*

$$(*) \quad \beta_{i,n} = x_i z^n - n x_j^2 x_k^2 y_i z^{n-1} +$$
$$+ \binom{n}{2}(x_i^2 x_j^4 x_k y_i y_k + x_i^2 x_j x_k^4 y_i y_j - x_i^5 x_j x_k y_j y_k) z^{n-2} +$$
$$+ \text{ terms of lower } z\text{-degree}$$

Proof. By symmetry it suffices to look at the case $i = 1$. We know from [Rob90, Lemma 3] that invariants $\beta_{1,n}$ with leading term $x_1 z^n$ exist, and we can clearly assume that they are multi-homogeneous of degree $(2n + 1, 2n, 2n)$, hence

$$\beta_{1,n} = x_1 z^n + f_1 z^{n-1} + f_2 z^{n-2} + \text{ terms of lower } z\text{-degree}$$

where $f_1, f_2 \in \Bbbk[x, y] := \Bbbk[x_1, x_2, x_3, y_1, y_2, y_3]$, $\deg f_1 = (3, 2, 2)$ and $\deg f_2 = (5, 4, 4)$. From $D(\beta_{1,n}) = 0$ we get the following differential equations

$$D(f_1) = -n x_1 D(z) = -n x_1^3 x_2^2 x_3^2$$
$$D(f_2) = -(n-1) f_1 D(z) = -(n-1) x_1^2 x_2^2 x_3^2 f_1,$$

which have the special solutions

$$h_1 := -n x_2^2 x_2^2 y_1, \text{ and } \quad h_2 := \binom{n}{2}(x_1^2 x_2^4 x_3 y_1 y_3 + x_1^2 x_2 x_3^4 y_1 y_2 - x_1^5 x_2 x_3 y_2 y_3).$$

An easy calculations shows that $\ker D \cap \Bbbk[x, y]_{(3,2,2)} = \Bbbk x_1^3 x_2^2 x_3^2$, and so $f_1 = h_1 + c x_1^3 x_2^2 x_3^2$ for some $c \in \Bbbk$. But then we may replace $\beta_{1,n}$ by $\beta_{1,n} - c x_1^2 x_2^2 x_3^2 \beta_{1,n-1}$, which has the form $x_1 z^n - n x_j^2 x_k^2 y_i z^{n-1} + $ terms of lower z-degree. Thus we can assume that $f_1 = h_1$, and hence $f_2 = h_2 + c_2$, where $D(c_2) = 0$. It is not difficult to see that

$$\ker D \cap \Bbbk[x, y]_{(5,4,4)} = \Bbbk x_1^5 x_2^4 x_3^4 \oplus \Bbbk x_1^2 x_2 x_3^4 u_{12} \oplus \Bbbk x_1^2 x_2^4 x_3 u_{13}$$

Subtracting from $\beta_{1,n}$ a suitable linear combination of the invariants $x_1^4 x_2^4 x_3^4 \beta_{1,n-2}$, $x_1 x_2 x_3^4 u_{12} \beta_{1,n-2}$ and $x_1 x_2^4 x_3 u_{13} \beta_{1,n-2}$, we can assume that $f_2 = h_2$, and the claim follows. $\square$

Define $S_N := \{u_{12}, u_{13}, u_{23}, \beta_{i,n} \mid i = 1, 2, 3, \text{ and } 0 \leq n \leq N\}$ and set $A_N := \Bbbk[S_N] \subset A$ for all $N \geq 0$, extending our definition of the subalgebras A_0 and A_1 above. One easily sees that A_0 is the ring formed by the invariants of z-degree 0, that is, the invariants of the induced $\mathbb{G}_a$-action on the hyperplane $\mathcal{V}_{\mathbb{A}^7}(z) \subset \mathbb{A}^7$. The A_N for $N \geq 1$ yield a family of separating morphisms $\varphi_N \colon \mathbb{A}^7 \to Y_N := \mathrm{Spec}(A_N)$, and, by KURODA's result mentioned above, we have $A = \bigcup_N A_N$.

The following lemma is crucial.

Lemma 9.5. *For all $N \geq 0$ the subalgebra $\Bbbk[\mathrm{LT}(A_N)] \subset A$ is generated by $\mathrm{LT}(S_N)$. Equivalently, S_N is a SAGBI-basis of A_N.*

Proof. Put $b_{i,n} := \mathrm{LT}(\beta_{i,n})$ and $m_{ij} := \mathrm{LT}(u_{ij})$:

$$b_{i,n} = x_i z^n, \quad m_{12} = x_1^3 y_2, \quad m_{13} = x_1^3 y_3, \quad m_{23} = x_2^3 y_3.$$

(a) We first claim that the relations between these leading term are generated by

$$b_{1,n}b_{1,m}b_{1,k}m_{23} - b_{2,n}b_{2,m}b_{2,k}m_{13} = 0 \text{ where } 0 \le n \le m \le k \le N, \text{ and}$$

$$b_{i,n}b_{j,m} - b_{i,n'}b_{j,m'} = 0 \text{ where } 0 \le n \le m \le N, \ m+n = m'+n' \ge 1, \ i,j \in \{1,2,3\}.$$

This is not difficult and we leave the details to the reader.

(b) It remains to show that, when we substitute the polynomials $\beta_{i,n}$ defined in Lemma 9.4 in the relations above, the leading term of the result belongs to $\Bbbk[\mathrm{LT}(S_N)]$, that is:

$$\mathrm{LT}(\beta_{1,n}\beta_{1,m}\beta_{1,k}u_{23} - \beta_{2,n}\beta_{2,m}\beta_{2,k}u_{13}) \in \Bbbk[\mathrm{LT}(S_N)], \text{ and}$$
$$\mathrm{LT}(\beta_{i,n}\beta_{j,m} - \beta_{i,n'}\beta_{j,m'}) \in \Bbbk[\mathrm{LT}(S_N)].$$

(b1) A simple computation shows that $\beta_{1,n}\beta_{1,m}\beta_{1,k}u_{23} - \beta_{2,n}\beta_{2,m}\beta_{2,k}u_{13}$ has z-degree $n + m + k$ and leading term $-x_1^3x_3^3y_2z^{m+n+k} = -b_{3,n}b_{3,m}b_{3,k}m_{12}$, which is indeed in $\Bbbk[\mathrm{LT}(S_N)]$.

(b2) A similar computation shows that $\beta_{1,n}\beta_{2,m} - \beta_{1,n'}\beta_{2,m'}$ has z-degree $n+m-1$ and leading term $(n-n')x_1^3x_3^2y_2z^{n+m-1} = (n-n')b_{3,n}b_{3,m-1}m_{12}$, which also belongs to $\Bbbk[\mathrm{LT}(S_N)]$.

(b3) It remains to consider $\beta_{1,n}\beta_{1,m} - \beta_{1,n'}\beta_{1,m'}$. For $m + n \le 1$ this expression is 0, and for $m + n \ge 2$ it has z-degree $m + n - 2$ and leading term $(nm - n'm')x_1^6x_2x_3y_2y_3z^{n+m-2}$ which is equal to $(mn-m'n')m_{12}m_{13}b_{2,n-1}b_{3,m-1}$ if $n > 0$ and to $(-m'n')m_{12}m_{13}b_{2,0}b_{3,n-2}$ if $n = 0$, with both belonging to $\Bbbk[\mathrm{LT}(S_N)]$. $\qquad\square$

For subalgebras $B_1 \subset B_2 \subset A$ the *conductor* is defined as usual by $[B_1 : B_2] := \{b \in B_2 \mid bB_2 \subset B_1\}$.

Lemma 9.6. (a) *If $f \in A$ and $\deg_z f \le N$, then $f \in A_N$.*
(b) *$(x_1, x_2, x_3)A_{N+1} \subseteq A_N$.*
(c) *$[A_N : A_{N+1}] \cap A_0 = (x_1, x_2, x_3)A_0$.*

Proof. (a) This statement is clear for $N = 0$. If $\deg_z f = N > 0$, then $\mathrm{LT}(f)$ is a monomial in $\mathrm{LT}(S)$ of z-degree N, and thus a monomial in $\mathrm{LT}(S_N)$. Now Lemma 9.5 implies that $\mathrm{LT}(f) = \mathrm{LT}(\tilde f)$ for some $\tilde f \in A_N$. Thus $\deg_z(f - \tilde f) < N$, and the claim follows by induction.

(b) We have $\mathrm{LT}(x_i\beta_{j,N+1}) = \mathrm{LT}(\beta_{i,1}\beta_{j,N})$, and so $\deg_z(x_i\beta_{j,N+1}-\beta_{i,1}\beta_{j,N}) \le N$, hence $(x_i\beta_{j,N+1} - \beta_{i1}\beta_{j,N}) \in A_N$ by (a), and thus $x_i\beta_{j,N+1} \in A_N$.

(c) Assume that $fA_{N+1} \subset A_N$ for some $f \in A_0$. Then $f\beta_{i,N+1} \in A_N$ for all i, hence $\mathrm{LT}(f\beta_{i,N+1}) \in \mathrm{LT}(A_N)$. Thus $\mathrm{LT}(f\beta_{i,N+1}) = \mathrm{LT}(f)x_iz^{N+1}$ is a monomial in $\mathrm{LT}(S_N)$. It follows that this monomial contains at least two factors of the form $x_jz^n = \mathrm{LT}(\beta_{j,n})$. This implies that $\mathrm{LT}(f)$, as a monomial in $\mathrm{LT}(S_0)$, contains a factor x_j. Hence, $\mathrm{LT}(f) = x_j\mathrm{LT}(\tilde f)$ for some $\tilde f \in A_0$, and so $f - x_j\tilde f \prec f$. Now the claim follows by induction since $x_jA_{N+1} \subset A_N$, by (b). $\qquad\square$

Lemma 9.7. *If $f \in A$ is a multi-homogeneous invariant whose multi-degree is not congruent to (k, k, k) modulo 3, then $f^2 \in (x_1, x_2, x_3)$. In particular, $\beta_{j,n}^2 \in (x_1, x_2, x_3)A$ for all $j \in \{1, 2, 3\}$, $n \ge 0$. Moreover, the radical $\mathfrak{p} := \sqrt{(x_1, x_2, x_3)A}$ is generated by $\{\beta_{i,n}\}$, and $A/\mathfrak{p}$ is a polynomial ring in 3 variables.*

Proof. By induction, it suffices to show that $\mathrm{LT}(f^2) = \mathrm{LT}(h)$ where $h \in (x_1, x_2, x_3)$. But $\mathrm{LT}(f)$, as a monomial in $\mathrm{LT}(S)$, must contain a factor of the form x_i or x_iz

since otherwise the multi-degree is congruent to (k, k, k) modulo 3. Hence, $\mathrm{LT}(f^2)$ contains a factor x_i, and so $\mathrm{LT}(f^2) = \mathrm{LT}(x_i p)$ for some $p \in A$.

Next we remark that $u_i \notin \mathfrak{p}$, for all i. In fact, if $u_i^k \in (x_1, x_2, x_3)A$, then $\mathrm{LT}(u_i^k) = \mathrm{LT}(u_i)^k$ is a monomial in $\mathrm{LT}(S_0)$ containing a factor x_j which is impossible. Since $\beta_{i,n} \in \mathfrak{p} := \sqrt{(x_1, x_2, x_3)A}$, by Lemma 9.7, it follows that $A/\mathfrak{p}$ is generated by the (non-zero) images of u_{12}, u_{13}, u_{23} which are algebraically independent, because their multi-degrees are linearly independent. Thus, $A/\mathfrak{p}$ is a polynomial ring in 3 variables, and $\mathfrak{p}$ is generated by $\{\beta_{i,n}\}$. $\qquad\square$

Proof of Proposition 9.2(d)–(f). For (d) we already know that $x_1, x_2, x_3 \in \mathfrak{f}_{X /\!/ \mathbb{G}_a}$, hence $\sqrt{(x_1, x_2, x_3)} \subseteq \mathfrak{f}_{X /\!/ \mathbb{G}_a}$, and, by Lemma 9.7, we have $\beta_{i,n} \in \mathfrak{f}_{X /\!/ \mathbb{G}_a}$ for all i, n. Now let $f \in \mathfrak{f}_{X /\!/ \mathbb{G}_a}$. Since $A = A_0 + (\beta_{i,n})A$ we can assume that $f \in A_0$. Since A_f is finitely generated there is an $N > 0$ such that $A_f = (A_N)_f$, and so $f^k \beta_{i,N+1} \in A_N$ for some $k > 0$ and all i. Now the claim follows from Lemma 9.6.

The first part of (e) follows from Lemma 9.7. For the second, we look at the chain of ideals in A

$$\mathfrak{p}_2 := \sqrt{(x_1)} \subset \mathfrak{p}_1 := \sqrt{(x_1, x_2)} \subset \mathfrak{p} = \sqrt{(x_1, x_2, x_3)}$$

and the corresponding closed subschemes

$$Z := \mathcal{V}_{\mathbb{A}^7 /\!/ \mathbb{G}_a}(\mathfrak{p}) \subset Z_1 := \mathcal{V}_{\mathbb{A}^7 /\!/ \mathbb{G}_a}(\mathfrak{p}_1) \subset Z_2 := \mathcal{V}_{\mathbb{A}^7 /\!/ \mathbb{G}_a}(\mathfrak{p}_2) \subset \mathbb{A}^7 /\!/ \mathbb{G}_a.$$

It follows that $U_1 := Z_1 \cap (\mathbb{A}^7 /\!/ \mathbb{G}_a)_{x_3}$ is irreducible of dimension 4, because its inverse image in $\mathbb{A}^7$ is $\mathcal{V}_{\mathbb{A}^7}(x_1, x_2) \cap (\mathbb{A}^7)_{x_3} \simeq (\mathbb{A}^5)_{x_3}$. Similarly, $U_2 := Z_2 \cap (\mathbb{A}^7 /\!/ \mathbb{G}_a)_{x_3}$ is irreducible of dimension 5. Now U_1 is affine and open in Z_1 and thus the complement $\overline{U_1} \setminus U_1$ has dimension $\dim U_1 - 1$, because Z_1 is a KRULL scheme. But $\overline{U_1} \setminus U_1 \subseteq Z$, hence equal to Z, and so $\overline{U_1}$ contains Z and is irreducible of dimension 4. Finally, $\overline{U_2}$ is irreducible of dimension 5 and contains $\overline{U_1}$. Thus we have the chain of irreducible closed subschemes

$$Z \subsetneq \overline{U_1} \subsetneq \overline{U_2} \subsetneq \mathbb{A}^7 /\!/ \mathbb{G}_a.$$

and the claim follows.

By (d) and (e) the quotient $\mathbb{A}^7 /\!/ \mathbb{G}_a$ is the disjoint union of an open and a closed algebraic variety, $(\mathbb{A}^7 /\!/ \mathbb{G}_a)_{alg}$ and $\mathcal{V}_{\mathbb{A}^7 /\!/ \mathbb{G}_a}(x_1, x_2, x_3) \simeq \mathbb{A}^3$, which clearly implies the claim. $\qquad\square$

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