ADDITIONAL MATERIAL

## An introduction the Atomic and Radiation Physics of Plasmas

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## 1 Introduction

Additional material related to the development of particle distribution functions (section 2) and the relationship of Maxwell's equations to ray treatments of light (section 4) for 'Tallents, G. (2018). An Introduction to the Atomic and Radiation Physics of Plasmas. Cambridge: Cambridge University Press' is presented here. The text of the book develops the physics of emission, absorption and interaction of light in astrophysics and in laboratory plasmas from first principles using the physics of various fields of study including quantum mechanics, electricity and magnetism, and statistical physics. This text can be regarded as an additional Appendix to the book. References to Equations and Sections given with decimal numbering refer to those presented in the book.

## 2 The Boltzmann and Fermi-Dirac distributions

In Section 8.1 we show that particles can be designated as fermions, where only a maximum of one particle can occupy a quantum state, or as bosons where any number of particles can occupy a quantum state. In a book on atomic and radiation physics, we are mainly concerned with electron quantum states (representing fermions) or the quantum states associated with photons (representing bosons).

The probability $P(E)$ of finding a particle in a quantum state of energy $E$ at a temperature $T$ is given by Equation 1.19. By including the number $N$ of particles allowed to occupy the quantum state of energy $E$, Equation 1.19 enables a presentation in the same expression of the probability of having either fermions or bosons in a quantum state. The Pauli exclusion principle (see section 8.1) allows fermions to have either $N=0$ or $N=1$ particles in the same quantum state, while bosons can have have $N=0,1,2, \ldots$ (up to any integer). Equation 1.19 is given as a probability proportionality:

$$
P(E) \propto \exp \left(\frac{N(\mu-E)}{k_{B} T}\right)
$$

where $\mu$ is the chemical potential. The chemical potential is the energy per particle required to add more particles into the available quantum states.

Equation 1.19 is used to determine the Maxwellian distribution of speeds in Section 1.2 and then to derive the Saha-Boltzmann ratio of the populations of
different ionisation stages at low density in Section 1.4.1. In later chapters, the probability of having a particle in a quantum state is employed in the derivation of the Planck black-body radiation distribution, enabling a calculation of the probability of the number of photons in a mode (Section 4.1.2). Finally, the Boltzmann ratio of populations is utilised to derive the high density form of the Saha-Boltzmann equation (Section 13.4).

To derive Equation 1.19, we need to consider the change $\Delta U$ of the total energy $U$ of an assumed large number $n_{\text {tot }}$ of particles distributed into a number of quantum states. From equilibrium thermodynamics, we have for a constant temperature $T$ and constant chemical potential $\mu$ that any change in the total energy $U$ of a collection of quantum states arises from a change in the entropy $S$ plus any energy change due to the addition of $\Delta n_{\text {tot }}$ more particles. We have

$$
\begin{equation*}
\Delta U=T \Delta S+\mu \Delta n_{t o t} . \tag{1}
\end{equation*}
$$

In thermodynamics, entropy change $\Delta S$ is defined by the first term on the right in Equation 1. There is additional energy associated with an increase in the 'disorder' represented by the entropy.

The total energy $U$ is determined by adding up the energy of each quantum state multiplied by the number of particles in the state, so that

$$
\begin{equation*}
U=\sum_{i} N_{i} n_{i} E_{i} \tag{2}
\end{equation*}
$$

where $N_{i}$ is the number of particles allowed in the $i$ th quantum state and $n_{i}$ is the number of quantum states in the system with an energy $E_{i}$. We discussed previously that for fermions, $N_{i}$ is either zero or one, where for bosons $N_{i}$ can range from $0,1,2, \ldots$ up to any integer. So as not violate the Pauli exclusion principle, we are assuming that the different quantum states with the same energy are separated into different atoms (for bound electrons) or satisfy the possible density of quantum states (for free electrons, see section 1.3).

The total number of particles is determined from a summation of the particle number over all the quantum states. We have

$$
\begin{equation*}
n_{t o t}=\sum_{i} N_{i} n_{i} . \tag{3}
\end{equation*}
$$

In statistical physics, the entropy $S$ of a system is determined by the number of ways $Q$ that particles can be arranged in the collection of quantum states of the
system. We have that

$$
\begin{equation*}
S=k_{B} \ln Q . \tag{4}
\end{equation*}
$$

We show below that this definition of entropy is consistent with the thermodynamic definition stated in terms of the change $\Delta U$ of total energy of the particles due to increasing entropy $\Delta S$ :

$$
\Delta S=\frac{\Delta U}{T}
$$

The logarithm in Equation 4 is useful in obtaining an approximate value for the entropy. Fortunately, the approximation we will use becomes very accurate when the number $n_{\text {tot }}$ of particles is large. The number of distinct arrangments of $n_{t o t}$ indistinguishable particles into quantum states (which we are labelling as $i$ ) is given by probability theory. We have that the number of distinct arrangements $Q$ is given by the factorial of the total number $n_{\text {tot }}$ of particles divided by the product of factorials of the number $n_{i}$ of quantum states $i$ of the system ${ }^{1}$ :

$$
\begin{equation*}
Q=\frac{n_{t o t}!}{\prod_{i} n_{i}!} . \tag{5}
\end{equation*}
$$

The total number of ways that $n_{\text {tot }}$ particles can be arranged is $n_{\text {tot }}!$. We are not concerned with the order of 'placing' the particles in each of the quantum states $i$ as we assume that the particles are indistinguishable, so Equation 5 divides the total number of arrangements by the number of ways $n_{i}$ ! that $n_{i}$ states can be arranged. This is done for each quantum state $i$ giving rise to the product in the denominator.

Taking the logarithm of $Q$ gives

$$
\ln Q=\ln \left(n_{t o t}!\right)-\sum_{i} \ln n_{i} .
$$

The Stirling approximation is a convenient simplification for the factorial of $n_{\text {tot }}$. For large $x$, we have

$$
\ln (x!) \approx x \ln x-x
$$

[^0]The value of $\ln Q$ becomes

$$
\begin{equation*}
\ln Q=n_{t o t} \ln n_{t o t}-n_{t o t}-\sum_{i} n_{i} \ln n_{i}+\sum_{i} n_{i} . \tag{6}
\end{equation*}
$$

Returning to the energy balance equation (Equation 1), we can divide throughout by a small increment $\Delta n_{j}$ of the population of $n_{j}$ quantum states associated with a quantum state $j$. Taking the limits of small increments and using $S=k_{B} \ln Q$, we have a differential equation

$$
\begin{equation*}
\frac{\partial U}{\partial n_{j}}=k_{b} T \frac{\partial \ln Q}{\partial n_{j}}+\mu \frac{\partial n_{\text {tot }}}{\partial n_{j}} . \tag{7}
\end{equation*}
$$

Each of these terms simplifies considerably. Using the summation of all quantum state energies given by Equation 2 we have

$$
\begin{equation*}
\frac{\partial U}{\partial n_{j}}=N_{j} E_{j} . \tag{8}
\end{equation*}
$$

All values in the summation of $U$ are zero in the partial derivative except when $i=j$. Similarly, using Equation 3 we have that

$$
\mu \frac{\partial n_{t o t}}{\partial n_{j}}=N_{j} \mu
$$

The partial derivative of $\ln Q$ also simplifies. We can differentiate $\ln Q$ using Equation 6. The total number of particles $n_{\text {tot }}$ is constant when considering the partial derivative with respect to the population of the $j$ th quantum state and again the summations are only non-zero upon partial differentiation when $i=j$. We have

$$
\begin{equation*}
\frac{\partial \ln Q}{\partial n_{j}}=-\ln n_{j} \tag{9}
\end{equation*}
$$

The result of these simplifications is

$$
N_{j} E_{j}=-k_{B} T \ln n_{j}+\mu N_{j} .
$$

Re-arranging gives

$$
\begin{equation*}
n_{j}=\exp \left(\frac{N_{j}\left(\mu-E_{j}\right)}{k_{B} T}\right) \tag{10}
\end{equation*}
$$

Dropping the sub-script $j$, we then have a probability for the occupation of a quantum state of energy $E$ determined by

$$
\begin{equation*}
P(E)=\exp \left(\frac{N(\mu-E)}{k_{B} T}\right) / n_{t o t} . \tag{11}
\end{equation*}
$$

### 2.0.1 Fermion and bosons

For fermions, the average occupancy $n(E)$ of a quantum state with energy $E$ is determined by the ratio of the probability of occupation when a particle can be present in the quantum state $(N=1)$ to the addition of the two probabilities when a further occupancy is not allowed $(N=1)$ and allowed ( $N=0$ ). We obtain Equation 1.20 with
$n(E)=\frac{P(N=0)}{P(N=1)+P(N=0)}=\frac{\exp \left((\mu-E) / k_{B} T\right)}{\exp \left((\mu-E) / k_{B} T\right)+1}=\frac{1}{1+\exp \left((-\mu+E) / k_{B} T\right)}$
We may have several quantum states with the same energy $E$ or an energy within a small range $E$ to $E+d E$. Equation 11 shows for fermions that if the energy $E$ is the same for different quantum states then the probability $P(E)$ of occupation is the same. We can allow for the effect on the populations of different quantum states with the same energy by multiplying the average occupancy by a 'degeneracy' $g$ (also known as a statistical weight) which is the number of distinct quantum states with the same energy. If the degeneracy or density of states in the energy range $E$ to $E+d E$ is $g(E)$, the population of all quantum states with energy $E$ to $E+d E$ is given by

$$
\begin{equation*}
f_{F D}(E) d E=\frac{g(E) d E}{1+\exp \left((-\mu+E) / k_{B} T\right)} \tag{12}
\end{equation*}
$$

where $g(E)$ is the degeneracy or density of quantum states at energy $E$. This distribution is known as the Fermi-Dirac distribution. It is used in Section 13.3.

The chemical potential $\mu$ for a collection of electrons is obtained by integrating Equation 12 over all energy and equating this integrated value to the known number of electrons in the system. Usually the number of electrons per unit volume, that is the electron density $n_{e}$ is known. The degeneracy $g(E)$ is then specified as the number of quantum states per unit energy per unit volume. In dealing with a bound quantum state, the electron degeneracy is specified as the number of quantum states with the same energy per atom or ion.

At low densities and high temperatures, the chemical potential $\mu$ is large and negative (see Section 13.3), so that the population of quantum states are proportional to $g(E) \exp \left(-E / k_{B} T\right)$. The ratio of two discrete quantum state populations $n$ per unit volume which we label for the higher energy state with subscript ' $u$ ' and for the lower energy state with subscript ' $l$ ' is given by Equation 1.27 and is
known as the Boltzmann ratio. Equation 1.27 is written as:

$$
\frac{n_{u}}{n_{l}}=\frac{g_{u}}{g_{l}} \exp \left(-\frac{\Delta E}{k_{B} T}\right)
$$

where the energy difference of the two quantum states is $\Delta E$ and $g_{u}$ and $g_{l}$ are respectively the upper and lower quantum state degeneracies.

In Section 4.1.2, we determined the Planck black body radiation distribution. Part of this calculation involved evaluating the probability of the presence of $n_{p}$ photons with energy $\hbar \omega$ for a radiation temperature of $T$. As photons are bosons, the chemical potential is zero. Adding a boson to a collection of bosons does not release or require any energy. In addition, the parameter $N$ in Equation 11 can represent any number of photons in the same mode. The probability of having $n_{p}$ photons per mode can be regarded as following Equation 11 so that

$$
P_{n_{p}}=\frac{\exp \left(-n_{p} \hbar \omega / k_{B} T\right.}{\sum_{n^{\prime}} \exp \left(-n^{\prime} \hbar \omega / k_{B} T\right.}
$$

where the summation in the denominator is from $n^{\prime}=0$ up to $n^{\prime}=\infty$. The average number of photons $n_{a v}$ per mode is then given by

$$
n_{a v}=\sum_{n_{p}} n_{p} P_{n_{p}}=\frac{1}{\exp \left(\hbar \omega / k_{B} T\right)-1} .
$$

The expression on the right hand side is derived in Section 4.1.2. This value of $n_{a v}$ can be generalised to represent the average occupancy of bosons in a quantum state of energy $E$ after replacing $\hbar \omega$ by $E$. There is a superficial resemblance to the average occupancy of fermions (given by Equation 1.20), except the minus in the denominator for bosons becomes a plus for fermions.

### 2.0.2 The thermodynamic and statistical physics entropy

The thermodynamic definition of an entropy change $\Delta S$ defines entropy in terms of the change of total energy $\Delta U$ by specifying that $\Delta S=\Delta U / T$. This definition assumes that the chemical potential remains constant and the temperature remains constant. The statistical physics definition is given by Equation 4 with $S=k_{B} \ln Q$. We need to show that the two definitions are consistent.

With constant chemical potential $\mu$ and constant temperature $T$, we can have an increase in entropy $S$ associated with an increase of the total energy $U$ by adding
the entropy increase due to an increase in 'disorder' (represented by $k_{B} \ln Q$ ) and the increase in entropy associated with the addition of particles. We have (see Equation 1) that the additional energy associated with an increase of the total particle number $n_{\text {tot }}$ is given by $\mu \Delta n_{\text {tot }}$. The increase of entropy associated with the increase of particle number is consequently $(\mu / T) \Delta n_{\text {tot }}$. Adding the differential of the entropy increase due to disorder plus the entropy increase due to particle number with respect to the total energy $U$ gives

$$
\frac{d S}{d U}=\frac{\partial\left(k_{B} \ln Q\right)}{\partial U}+\frac{\mu}{T} \frac{\partial n_{t o t}}{\partial U} .
$$

We can write for any of the quantum states $j$ that

$$
\frac{\partial \ln Q}{\partial U}=\frac{\partial \ln Q}{\partial n_{j}} \frac{\partial n_{j}}{\partial U}=\left(-\ln n_{j}\right) \frac{1}{N_{j} E_{j}}
$$

upon using Equation 9 for the differentiation of $\ln Q$ and Equation 8 for the differentiation of the total energy $U$. For the term involving the chemical potential $\mu$, we write that

$$
(\mu / T) \frac{\partial n_{\text {tot }}}{\partial U}=(\mu / T) \frac{\partial n_{\text {tot }}}{\partial n_{j}} \frac{\partial n_{j}}{\partial U}=\frac{\mu / T}{E_{j}} .
$$

Using the Boltzmann factor for $n_{j}$ with $n_{j}=\exp \left(N_{j}\left(\mu-E_{j}\right) / k_{B} T\right)$ gives the required result that

$$
\frac{d S}{d U}=\frac{\partial\left(k_{B} \ln Q\right)}{\partial U}+\frac{\mu}{T} \frac{\partial n_{t o t}}{\partial U}=\left(-\ln n_{j}\right) \frac{k_{B}}{N_{j} E_{j}}+\frac{\mu / T}{E_{j}}=\frac{1}{T} .
$$

## 3 Special relativity and the Thomas correction

In section 7.4, the effects of spin-orbit coupling on the fine structure energies of hydrogen and hydrogen-like ions are examined. The magnetic field in the frame of reference of the electron is determined using the Bio-Savart law by considering the $+Z e$ charged nucleus orbiting in the frame of the electron. This value of the magnetic field and consequently the energy associated with the intrinsic electron magnetic moment due to the electron spin in the magnetic field is reduced by a factor $1 / 2$ due to the necessity to measure magnetic fields in the frame of reference of the electron.

If an electron in an atom has an angular momentum, it is in a frame of reference rotating with respect to the electric field produced by the nucleus. In special
relativity, the electric field is equivalent to a magnetic field in the electron frame. In hydrogen-like ions, the electron motion perpendicular to the nuclear electric field creates a magnetic field in the electron frame which is opposite to and with a magnitude of half the value of the magnetic fiield created by the orbiting nuclear charge (in the electron frame). The total magnetic field exerienced by the electron is consequently half the magnetic field expected for the orbiting nuclear charge. The factor $1 / 2$ is known as the 'Thomas correction' after Llewellyn Hilleth Thomas (1903-1992) who first explained the reduction in spin-orbit coupling energy. The origins of the Thomas correction are explored in this appendix.

Special relativity provides equations for the changes in electric and magnetic fields observed when considering two frames of reference with a relative velocity between them. Transforming electric and magnetic fields between inertial frames moving relative to each other at velocity $\mathbf{v}$ results in identical electric and magnetic field components parallel to the direction of the velocity. The electric and magnetic field components perpendicular to the velocity are, however, changed so that in the frame moving at velocity $\mathbf{v}$ relative to a rest frame, we have

$$
\begin{gather*}
\mathbf{E}^{\prime}=\frac{\mathbf{E}+\mathbf{v} \times \mathbf{B}}{\sqrt{1-v^{2} / c^{2}}}  \tag{13}\\
\mathbf{B}^{\prime}=\frac{\mathbf{B}-(\mathbf{v} \times \mathbf{E}) / c^{2}}{\sqrt{1-v^{2} / c^{2}}} \tag{14}
\end{gather*}
$$

where the primes indicate the value of electric and magnetic fields in the frame moving at velocity $\mathbf{v}$ and unprimed values are the electric and magnetic fields in the rest frame. Equation 13 for the effective electric field induced when moving through a magnetic field is the basis for the motional Stark effect discussed in section 10.4. Using the motional Stark effect, magnetic fields are measured by recording the Stark shift of atomic energy levels in atomic beams moving across the magnetic field in, for example, a tokamak. Stark shifts of atomic energy levels are energy changes normally produced by an electric field external to the atom.

To examine the magnetic field producing spin-orbit coupling we need to use equation 14 showing a different magnetic field in a frame moving perpendicular to an electric field. The treatment follows that of H. Kroemer [Am. J. Phys. 72, 51 (2004) 'The Thomas precession factor in spin-orbit interaction' doi: 10.1119/1.1615526]. Consider an electron moving in a straight line parallel to the
$x$-axis in Cartesian co-ordinates with velocity $v_{x}$ with an electric field $E_{y}$ parallel to the $y$-axis and a magnetic field $B_{z}$ parallel to the $z$-axis. For the electron to travel in a straight line, the forces on the electron due to the electric and magnetic fields must be equal in magnitude and opposite in direction, so that

$$
\mathbf{E}=-\mathbf{v} \times \mathbf{B}
$$

In our Cartesian co-ordinates this means that $E_{y}=v_{x} B_{z}$.
We need to expand the denominator in equation 14 as a series. Assuming that $v \ll c$ and expanding up to terms varying as $v^{2}=v_{x}^{2}$ gives

$$
\frac{1}{\sqrt{1-v^{2} / c^{2}}} \approx 1+\frac{1}{2}\left(\frac{v_{x}}{c}\right)^{2}+\ldots
$$

The magnetic field in the moving frame given by equation 14 becomes

$$
B_{z}^{\prime}=B_{z}-\frac{E_{y} v_{x}}{c^{2}}+\frac{1}{2} B_{z}\left(\frac{v_{x}}{c}\right)^{2}+\ldots
$$

Using the assumption that the electric and magnetic forces are balanced ( $E_{y}=$ $v_{x} B_{z}$ ) and neglecting terms greater than $v_{x}^{2}$, we obtain

$$
B_{z}^{\prime}=B_{z}-\frac{1}{2} \frac{v_{x} E_{y}}{c^{2}}
$$

In vector notation, we have

$$
\begin{equation*}
\mathbf{B}^{\prime}=\mathbf{B}-\frac{1}{2} \frac{\mathbf{v} \times \mathbf{E}}{c^{2}} . \tag{15}
\end{equation*}
$$

This equation is similar to equation 14 , but has the addition of a factor $1 / 2$ on the term correcting the magnetic field in the moving frame.

In section 7.4, the magnetic field in the frame of the electron in a hydrogen-like ion is deduced using the Biot-Savart law by considering the nuclear charge $+Z e$ orbiting around the electron. Assuming the electron orbital angular momentum is in the $z$-direction, we have a magnetic field in the electron frame in the $z$-direction at radius $r$ from the nucleus given by

$$
B_{z}^{\prime}=B_{z}-\frac{1}{2} \frac{v E}{c^{2}}=\frac{\mu_{0}}{4 \pi} \frac{Z e v}{r^{2}}-\frac{1}{2} \frac{v E}{c^{2}}
$$

upon using equation 15 . Here $v$ is now the notional rotation velocity of the electron perpendicular to the magnetic field at radius $r$. The velocity $v$ is also perpendicular
to the radially directed electric field $E$. The electric field due to the nuclear charge is given by

$$
E=\frac{1}{4 \pi \epsilon_{0}} \frac{Z e}{r^{2}} .
$$

Consequently, we can use the identity $c^{2}=1 /\left(\epsilon_{0} \mu_{0}\right)$ to show that

$$
\frac{v E}{c^{2}}=\frac{\mu_{0}}{4 \pi} \frac{Z e v}{r^{2}}
$$

and hence for all radial distances $r$ from the nucleus

$$
\begin{equation*}
B_{z}^{\prime}=\frac{1}{2} B_{z} . \tag{16}
\end{equation*}
$$

The factor of $1 / 2$ is the Thomas correction to the magnetic field strength created by orbital angular momentum and the correction to the spin-orbit coupling energy used, for example, to develop equation 7.16. We have shown that the Thomas correction factor $1 / 2$ arises due to the reduction in magnetic field seen by the electron due to the electron motion perpendicular to the nuclear electric field creating a magnetic field in the electron frame of reference.

## 4 Light rays and Fermat's principal

In section 3.4.2, we discuss refraction of light in terms of light rays and use Fermat's principal to obtain an expression (equation 3.47) for the path of light rays in a plasma medium with a continuous variation of refractive index. Fermat's principal states that, compared to nearby paths, light travels between two points along the path that requires the least time. In the treatment here, we develop a 'bridge' between the wave equation (derived from Maxwell's equations) and ray treatments of light optics, then show how Fermat's principle can be derived.

The concept of a ray of light can be regarded as arising from the plane wave solutions of the wave equation (see section 2.1). In a plane light wave, the electric field $\mathbf{E}$ oscillates with angular frequency $\omega$ in a direction perpendicular to the wavevector $\mathbf{k}$ representing the direction and spatial period of the rapid spatial variation of the field. The electric field is assumed constant in all planes perpendicular to $\mathbf{k}$ and the wave propagates in the $\mathbf{k}$ direction. The planes with constant electric field implicitly extend to infinity as no boundary is set in treatments such
as given in section 2.1. If the variation in the refractive index $n(\mathbf{r})$ with position $\mathbf{r}$ occurs on length scales much greater than the wavelength $\lambda$ of the light, the concept of the plane wave solution can be applied to a local plane associated with a light ray. Differences in refractive index can be accommodated by considering other nearby light rays propagating in approximately parallel directions at slightly different spatial positions. Such a treatment seems heuristically feasible, but we now show that it is possible to be more precise and to determine the appropriate equations for ray propagation from the wave equation.

Using the refractive index $n(\mathbf{r})$ to allow for the effects of the medium on the electric field $\mathbf{E}$, the wave equation was determined from Maxwell's equations in section 2.1. We can write

$$
\begin{equation*}
\nabla^{2} \mathbf{E}-\frac{(n(\mathbf{r}))^{2}}{c^{2}} \frac{\partial^{2} \mathbf{E}}{\partial t^{2}}=0 \tag{17}
\end{equation*}
$$

where the phase velocity of the light is given by the vacuum speed of light divided by the real part of the refractive index $(c / \eta(\mathbf{r}))$. Neglecting absorption of light, we make the assumption that solutions of the wave equation are of the form

$$
\begin{equation*}
\mathbf{E}(\mathbf{r}, t)=\mathbf{E}_{0}(\mathbf{r}) \exp \left(i\left(k_{0} R(\mathbf{r})-\omega t\right)\right) \tag{18}
\end{equation*}
$$

where $k_{0}=\omega / c$ is the wavevector amplitude in vacuum and $R(\mathbf{r})$ is a real function varying with position $\mathbf{r}$ and having the dimensions of length. The parameter $R(\mathbf{r})$ determines the phase of the electric field as a function of position $\mathbf{r}$. Points of constant phase for an extended light field are referred to as 'wavefronts'.

Substituting equation 18 into the wave equation (equation 17) and using $k_{0}=$ $\omega / c$ gives

$$
\nabla^{2}\left[\mathbf{E}_{0}(\mathbf{r}) \exp \left(i k_{0} R(\mathbf{r})\right)\right]+(\eta(\mathbf{r}))^{2} k_{0}^{2} \mathbf{E}_{0}(\mathbf{r}) \exp \left(i k_{0} R(\mathbf{r})\right)=0
$$

where $\eta(\mathbf{r})$ is the real component of the refractive index as we are not considering absorption (involving the imaginary component of the refractive index). Evaluating and re-arranging the wave equation, we obtain

$$
\begin{equation*}
\left[\nabla R(\mathbf{r}) \cdot \nabla R(\mathbf{r})-(\eta(\mathbf{r}))^{2}\right] \mathbf{E}_{0}(\mathbf{r})=\frac{\nabla^{2} \mathbf{E}_{0}(\mathbf{r})}{k_{0}^{2}}+\frac{i}{k_{0}} \mathbf{E}_{0}(\mathbf{r}) \nabla^{2} R(\mathbf{r})+\frac{2 i}{k_{0}} X(\mathbf{r}) \tag{19}
\end{equation*}
$$

where using Cartesian co-ordinates

$$
X(\mathbf{r})=\hat{\mathbf{x}}\left(\frac{\partial \mathbf{E}_{0}(\mathbf{r})}{\partial x}\right) \cdot \nabla R(\mathbf{r})+\hat{\mathbf{y}}\left(\frac{\partial \mathbf{E}_{0}(\mathbf{r})}{\partial y}\right) \cdot \nabla R(\mathbf{r})+\hat{\mathbf{z}}\left(\frac{\partial \mathbf{E}_{0}(\mathbf{r})}{\partial z}\right) \cdot \nabla R(\mathbf{r}) .
$$

The wavevector $k_{0}$ is a large number compared to the gradients $(\nabla)$ and Laplacians $\left(\nabla^{2}\right)$ of the electric field and wavefront parameter $R(\mathbf{r})$ provided the refractive index does not change rapidly over the length scale of the wavelength. For approximately uniform media, the right hand side of equation 19 can be accurately equated to zero, so that

$$
\nabla R(\mathbf{r}) \cdot \nabla R(\mathbf{r})=(\eta(\mathbf{r}))^{2} .
$$

This equation is more simply written as

$$
\begin{equation*}
\nabla R(\mathbf{r})=\eta(\mathbf{r}) \hat{\mathbf{s}}(\mathbf{r}) \tag{20}
\end{equation*}
$$

where $\hat{\mathbf{s}}(\mathbf{r})$ is a unit vector in the direction of $\nabla R(\mathbf{r})$. Equation 20 is known as the Eikonal equation.

Points of constant phase are represented by constant values of $R(\mathbf{r})$ for an extended light field. As the value of $\nabla R(\mathbf{r})$ represents the steepest change of $R(\mathbf{r})$, points of constant phase (with zero change of $R(\mathbf{r})$ ) are perpendicular to $\nabla R(\mathbf{r})$ and hence perpendicular to $\hat{\mathbf{s}}(\mathbf{r})$. In a plane wave solution of the wave equation, the wavefronts are perpendicular to the direction of propagation (represented by the wavevector $\mathbf{k}$ direction), so assuming the plane wave solution to the wave equation has validity, we have that the direction of $\nabla R(\mathbf{r})$ and $\hat{\mathbf{s}}(\mathbf{r})$ represent the direction of propagation of a light ray. More detailed calculations show that $\hat{\mathbf{s}}(\mathbf{r})$ gives the direction of propagation of a light ray provided the assumption made in simplifying equation 19 is valid: the changes in the refractive index are small over the length scale of a wavelength.

### 4.1 Fermat's principle

Fermat's principal states that, compared to nearby paths, light travels between two points along the path that requires the least time. We can derive Fermat's principle from equation 20. Taking the curl of both sides of equation 20 gives

$$
\nabla \times[\eta(\mathbf{r}) \hat{\mathbf{s}}(\mathbf{r})]=\nabla \times \nabla R(\mathbf{r})=0
$$

as $\nabla \times \nabla$ of a scalar is identically equal to zero. If we integrate the left hand side over an area $A$, we can convert the integral over area to a line integral around the edge of the area using Stokes' theorem (discussed in Appendix A.2). We have

$$
\int_{A} \nabla \times[\eta(\mathbf{r}) \hat{\mathbf{s}}(\mathbf{r})] \cdot d \mathbf{A}=\int_{C} \eta(\mathbf{r}) \hat{\mathbf{s}}(\mathbf{r}) \cdot d \mathbf{l}=0
$$

The line integral is around a closed contour which we can imagine is from a point 1 to point 2 in the propagation of a ray, and then back from point 2 to 1 . The return from 2 to 1 in the line integral along a different path has the same absolute value (but opposite sign) as the integral 1 to 2 because the closed integral 1 to 2 then back to 1 is zero. This means that a line integral from point 1 to point 2 given by

$$
\begin{equation*}
\int_{1}^{2} \eta(\mathbf{r}) \hat{\mathbf{s}}(\mathbf{r}) \cdot d \mathbf{l} \tag{21}
\end{equation*}
$$

has a constant value independent of the path taken to evaluate the line integral between points 1 and 2 .

We now consider a simple integration $L$ of the refractive index along a path between points 1 and 2 and compare this to integrals of the form of equation 21. We have an optical pathlength $L$ given by

$$
\begin{equation*}
L=\int_{1}^{2} \eta(\mathbf{r}) d l \geq \int_{1}^{2} \eta(\mathbf{r}) \hat{\mathbf{s}}(\mathbf{r}) \cdot d \mathbf{l} \tag{22}
\end{equation*}
$$

with equality only when the right hand side integral is evaluated for a path where $d \mathbf{l}$ is always along the direction of propagation $\hat{\mathbf{s}}(\mathbf{r})$. The equality in equation 22 occurs only when $d \mathbf{l}$ is always along the direction of propagation $\hat{\mathbf{s}}(\mathbf{r})$ as then $\hat{\mathbf{s}}(\mathbf{r}) \cdot d \mathbf{l}=d l$, otherwise $\hat{\mathbf{s}}(\mathbf{r}) \cdot d \mathbf{l}<d l$. The path taken by light is seen to follow the minimum of all the optical pathlengths $L$. From equation 22, all pathlengths other than the one taken by the light have higher values of $L$. The time for propagation of light along a pathlength $L$ is given by $L / c$, so the minimum value of $L$ gives the minimum propagation time and proves Fermat's principle that, compared to nearby paths, light travels between two points along the path that requires the least time.

Fermat's principle enables procedures for determining the path taken by light rays in optical systems. In plasmas, the minimisation of the optical pathlength $L$ for propagation along the $z$-direction gives rise to a general differential equation for the ray path. This is listed as equation 3.47 and can be written

$$
\frac{d}{d z}\left(\eta(\mathbf{r}) \frac{d \mathbf{r}}{d z}\right)=\nabla \eta(\mathbf{r})
$$

### 4.2 An analogy between light paths and particle trajectories

The trajectory of a particle of kinetic energy $T(\mathbf{r})$ in a potential energy field $V(\mathbf{r})$ is well-known to be determined by minimising the integral of the Lagrangian parameter $L_{G}$ with respect to time. The Lagrangian parameter is given by

$$
L_{G}=T(\mathbf{r})-V(\mathbf{r}) .
$$

If the total energy $E$ of the particle is constant, then $E=T(\mathbf{r})+V(\mathbf{r})$ for all positions $\mathbf{r}$ and the Lagrangian is given by

$$
L_{G}=2 T(\mathbf{r})-E .
$$

The path of a particle of mass $m$ and velocity $v(\mathbf{r})$ is determined by minimising the integral of this Lagrangian with respect to time. The particle will follow the path in the potential field $V(\mathbf{r})$ which minimises the integral $\int L_{G} d t$. For convenience, we determine to minimise an integral $\int\left(L_{G}+E\right) d t$, so consider

$$
\begin{equation*}
\int\left(L_{G}+E\right) d t=\int 2 T(\mathbf{r}) d t=\int m v(\mathbf{r})(v(\mathbf{r}) d t)=\int m v(\mathbf{r}) d l=\int p(\mathbf{r}) d l \tag{23}
\end{equation*}
$$

where we introduce an increment of distance $d l=v(\mathbf{r}) d t$ and the momentum of the particle $p(\mathbf{r})=m v(\mathbf{r})$. The integral $\int\left(L_{G}+E\right) d t$ is minimised over the same path as $\int L_{G} d t$ as the energy $E$ is constant.

Our manipulations in equation 23 have shown that the path of the particle of total energy $E$ is along the path which minimises the integral of the amplitude of the particle momentum. We can equate this minimisation to a ray path in a plasma by relating momentum and refractive index. Kinetic energy is related to the potential energy field $V(\mathbf{r})$ by

$$
\frac{T(\mathbf{r})}{E}=1-\frac{V(\mathbf{r})}{E} .
$$

Converting the kinetic energy $T(\mathbf{r})$ to momentum $p$, we have

$$
\begin{equation*}
\frac{p^{2}(\mathbf{r})}{2 m E}=1-\frac{V(\mathbf{r})}{E} . \tag{24}
\end{equation*}
$$

This relationship between momentum $p(\mathbf{r})$ and potential energy $V(\mathbf{r})$ has an analogy with the relationship with the refractive index $\eta(\mathbf{r})$ in a plasma and the electron
density $n_{e}(\mathbf{r})$. The plasma refractive index and electron density are related by

$$
\eta^{2}(\mathbf{r})=1-\frac{n_{e}(\mathbf{r})}{n_{\text {crit }}}
$$

where $n_{\text {crit }}$ is the critical density (see equation 2.12). If we equate $p(\mathbf{r}) / \sqrt{2 m E}$ to the refractive index $\eta(\mathbf{r})$, then the ratio of the potential energy to the total particle energy $(V(\mathbf{r}) / E)$ is analogous to the plasma electron density divided by the critical density $\left(n_{e}(\mathbf{r}) / n_{\text {crit }}\right)$.

We have seen that minimising an integral of the value of momentum over different paths gives the path taken by a particle in a conservative (no energy loss) potential field (see equation 23). Fermat's principle states that minimising the integral of the refractive index over different paths gives the path taken by a light ray (see equation 22). Consequently, a light ray will follow the path taken by a particle of total energy $E$ in potential field $V(\mathbf{r})$ if we imagine that $V(\mathbf{r}) / E$ replaces $n_{e}(\mathbf{r}) / n_{\text {crit }}$.

A simple one-dimensional scenario can illustrate the analogy between a light ray and the motion of a particle. If a particle of initial kinetic energy $E$ is incident into a potential $V(z)$ from zero potential, the particle will rise up the potential until $E=V(z)$, then stop so that it has zero momentum and finally return back along the path, exiting with kinetic energy $E$. A ray of light incident into a one dimensional plasma with an increasing electron density with increasing $z$ will penetrate up to the critical density, then reflect and follow a path in the $-z$ direction.

More complicated ray paths can be evaluated by considering the ballistic motion of a particle. For example, the angular momentum $L_{M}$ of a particle with momentum $\mathbf{p}$ around a point is given by

$$
L_{M}=\mathbf{r} \times \mathbf{p}=r p \sin \theta
$$

where $\mathbf{r}$ is a vector from the point to the particle and $\theta$ is the angle of the particle momentum $\mathbf{p}$ to the position vector $\mathbf{r}$. Conservation of angular momentum for particle motion without dissipation means that $r p \sin \theta$ is constant. Our analogy developed above between the path of a light ray and particle motion has the refractive index $\eta$ as an analogous quantity to the particle momentum $p$, so we have that the quantity $r \eta \sin \theta$ is constant for a light ray. This is immediately useful
if we consider a circle of material of refractive index $\eta_{2}$ of radius $r_{0}$ surrounded by material of refractive index $\eta_{1}$. A light ray incident in the plane of the circle onto the material with refractive index $\eta_{2}$ with an angle of incidence $\theta_{1}$ must have $r \eta \sin \theta$ constant, so that the angle $\theta_{2}$ of the ray in the material with refractive index $\eta_{2}$ is given by

$$
\eta_{2} \sin \theta_{2}=\eta_{1} \sin \theta_{1}
$$

which is Snell's law (given as equation 3.41). The ray will continue to propagate within the circle of refractive index $\eta_{2}$ with $r \eta_{2} \sin \theta$ constant. The closest approach $r_{\text {min }}$ of the ray to the centre of the circle will occur when $\theta=\pi / 2$, so that

$$
r_{\min }=\frac{\eta_{1} r_{0} \sin \theta_{1}}{\eta_{2}} .
$$

This relationship for the closest approach of a ray to the centre of a circular medium holds even if the refractive index $\eta_{2}$ is continuously varying with radius $r$ (as it does for a circle of plasma).


[^0]:    ${ }^{1}$ Equation 5 assumes that particles have sufficient time and sufficient interaction for all the quantum states to be accessible. For fermions, collisions between particles often ensure equal accessibility, leading to equilibrium distributions where, for example, Equation 1.19 is valid. For photons and other bosons, the particles need to interact with electrons (or other fermions).

