Electron motion in an intense electromagnetic field

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Abstract

The motion of an electron in an electromagnetic field of arbitrarily high intensity is examined. Relativistic effects apparent in the rest frame are taken into account for linearly and elliptically polarized light. It is shown that the light intensity and frequency in the frame of the electron are Lorentz invariant. For linear polarization, an electron exhibits a figure-of-eight motion in the plane of the electric field and the direction of light propagation which is superimposed on a drift velocity in the direction of the light propagation. With circular polarization, the electron exhibits a drift velocity, but does not oscillate in the direction of light propagation. Electron motions strongly influence the energy transfer from light to plasma in experiments where high power laser light is focused onto solid, liquid or gaseous targets.

1 Introduction

The interaction of focused high intensity laser light with targets is of relevance to inertial fusion experiments and the production of high energy electron, protons and x-rays from laser-irradiated targets. At high laser irradiance, the electromagnetic forces on individual electrons from light increase the momentum and energy of the electrons and are consequentially important for the expansion behavior and radiation emission of the plasmas formed in laser-irradiated targets. In this paper, the forces, momenta and trajectories of an electron in an arbitrarily large, uniform electromagnetic field are determined.

2 Momentum scaling of electrons in a light field

For a linearly polarized electromagnetic wave, we consider that the wave propagates in the z-direction with the electric field $\mathbf{E}$ directed in the x-direction and the magnetic field $\mathbf{B}$ directed in the y-direction. It is reasonable to assume light pulse durations with many electric and magnetic field oscillations of frequency ω with the electric field given by

$$E_x = E_0 \cos(\omega t)$$

where the amplitude E_0 is constant in time. The accompanying magnetic field varies such that

$$B_y = B_0 \cos(\omega t) = \frac{E_0}{c} \cos(\omega t).$$

In the rest frame, the forces on an electron of charge $-e$ in the x- and z-directions can be written as

$$\frac{dp_x}{dt} = -e\left(E_x - v_z B_y\right) = -eE_x\left(1 - \frac{p_x}{\gamma m_0 c}\right)$$

$$\frac{dp_z}{dt} = -ev_x B_y = -eE_x\left(\frac{p_x}{\gamma m_0 c}\right) \tag{1}$$

upon converting the x- and z-directed velocities (v_x and v_z) to momenta using $p_x = \gamma m_0 v_x$ and $p_z = \gamma m_0 v_z$, where γ is the Lorentz parameter and m_0 is the electron rest mass. Eliminating the electric field E_x from Equations 1, we find that

$$\frac{dp_x}{dt} = \frac{m_0 c}{p_x}\frac{dp_z}{dt}\left(\gamma - \frac{p_z}{m_0 c}\right) \tag{2}$$

It is convenient to use reduced momenta written in units of $m_0 c$ so that $\hat{p}_x = p_x/(m_0 c)$ and $\hat{p}_z = p_z/(m_0 c)$. Equation 2 becomes

$$\frac{d\hat{p}_x}{dt} = \frac{1}{\hat{p}_x}\frac{d\hat{p}_z}{dt}\left(\gamma - \hat{p}_z\right). \tag{3}$$

The Lorentz parameter can be evaluated using

$$\gamma^2 = 1 + \hat{p}_x^2 + \hat{p}_z^2. \tag{4}$$

Differentiating the Lorentz parameter

$$\gamma\frac{d\gamma}{dt} = \hat{p}_x\frac{\hat{p}_x}{dt} + \hat{p}_z\frac{\hat{p}_z}{dt}.$$

Using Equation 3, we have that

$$\frac{d\hat{p}_z}{dt} = \frac{d\gamma}{dt}. \tag{5}$$

Integrating with appropriate limits shows that the z-directed momentum and Lorentz parameter vary proportionally in time such that

$$\hat{p}_z(t) = \gamma(t) - 1. \tag{6}$$

Substituting Equations 5 and 6 into Equation 3, the momentum in the electric field direction (x-direction) is related to the Lorentz factor by

$$\hat{p}_x\frac{d\hat{p}_x}{dt} = \frac{d\gamma}{dt}$$

Integrating the momentum and Lorentz factor with appropriate limits means that

$$\frac{1}{2}\hat{p}_x^2(t) = \gamma(t) - 1$$

We already found that $\hat{p}_z(t) = \gamma(t) - 1$ (Equation 6), so in summary:

$$\hat{p}_z(t) = \frac{1}{2}\hat{p}_x^2(t) = \gamma(t) - 1. \tag{7}$$

We can evaluate the angle θ of the electron trajectory to the z-axis ($\mathbf{k}$-direction of the light). At any time:

$$\cos\theta = \frac{\hat{p}_z}{(\hat{p}_x^2 + \hat{p}_z^2)^{1/2}} \tag{8}$$

The instantaneous velocity v of the electron is given by

$$\begin{aligned}\left(\frac{v}{c}\right) &= \left(\left(\frac{v_x}{c}\right)^2 + \left(\frac{v_z}{c}\right)^2\right)^{1/2} \\ &= \frac{1}{\gamma}(\hat{p}_x^2 + \hat{p}_z^2)^{1/2}. \end{aligned} \tag{9}$$

The standard expression for the Doppler shifted frequency ω' of the incident light in the frame of the electron is given by

$$\omega' = \omega\,\gamma\left(1 - \frac{v}{c}\cos\theta\right) \tag{10}$$

where ω is the light frequency in the rest frame. Substituting Equations 8 and 9, the Doppler shifted frequency in the electron frame is given by

$$\omega' = \omega\,(\gamma - \hat{p}_z) \tag{11}$$

Using Equation 6 we see that $\omega' = \omega$. There is no Doppler shift of the light frequency in the frame of the electron. The 'time dilation' effect on the light frequency (represented by γ) is balanced by the momentum component $\hat{p}_z$ of the electron away from the light source (in the $\mathbf{k}$-direction along the z-axis).

The reduced vector potential a_0 is useful in reducing the apparent complexity of equations of electron motion in a light field. The reduced vector potential is defined by

$$a_0 = \frac{eE_0}{m_0 c\,\omega}. \tag{12}$$

The reduced vector potential a_0 is related to the laser intensity $I = (1.2)\epsilon_0 c E_0^2$ (power per unit area) when measured in W cm^{-2} by

$$a_0 = \sqrt{\frac{I\lambda^2}{1.37\times 10^{18}}} \tag{13}$$

when the light wavelength $\lambda = 2\pi c/\omega$ is measured in microns. Quantities of form I/ω^3 are Lorentz invariant (see Appendix A [1]). As I/ω^3 and the frequency ω are Lorentz invariant, the vector potential a_0 and the intensity I are Lorentz invariant for an electron in an electromagnetic field.

3 Electron motion in linearly polarized light fields

In this Section, we calculate the motion of a single electron that is assumed to be initially stationary in a linearly polarized high intensity light field. At high light intensities, electrons can be accelerated to velocities close to the speed of light c so any initial thermal or other electron velocity can be ignored, or the calculations can be regarded as being undertaken in the initial frame of reference of the electron before any light impinges. Due to electron motion in the z-direction, the electron moves relative to the oscillating electromagnetic field. The electromagnetic field oscillations in the frame of the electron need to be calculated using a retarded time τ' such that

$$\tau' = t - \frac{z}{c} \tag{14}$$

where z represents the z-position of the electron relative to its initial position in the electron frame. The electric field in the frame of the electron can be written as

$$E' = E_0' \cos(\omega' \tau')$$

Quantities calculated in the moving frame of the electron are here designated with an apostrophe and without an apostrophe, quantities are relevant to the rest frame or are Lorentz invariant. For a list of Lorentz invariant quantities related to the propagation of light, see Appendix A of Tallents [1].

We now initially calculate the x-directed electron momentum and velocity in a frame of reference moving along the z-axis with the electron. In this frame, the electric field E is Lorentz invariant in calculations of the x-directed electron momentum p_x as the electric field is parallel to the x-directed electron velocity. Similarly, the light phases ωt and $\omega\tau$ are Lorentz invariant. In the frame moving along z with the electron, the velocity v_z' in the z-direction is zero, so that the Lorentz force $|\mathbf{v} \times \mathbf{B}| = v_z' B_0' \cos(\omega\tau) = 0$. The force parallel to the electric field $\mathbf{E}$ in the frame of the electron is given by

$$\frac{dp_x}{d\tau'} = -eE_0 \cos(\omega\tau). \tag{15}$$

Integrating, we have the electron momentum component in the x-direction:

$$\hat{p}_x = \frac{p_x}{m_o c} = -\frac{eE_0}{m_0 c\, \omega} \sin(\omega\tau) \tag{16}$$

where m_0 is the electron rest mass. Using the reduced vector potential

$$a_0 = \frac{eE_0}{m_0 c\, \omega}$$

we have

$$\hat{p}_x = \frac{p_x}{m_o c} = -a_0 \sin(\omega\tau). \tag{17}$$

The rest frame momentum p_x is related to the x-directed electron velocity v_x by $p_x = \gamma m_0 v_x$, where γ is the Lorentz parameter for the moving electron. The electron velocity in the x-direction in the rest frame is given by

$$\frac{v_x}{c} = -\frac{a_0}{\gamma} \sin(\omega\tau). \tag{18}$$

The force $dp_z/d\tau$ in the frame moving along the z-axis parallel to the z-directed velocity v_z and the velocity v_x are the same in the rest frame and the frame moving along the z-axis with the electron. Due to the $-e\mathbf{v} \times \mathbf{B}$ force arising from the $\mathbf{B}$ field of the electromagnetic wave, the force on the electron in the direction of wave propagation (the z-direction) is given by

$$\frac{dp_z}{d\tau} = -ev_x B_0' \cos(\omega\tau). \tag{19}$$

The magnetic field $\mathbf{B}$ is directed perpendicularly to the frame velocity following the electron along the z-axis, so that the magnetic field amplitude B_0' in the frame moving along the z-axis is related to the rest frame magnetic field amplitude B_0 by $B_0' = \gamma B_0$. Substituting for v_x (Equation 18) and noting that $B_0 = E_0/c$ we can write that the force in the rest frame is given by

$$\frac{dp_z/(m_0 c)}{d\tau} = \frac{eE_0\, a_0}{m_0 c} \sin(\omega\tau)\cos(\omega\tau). \tag{20}$$

Integrating we obtain the momentum in the z-direction in the rest frame

$$\hat{p}_z = \frac{p_z}{m_0 c} = \frac{a_0^2}{2} \sin^2(\omega\tau). \tag{21}$$

Using $p_z = \gamma m_0 v_z$, we have for the electron velocity in the z-direction in the rest frame

$$\frac{v_z}{c} = \frac{a_0^2}{2\gamma} \sin^2(\omega\tau). \tag{22}$$

Integrating again gives an expression for the distance z of travel of an electron in an electromagnetic field as a function of retarded time τ. Introducing the wavevector amplitude $k = \omega/c$ for the field, we have that the position z of the electron in the rest frame is given by

$$k\, z = \frac{a_0^2}{8\gamma} \left(-\sin(2\omega\tau) + 2\omega\tau\right). \tag{23}$$

The electron oscillates in time in the z-direction at a frequency close to 2ω with a superimposed "drift velocity". The drift velocity v_d of the electron can be regarded as given by

$$\frac{v_d}{c} = \frac{z}{c\,t} = \left(\frac{a_0^2}{4\gamma}\right)\frac{\tau}{t} = \left(\frac{a_0^2}{4\gamma}\right)\frac{t - z/c}{t} = \left(\frac{a_0^2}{4\gamma}\right)\left(1 - \frac{v_d}{c}\right)$$

Re-arranging

$$\frac{v_d}{c} = \frac{a_0^2}{4\gamma + a_0^2}. \tag{24}$$

The phase of the light field in the rest frame is obtained using Equations 14 and 23:

$$\omega\, t = \omega\, \tau + k\, z = \omega\, \tau + \frac{a_0^2}{8\gamma} \left(-\sin(2\omega\tau) + 2\omega\tau\right). \tag{25}$$

We need to evaluate an expression for the electron Lorentz parameter γ appropriate to the rest frame. The total electron velocity v in the rest frame comprises two components in the x- and z-directions such that

$$v^2 = v_x^2 + v_z^2.$$

The Lorentz factor γ for the electron in the rest frame can be related to the electron momenta in the x- and z-directions:

$$\gamma^2 = 1 + \left(\frac{p_x}{m_0 c}\right)^2 + \left(\frac{p_z}{m_0 c}\right)^2. \tag{26}$$

Substituting expressions for the x-momentum (Equation 16) and z-momentum (Equation 21) gives

$$\gamma^2 = 1 + a_0^2 \sin^2(\omega\tau) + \frac{a_0^4}{4}\sin^4(\omega\tau) = \left(1 + \frac{a_0^2}{2}\sin^2(\omega\tau)\right)^2.$$

The time varying Lorentz parameter in the rest frame for the electron becomes

$$\gamma = 1 + \frac{a_0^2}{2}\sin^2(\omega\tau). \tag{27}$$

The cycle-average for the square of the sine here is 1/2 suggesting that the cycle-averaged Lorentz parameter has a value

$$<\gamma>= 1 + \frac{a_0^2}{4}. \tag{28}$$

For a_0 less than approximately 2, this cycle-averged Lorentz parameter can be approximated by

$$<\gamma>\approx \left(1 + \frac{a_0^2}{2}\right)^{1/2}. \tag{29}$$

As well as being appropriate at low intensity ($a_0 < 2$) in the rest frame, Equation 29 represents the average Lorentz factor in the frame of reference moving with the electron along the z-axis.

Substituting the time-varying Lorentz factor γ (Equation 27) into Equation 22, the instantaneous velocity in the z-direction in the rest frame becomes

$$\frac{v_z}{c} = \frac{a_0^2 \sin^2(\omega\tau)}{2 + a_0^2 \sin^2(\omega t)}. \tag{30}$$

Using the cycle-average for the squares of the sinusoidal variation, the cycle-averaged drift velocity can be regarded as

$$\left(\frac{v_z}{c}\right)_{av} = \frac{a_0^2}{4 + a_0^2}. \tag{31}$$

A different cycle averaged relationship is obtained if we substitute the cycle-averaged Lorentz parameter (Equation 28) into Equation 24, namely

$$\frac{v_d}{c} = \frac{a_0^2}{4 + 2a_0^2}. \tag{32}$$

Equation 30 shows that the instantaneous velocity in the z-direction varies from zero to a maximum drift velocity given by

$$\left(\frac{v_z}{c}\right)_{max} = \frac{a_0^2}{2 + a_0^2}. \tag{33}$$

The maximum velocity in the direction of light propagation at low a_0 is $a_0^2 c/2$, while at high a_0, the maximum velocity approaches the speed of light c.

Our different cycle-averaging procedure here show that a cycle-average for electron velocities in an electromagnetic field is difficult to define as, for example, the instantaneous velocity in the direction of light propagation varies from zero up to $(v_z/c)_{max}$. However, the cycle-averaged electron velocity is always in the direction of light propagation which causes light to exert a force on free electrons in the direction of the wave vector $\mathbf{k}$. Using the equations determined in this Section, we can plot the position of an electron in a constant electromagnetic field as a function of time (see Figure 1).

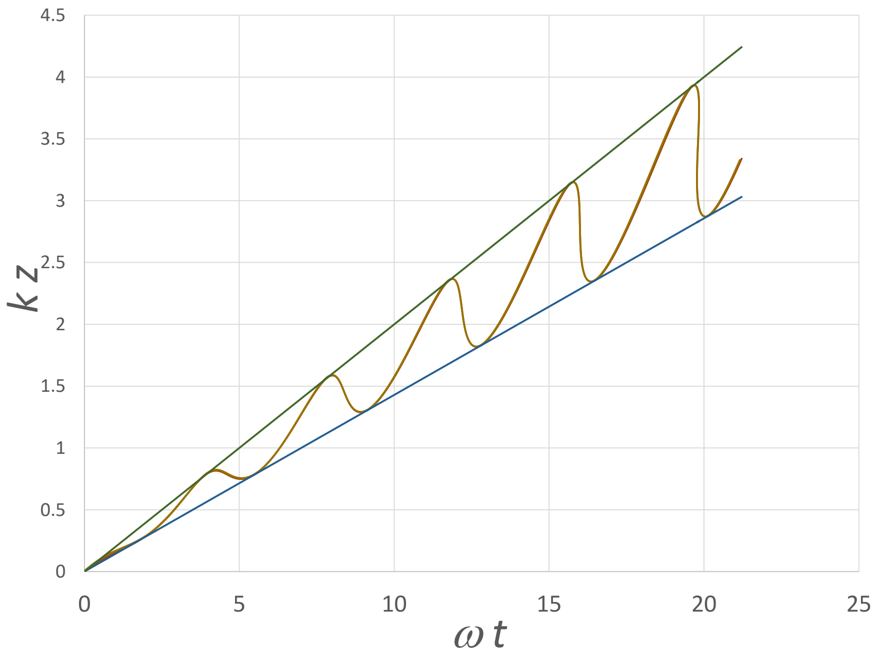

Figure 1: The electron position along the direction of light propagation as a function of time in the rest frame. The results are plotted in terms of the frequency ω and wavenumber k of the light for a constant reduced vector potential of $a_0 = 1$ ($I\lambda^2 = 1.37 \times 10^{18}$ W cm$^{-2}\mu$m^2). The higher line fitting the maximum points of oscillation is determined using Equation 31, while the lower line fitting the minimum points of oscillation is determined using $v_d/c = a_0^2/(4 + 3a_0^2)$.

The cycle-averaged electron velocity in the x-direction (the direction of the electric field) is, of course, zero. Substituting Equation 27 in Equation 18, we find that the maximum electron velocity in the x-direction is given by

$$\left(\frac{v_x}{c}\right)_{max} = \frac{a_0}{1 + a_0^2/2}. \tag{34}$$

Equation 34 suggests that the maximum electron velocity in the direction of the electric field is produced when $a_0 = \sqrt{2}$ with a maximum velocity at this reduced vector potential of $c/\sqrt{2}$. The maximum electron velocity in the $\mathbf{k}$-direction when $a_0 = \sqrt{2}$ is $c/2$ with $\gamma = 2$. The variation of the maximum velocities parallel to the electric field and parallel to the direction of electromagnetic beam propagation as a function of light intensity are shown in Figure 2.

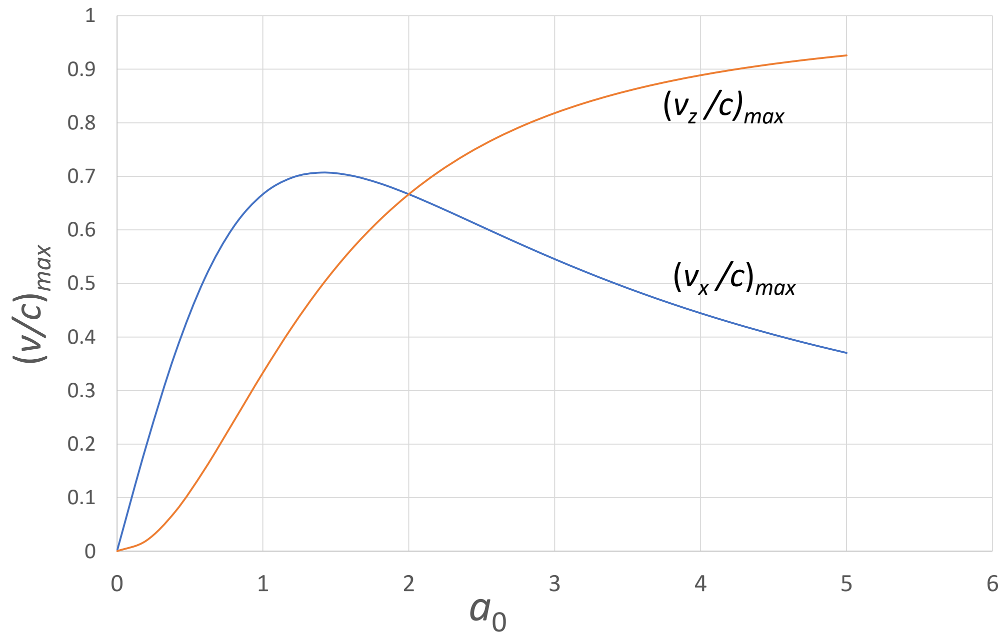

Figure 2: The maximum electron velocities in the rest frame for an electron in a linearly polarized electromagnetic field at the peak of the field oscillation as a function of the reduced vector potential a_0. The velocity v_x is in the direction of the electric field, while the velocity v_z is in the direction of beam propagation.

4 Elliptical polarization

An elliptically polarized light wave can be represented by two oscillating electromagnetic fields with a phase difference of $\pi/2$ between them. In a similar co-ordinate system as employed for linear polarization, we can write for the electric fields

$$E_x = E_{x0}\cos(\omega t)$$
$$E_y = E_{y0}\sin(\omega t) \tag{35}$$

The magnetic fields for elliptically polarized light can be written as

$$B_y = B_{y0}\cos(\omega t) = \frac{E_x}{c}\cos(\omega t)$$
$$B_x = B_{x0}\sin(\omega t) = \frac{E_y}{c}\sin(\omega t). \tag{36}$$

In the rest frame, the forces on an electron of charge $-e$ in the x-, y- and z-directions can be written as

$$\frac{dp_x}{dt} = -e\left(E_x - v_z B_y\right) = -eE_x\left(1 - \frac{p_x}{\gamma m_0 c}\right)$$
$$\frac{dp_y}{dt} = -e\left(E_y - v_z B_x\right) = -eE_y\left(1 - \frac{p_y}{\gamma m_0 c}\right)$$

$$\frac{dp_z}{dt} = -e(v_x B_y + v_y B_x) = -e\left(\frac{E_x p_x + E_y p_y}{\gamma m_0 c}\right). \tag{37}$$

These equations can be solved using a similar method to the solution for linear polarization by introducing the degree of ellipticity ϵ of the incident light polarization such that

$$\epsilon = \frac{E_y^2}{E_x^2} = \frac{E_{y0}^2}{E_{x0}^2}\cot^2(\omega t). \tag{38}$$

The electron momenta p_x and p_y are similarly related with $\epsilon = p_y^2/p_x^2$. The force on the electron in the z-direction becomes

$$\frac{dp_z}{dt} = -\frac{e}{\gamma m_0 c}(E_x p_x + E_y p_y) = -\frac{eE_x p_x}{\gamma m_0 c}(1+\epsilon) \tag{39}$$

Eliminating the electric field E_x from Equations 37 and 39, we find that

$$\frac{dp_x}{dt} = \left(\frac{1}{1+\epsilon}\right)\left(\frac{m_0 c}{p_x}\right)\frac{dp_z}{dt}\left(\gamma - \frac{p_z}{m_0 c}\right) \tag{40}$$

It is convenient to use reduced momenta written in units of $m_0 c$. Equation 2 becomes

$$\frac{d\hat{p}_x}{dt} = \left(\frac{1}{1+\epsilon}\right)\frac{1}{\hat{p}_x}\frac{d\hat{p}_z}{dt}\left(\gamma - \hat{p}_z\right). \tag{41}$$

The Lorentz parameter can be evaluated using

$$\gamma^2 = 1 + \hat{p}_x^2 + \hat{p}_x^2 + \hat{p}_z^2 = 1 + \hat{p}_x^2(1+\epsilon) + \hat{p}_z^2. \tag{42}$$

Differentiating the Lorentz parameter

$$\gamma\frac{d\gamma}{dt} = \hat{p}_x\frac{\hat{p}_x}{dt}(1+\epsilon) + \hat{p}_z\frac{\hat{p}_z}{dt}.$$

Using Equation 41, we have that

$$\frac{d\hat{p}_z}{dt} = \frac{d\gamma}{dt}. \tag{43}$$

As for linear polarization, integrating with appropriate limits shows that the z-directed momentum and Lorentz parameter vary proportionally in time such that

$$\hat{p}_z(t) = \gamma(t) - 1. \tag{44}$$

We can evaluate the angle θ of the electron trajectory to the z-axis (**k**-direction of the light). At any time:

$$\cos\theta = \frac{\hat{p}_z}{(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2)^{1/2}} \tag{45}$$

The instantaneous velocity v of the electron is given by

$$\left(\frac{v}{c}\right) = \left(\left(\frac{v_x}{c}\right)^2 + \left(\frac{v_y}{c}\right)^2 + \left(\frac{v_z}{c}\right)^2\right)^{1/2}$$

$$= \frac{1}{\gamma}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2)^{1/2}. \tag{46}$$

The expression for the Doppler shifted frequency ω' of the incident light in the frame of the electron is given by Equation 11. Substituting Equations 45 and 46, the Doppler shifted frequency in the electron frame is given by the same expression as found for linear polarization

$$\omega' = \omega\ (\gamma - \hat{p}_z) \tag{47}$$

Using Equation 44 we see that $\omega' = \omega$. As found for linearly polarized light, there is no Doppler shift of the light frequency in the frame of the electron for elliptically polarized light.

The reduced vector potentials for elliptically polarized light can be defined by

$$a_x = \frac{eE_{x0}}{m_0 c\,\omega}$$

$$a_y = \frac{eE_{y0}}{m_0 c\,\omega}. \tag{48}$$

The reduced vector potential magnitudes a_x and a_y are related to the laser intensity $I = (1/2)\epsilon_0 c(E_{x0}^2 + E_{y0}^2)$ (power per unit area) when measured in W cm^{-2} by

$$a_x^2 + a_y^2 = \frac{I\lambda^2}{1.37 \times 10^{18}} \tag{49}$$

when the light wavelength $\lambda = 2\pi c/\omega$ is measured in microns. As discussed for linear polarization, quantities of form I/ω^3 are Lorentz invariant. As I/ω^3 and the frequency ω are Lorentz invariant, the vector potentials a_x and a_y and the intensity I are Lorentz invariant for an electron in an elliptically polarized electromagnetic field.

5 Electron motion in elliptically polarized light fields

In this Section, we calculate the motion of a single electron in an elliptically polarized light field. In the frame moving along the direction of light propagation z with the electron, the velocity v_z' in the z-direction is zero, so that the Lorentz forces $|\mathbf{v} \times \mathbf{B}| = v_z' B_y' = v_z' B_x' = 0$. The forces parallel to the two electric fields in the frame of the electron are given by

$$\frac{dp_x}{d\tau} = -eE_{x0}\cos(\omega\tau)$$

$$\frac{dp_y}{d\tau} = -eE_{y0}\sin(\omega\tau) \tag{50}$$

Integrating, we have the electron momentum component in the x- and y-directions:

$$\frac{p_x}{m_o c} = -\frac{eE_{x0}}{m_0 c\,\omega}\sin(\omega\tau) = -a_x\sin(\omega\tau)$$

$$\frac{p_y}{m_o c} = \frac{eE_{y0}}{m_0 c\,\omega}\left(1 - \cos(\omega\tau)\right) = a_y\left(1 - \cos(\omega\tau)\right) \tag{51}$$

upon using reduced vector potentials $a_x = eE_{x0}/(m_0c\omega)$ and $a_y = eE_{y0}/(m_0c\omega)$. The momenta p_x and p_y in the rest frame are related to the x-and y-directed electron velocity by $p_x = \gamma m_0 v_x$ and $p_y = \gamma m_0 v_x$, where γ is the Lorentz parameter for the moving electron in the rest frame. Using $p_x = \gamma m_0 v_x$ and $p_y = \gamma m_0 v_y$, we have for the electron velocity in the x and y-direction in the rest frame

$$\frac{v_x}{c} = -\frac{a_x}{\gamma}\sin(\omega\tau)$$

$$\frac{v_y}{c} = \frac{a_y}{\gamma}(1-\cos(\omega\tau)). \tag{52}$$

The electron oscillates in time in the x and y-directions at a frequency close to ω, but with a phase difference of $\pi/2$ between the x and y-directions.

Due to the $-e\mathbf{v}\times\mathbf{B}$ force arising from the $\mathbf{B}$ field of the electromagnetic wave, the force on the electron in the direction of wave propagation (the z-direction) with elliptical polarization is given by

$$\frac{dp_z}{d\tau} = -e(v_x B'_y + v_y B'_x). \tag{53}$$

The magnetic fields B'_x and B'_y are directed perpendicularly to the frame velocity along the z-axis, so that the magnetic fields B'_x and B'_y in the frame moving along the z-axis with the electron are related to the rest frame magnetic fields by $B'_x = \gamma B_x$ and $B'_y = \gamma B_y$. Substituting for v_x and v_y (Equation 52) and noting that $B_x = E_y/c$ and $B_y = E_x/c$ we can write that the force in the rest frame is given by

$$\frac{dp_z/(m_0c)}{d\tau} = \left(\frac{e(E_x\,a_x - E_y\,a_y)}{m_0c}\right)\sin(\omega\tau)\cos(\omega\tau) + \frac{eE_ya_y}{m_0c}\sin(\omega\tau). \tag{54}$$

Integrating we obtain the momentum in the z-direction

$$\frac{p_z}{m_0c} = \left(\frac{a_x^2 - a_y^2}{2}\right)\sin^2(\omega\tau) - (\cos(\omega\tau))\,a_y^2. \tag{55}$$

assuming that the initial z-directed momentum is zero. Re-writing Equation 55 with the approximation that the average of $1-\cos(\omega\tau)$ is $1/2$ gives that

$$\frac{p_z}{m_0c} \approx -\left(\frac{a_x^2 - a_y^2}{4}\right)\cos(2\omega\tau) + \frac{a_x^2 + a_y^2}{4}. \tag{56}$$

Using $p_z = \gamma m_0 v_z$, we have for the electron velocity in the z-direction in the rest frame

$$\frac{v_z}{c} \approx -\left(\frac{a_x^2 - a_y^2}{4\gamma}\right)\cos(2\omega\tau) + \frac{a_x^2 + a_y^2}{4\gamma} \tag{57}$$

Integrating the z-directed velocity results in the evaluation of the position z of the electron in terms of the wavevector k for the incident light:

$$k\,z = \left(\frac{a_x^2 - a_y^2}{8\gamma}\right)\sin(2\omega\tau) + \left(\frac{a_x^2 + a_y^2}{4\gamma}\right)\omega\tau \tag{58}$$

The phase ωt of the elliptically polarized light field in the rest frame is obtained using Equations 14 and 58:

$$\omega\, t = \omega\, \tau + k\, z = \omega\, \tau + \left(\frac{a_x^2 - a_y^2}{8\gamma}\right)\sin(2\omega\tau) + \left(\frac{a_x^2 + a_y^2}{4\gamma}\right)\omega\tau. \tag{59}$$

Equations 57 and 58 shows that the electron oscilates at a frequency of 2ω in the direction of light propagation (the z-direction) superimposed on a drift velocity. For circular polarization ($E_x = E_y$), the z-oscillation disappears with the electron still exhibiting a drift velocity in the direction of light propagation.

The value of the Lorentz parameter γ is found using

$$\gamma^2 = 1 + a_x^2 \sin^2(\omega\tau) + a_y^2(\cos(\omega\tau) - 1)^2 + \left(\frac{p_z}{m_0 c}\right)^2$$

$$= 1 + a_x^2 \sin^2(\omega\tau) + a_y^2(\cos(\omega\tau) - 1)^2 + (\gamma - 1)^2.$$

using $p_z/m_0c = \gamma - 1$ (Equation 44). Simplifying, we find for the Lorentz paramter

$$\gamma = 1 + \frac{a_x^2}{2}\sin^2(\omega\tau) + \frac{a_y^2}{2}(\cos(\omega\tau) - 1)^2 \tag{60}$$

The cycle-averaged Lorentz parameter becomes

$$<\gamma>= 1 + \frac{a_x^2}{4} + \frac{a_y^2}{4} \tag{61}$$

For circular polarization, setting $a_x = a_y$ means that the Lorentz parameter simplifies to

$$\gamma = 1 + a_x^2(1 - \cos(\omega\tau)) \tag{62}$$

The cycle-averaged drift velocity of the electron along the z-axis is found by substituting the value of the cycle-averaged Lorentz parameter (Equation 61) into the drift velocity obtained from Equation 58:

$$\left(\frac{v_z}{c}\right)_{av} = \frac{(a_x^2 + a_y^2)}{4\gamma} = \frac{(a_x^2 + a_y^2)}{4 + (a_x^2 + a_y^2)}. \tag{63}$$

If $a_x^2 + a_y^2$ for elliptical polarization is replaced by a_0^2 as used for linear polarization, the drift velocity is the same as found for linear polarization (see Equation 31).

References

[1] G. J. Tallents. *An introduction to special relativity for radiation and plasma physics.* Cambridge University Press, Cambridge, U.K., 2022.