Additional Exercises

An Introduction to the Atomic and Radiation Physics of Plasmas

Author:

Greg Tallents

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1 Introduction

Additional Exercises for 'Tallents, G. (2018). An Introduction to the Atomic and Radiation Physics of Plasmas. Cambridge: Cambridge University Press' are presented here. The text of this book develops the physics of emission, absorption and interaction of light in astrophysics and in laboratory plasmas from first principles using the physics of various fields of study including quantum mechanics, electricity and magnetism, and statistical physics. The book links undergraduate level atomic and radiation physics with the advanced material required for postgraduate study and research. Additional Exercises are presented here, sometimes along with a comment added in brackets indicating a numerical answer, or in some cases, wider implications of the Exercise.

Exercises relevant to each chapter are included at the end of each chapter in the book. Many of the Additional Exercises presented here use the physics developed over more than one chapter. References to Equations, Sections and Exercises refer to the text of the book, while questions presented in the following pages are referred to as Additional Exercises.

2 Fundamentals and the hydrogen atom

2.1

The Stern-Gerlach experiment used a non-uniform magnetic field to accelerate a beam of silver atoms transversely. If the magnetic field gradient $\nabla \mathbf{B}$ is uniform across the atomic beam and extends along the beam a length L, show that the atoms are deflected by angles θ, such that

$$\theta = \pm \frac{g_s \mu_B}{2E} L \nabla B$$

where $g_s = 2.0023$ is the g-factor for electron spin, μ_B is the Bohr magneton and E is the kinetic energy of the silver atoms. Evaluate a numerical deflection angle if the magnetic field gradient is 10 Tm^{-1}, the atoms have a kinetic energy of 1 eV and the length of the field along the atomic beam $L = 1$ m. [±0.6 mrad.]

2.2

The ground state of the hydrogen atom has a wavefunction given by

$$\psi_1 = \frac{1}{\sqrt{\pi} a_0^{3/2}} \exp(-r/a_0)$$

where r is the distance from the nucleus and a_0 is the Bohr radius. The nuclear potential $V(r)$ of the hydrogen atom is given by

$$V(r) = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{r}.$$

Show that the expectation value $\int \psi_1 V(r) \psi_1 dV$ of the potential energy for the hydrogen ground state is such that

$$\int \psi_1^* V(r) \psi_1 dV = -\frac{e^2}{4\pi\epsilon_0 \, a_0}.$$

[This potential energy is equal to $-2R_d$, where R_d is the ionisation energy of the hydrogen ground state and is identical to the value predicted by the Bohr model, see Equation 1.30].

2.3

The kinetic energy operator $\hat{T}$ is given by

$$\hat{T} = \frac{\hat{p}^2}{2m_0} = -\frac{\hbar^2}{2m_0} \nabla^2$$

where m_0 is the electron mass. Using the wavefunction ψ_1 given for the above Additional Exercise 2.2, show for the ground state of hydrogen that

$$\nabla^2 \psi_1 = \frac{1}{\sqrt{\pi} a_0^{3/2}} \frac{1}{r^2} e^{-r/a_0} \frac{r}{a_0} \left(\frac{r}{a_0} - 2 \right)$$

and that the expectation value T for the kinetic energy of the ground state electron is given by

$$T = \int \psi_1^* \hat{T} \psi_1 dV = \frac{\hbar^2}{2m_0 \, a_0^2}.$$

2.4

As the total ground state energy for hydrogen comprises the addition of the kinetic and potential energy, use the results from the Additional Exercises 2.2 and 2.3, to show that the ionisation energy R_d of the hydrogen atom has a value

$$R_d = \frac{\hbar^2}{2m_0\, a_0^2}.$$

3 Quantum mechanics

3.1

Using Equation 7.20, show that the first order relativistic correction to the kinetic energy operator $\hat{T}$ leads to an expression

$$\hat{T} = \hat{T}_0 - \frac{1}{2m_0c^2}\hat{T}_0^2$$

where $\hat{T}_0 = \hat{p}^2/2m_0$ is the kinetic energy operator neglecting relativity. Here $\hat{p}$ is the momentum operator.

3.2

Setting

$$\hat{T}_R = -\frac{1}{2m_0c^2}\hat{T}_0,$$

for the relativstic correction to the kinetic energy operator, prove the following commutator relationship

$$[\hat{x}, [\hat{x}, \hat{H}_0 + \hat{T}_R]] = -\frac{\hbar^2}{m_0}\left(1 - \frac{3}{m_0c^2}\hat{T}_0\right)$$

where $\hat{H}_0 = \hat{T}_0 + V(r)$ is the Hamiltonian for the electron energy in a central potential $V(r)$ and $\hat{x}$ is an operator for the component of a position vector along the x-axis.

3.3

The general sum rule for Hermitian operators $\hat{f}$ has the following relationship

$$\frac{1}{2}\int \psi_1^*[\hat{f},[\hat{f},H_0]]\psi_1 dV = -\sum_j (E_j - E_1)\left|\int \psi_1^* \hat{f}\psi_j dV\right|^2.$$

where the sum is over all wavefunctions ψ_j with energy E_j and $\hat{H}_0$ is the energy Hamiltonian (see Equation A.33 in Appendix D). If this relationship is valid[1] when $\hat{H}_0$ is replaced by $\hat{H}_0 + \hat{T}_R$, use the result of the Additional Exercise 3.2 to show that

$$\sum_j (E_j - E_1)\left|\int \psi_1^* \hat{x}\psi_j dV\right|^2 = \frac{\hbar^2}{2m_0}\left(1 - \frac{3}{m_0 c^2}T_1\right)$$

where T_1 is the expectation value of the kinetic energy for wavefunction ψ_1.

3.4

Use the definition of an oscillator strength f_{1j} (Equation 10.29) and the result of the Additional Exercise 3.3 to show that the oscillator strength sum rule for absorption from a quantum state subject to a small relativistic kinetic energy correction becomes

$$\sum_j f_{1j} = 1 - \frac{3}{m_0 c^2}T_1$$

where T_1 is the kinetic energy expectation value of the state 1. [Caveat: The general sum rule with the relativistic correction is not exact as assumed in the Additional Exercise 3.3, so that the correct relativistic oscillator strength sum rule becomes:

$$\sum_j f_{1j} = 1 - \frac{5}{3m_0 c^2}T_1,$$

according to H. Sinky and P. T. Leung 2006 Phys. Rev. A74, 034703 'Relativistic corrections to a generalized sum rule'.]

[1] Unfortunately, the relationship is not exact with the extra $\hat{T}_R$ term, but assuming that the extra $\hat{T}_R$ term is negligible gives an answer close to that of more complete treatments, see the caveat for the Additional Exercise 3.4.

4 Radiative processes

4.1

In section 5.1, the time taken for cyclotron radiation to cause the electron kinetic energy to decrease was examined assuming the electron velocity distribution is isotropic relative to a magnetic field $\mathbf{B}_0$. The electron kinetic energy E_{kin} was shown to decrease with time t such that $E_{kin}(t) = E_{kin}(0)\exp(-t/t_0)$ with $t_0 = 0.3/B_0^2$ seconds, when the magnetic field B_0 is measured in Tesla. Show that electrons with velocity $\mathbf{v}$ at an angle α to the magnetic field $\mathbf{B}_0$ radiate so that their kinetic energy decreases with a time constant $t_0 = 2.7\sin^2\alpha/B_0^2$ seconds.

4.2

Formulas for the black-body radiation discussed in Chapter 4 in terms of angular frequency ω are often expressed in terms of the wavelength λ of the radiation. We have shown in Chapter 4 (equation 4.9) that inside a black-body the radiation intensity in units of power per unit area per unit steradian per unit angular frequency is given by

$$J_p(\omega) = \frac{\hbar\omega^3}{4\pi^3 c^2}\frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1}.$$

Show that the black body intensity $B(\lambda)$ in units of power per unit area per unit steradian per unit of wavelength is given by

$$B(\lambda) = \frac{2hc^2}{\lambda^5}\frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}.$$

Hence show that the Rayleigh-Jeans expression valid for long wavelengths can be written as

$$B(\lambda) \approx \frac{2c}{\lambda^4}k_B T.$$

4.3

The cross-section σ_{12} for the absorption of radiation by a transition from a bound quantum state 1 to another bound state 2 alters the intensity I of radiation travelling in the direction z following

$$\frac{dI}{dz} = N_1\sigma_{12}I.$$

By considering the Einstein B-coefficient (see Exercise 10.4), show that

$$\sigma_{12} = \frac{\hbar\omega}{c}\frac{\pi e^2}{3\epsilon_0\hbar^2}\left|\int \psi_1 x \psi_2 dV\right|^2 f(\omega - \omega_{21})$$

where the integral is over the initial (1) and final (2) bound quantum states and $f(\omega - \omega_{21})$ is the lineshape variation in frequency ω from the line centre frequency ω_{21}.

4.4

From Exercise 10.2, the oscillator strength f_{12} for a bound-bound radiative transtion is given by

$$f_{12} = \frac{2\omega_{21} m_0}{\hbar}\left|\int \psi_1 x \psi_2 dV\right|^2 .$$

Using the result from the above Additional Exercise 4.3, show that

$$\sigma_{12} = 2\pi^2 \frac{\hbar}{m_0}\alpha\, f_{12}\ f(\omega - \omega_{21})$$

where $\alpha = 1/137$ is the fine structure constant.

4.5

The value $\sigma_{bf} d\omega$ for bound-free absorption from the principal quantum number n states of hydrogen-like ions of atomic number Z gives the cross-section for the absorption in the frequency range ω to $\omega + d\omega$. Karzas and Latter give an expression for the cross-section per frequency unit (see Equation 5.27) as follows:

$$\sigma_{bf} = \frac{16}{3\pi}\alpha^3 \frac{\pi^2 c^2}{\hbar\omega^3}\frac{R_d Z^4}{n^5}\frac{G_{bf}}{\sqrt{3}}$$

where $G_{bf} \approx 1$ is the Gaunt factor. Show that the oscillator strength $f_{bf} d\omega$ for bound-free absorption in a hydrogen-like ion over the frequency range ω to $\omega + d\omega$ is given by

$$f_{bf} d\omega = \frac{8}{3\sqrt{3}\pi}\alpha^2 \frac{m_0 c^2}{\hbar^2\omega^3}\frac{R_d Z^4}{n^5} G_{bf} d\omega.$$

4.6

Using photon energy units E_R of $Z^2 R_d$, evaluate the result from the above Additional Exercise 4.5 to show that the bound-free oscillator strength for the ground state $n = 1$ of a hydrogen-like ion is given by

$$f_{bf}\, dE_R = 0.98107\, G_{bf} \frac{dE_R}{E_R^3}.$$

4.7

The bound-bound absorption oscillator strengths f_{1n} from the ground state of hydrogen and hydrogen-like ions can be evaluated and summed. We have

$$\sum_{n=2}^{\infty} f_{1n} = 0.565.$$

The Thomas-Reiche-Kuhn sum rule requires the sum of all oscillator strengths both bound-bound and bound-free from the ground state to be equal to unity:

$$\int_{E_R=1}^{\infty} f_{bf}\, dE_R + \sum_{n=2}^{\infty} f_{1n} = 1.$$

Considering the result from the Additional Exercise (4.6), evaluate an appropriate frequency averaged Gaunt factor $\tilde{G}_{bf}$ so that the Thomas-Reiche-Kuhn sum rule is satisfied. [0.887]

4.8

In section 3.5.2 we noted that the frequency ω_s of scattered light is related to the incident frequency ω_i of light by

$$\omega_s = \omega_i \frac{1 - \hat{\mathbf{i}} \cdot \mathbf{v}/c}{1 - \hat{\mathbf{s}} \cdot \mathbf{v}/c}$$

where $\hat{\mathbf{i}}$ is a unit vector in the incident light direction, $\hat{\mathbf{s}}$ is a unit vector in the scattered light direction and $\mathbf{v}$ is the velocity of the scattering object. Show that a plane mirror moving at speed v perpendicular to the mirror plane towards the light source reflects light incident at an angle of incidence θ with a frequency shift $\omega_s - \omega_i$ in the laboratory frame given by

$$\omega_s - \omega_i = \frac{2\,\omega_i\, v}{c} \cos\theta,$$

provided the speed v is small compared to the speed of light c. [A 'plane mirror' could be the critical density in a laser-produced plasma, see e.g. A. Adak et al 2014 Phys. Plasmas 21, 062704. 'Ultrafast dynamics of a near-solid-density layer in an intense femtosecond laser-excited plasma'.]

4.9

Use Equation 4.33 to show that in LTE the source function S employed in radiative transfer calculations is given by

$$S = \frac{\hbar\omega^3}{\pi^2 c^2}\frac{1}{\exp(\hbar\omega/k_B T) - 1}.$$

4.10

Consider a steady-state balance of the populations N_2 and N_1 of two bound quantum states with collisional excitation at rate $N_1 K_{12} n_e$, de-excitation at rate $N_2 K_{21} n_e$ and spontaneous radiative decay at rate $A_{21} N_2$ added to rates of photo-excitation and stimulated emission included in Equation 4.26. If only collisional and radiative processes between the two states 1 and 2 are significant, show that the ratio of the populations N_2 and N_1 in steady state are given by

$$\frac{N_2}{N_1} = \frac{K_{12} n_e + (g_2/g_1)(\pi^2 c^3/\hbar\omega_{21}^3) A_{21} \int_0^\infty (I(\omega)/c)\, f(\omega)\, d\omega}{K_{21} n_e + A_{21} + (\pi^2 c^3/\hbar\omega_{21}^3) A_{21} \int_0^\infty (I(\omega)/c)\, f(\omega)\, d\omega}$$

where $f(\omega)$ is the line shape profile and $I(\omega)$ is the radiation intensity at frequency ω within the plasma. The parameters g_1 and g_2 are the degeneracies of the quantum states 1 and 2 and ω_{21} is the frequency of the line centre for radiative transitions between states 1 and 2.

4.11

(a) Using the results of the Additional Exercise 4.10, show in the limit of very large radiation intensity $I(\omega)$ that

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}.$$

(b) In the limit where radiative processes become negligible compared to collisional processes show that

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \exp\left(-\frac{\hbar\omega_{21}}{k_B T}\right).$$

[For (b) refer to Equation 12.4].

4.12

Using the Equations of section 4.3, show for a photon energy $\hbar\omega_{21}$ corresponding to the energy difference between two quantum states that the source function S employed in radiative transfer calculation is given by

$$S = \frac{\hbar\omega_{21}^3}{\pi^2 c^2} \frac{1}{(g_2 N_1)/(g_1 N_2) - 1}$$

where N_1 and N_2 are the population densities of the two states 1 and 2. [This result along with the Equation found for Additional Exercise 4.10 can be used to determine the source function at the frequency ω_{21} corresponding to a transition between two quantum states where only collisional and radiative processes between the two states (and no other states) are significant.]

4.13

Consider the steady-state population balance between two quantum states as proposed in Additional Exercise 4.10. Show that the probability P_R of an absorbed photon being re-emitted is given by

$$P_R = \frac{A_{21} + (\pi^2 c^3/\hbar\omega_{21}^3) A_{21} \int_0^\infty (I(\omega)/c)\, f(\omega)\, d\omega}{A_{21} + n_e K_{21} + (\pi^2 c^3/\hbar\omega_{21}^3) A_{21} \int_0^\infty (I(\omega)/c)\, f(\omega)\, d\omega}.$$

4.14

Consider the scenario for Additional Exercise 4.13 with the radiation of intensity $I(\omega)$ incident in a single direction from a backlighter through a thin, uniform plasma. Show that the fraction f_S of the absorbed radiation which is re-emitted into 4π steradian (angularly re-distributed) is given by

$$f_S = \frac{1}{1 + (\pi^2 c^3/\hbar\omega_{21}^3) \int_0^\infty (I(\omega)/c)\, f(\omega)\, d\omega}.$$

4.15

In the discussion of opacity in section 6.5, the second equation (un-numbered) obtained from equation 6.13 shows that the radiation power F_A per unit area for a planar geometry after integration in angle is given by

$$F_A = -\frac{4\pi}{3}\frac{1}{K}\frac{dJ}{dz}$$

where J is the radiation intensity per unit solid angle, K is an absorption coefficient and z represents distances perpendicular to the planes of symmetry in the one-dimensional geometry. Assume a sharp radiation 'heat front' propagates into a plasma as described in section 6.4 so that the radiation F_A at time t heats a thin plasma layer of density ρ, heat capacity C_V per unit density and thickness z to $z + \Delta z$ to a temperature T. If the temperature before the heat front impinges on the plasma can be regarded as zero and we have a Planck black body radiation distribution in the heat front, show that

$$z^2 = \frac{8}{3}\frac{\sigma_{SB}T^3}{K_{Ros}C_V\ \rho}t$$

where σ_{SB} is the Stefan-Boltzmann constant, K_{Ros} is the Rosseland mean absorption coefficient and t is the time for the heat front to propagate a distance z.

5 Astrophysics and space

5.1

The solar surface is typically 5800 K, while sunspots are regions appearing darker with lower temperatures. If a sunspot has a temperature of 4200 K, determine the relative spectrally integrated intensity of the sunspot compared to other parts of the solar photosphere. [0.275].

5.2

The Sun has a regular 11 year cycle where the number of sunspots drops to close to zero and then increases to 100 - 200 per year at a solar maximum. However, the spectrally integrated solar output only varies by approximately 0.1%. Estimate the relative area of the solar photosphere taken up by sunspots at the time of solar maximum. [0.14%].

5.3

Given the solar photosphere temperature of 5800 K and that the radius of the Sun is 6.957×10^5 km, determine (a) the mass of solar material converted to energy per second. (b) The present mass of the Sun is 1.989×10^{30} kg and the Sun is 4.6×10^9 years old. Estimate the fraction of solar mass that has been converted to energy in the life of the Sun. [(a) 4.3×10^9 kg s^{-1}, (b) 3×10^{-4}].

5.4

The Hertzsprung-Russell diagram effectively plots the total power (luminosity) radiated by a star as a function of the photosphere temperature. The temperatures of the photosphere of stars range from 3000 K (star type M) to 30000 K (star type O) with star radii varying from 0.084 to 1708 compared to the solar radius. Estimate the maximum and minimum total power (luminosity) relative to that of the Sun that could be expected from this temperature and size range of the stars. [5×10^{-4} - 2×10^9. The maximum reached in practise is $\approx 10^6$ as the largest stars (supergiants) only have temperatures 3000 - 10000 K.]

5.5

The space shuttle communicated to earth via several radio channels: UHF 289.7 - 296.8 MHz, S-band 1.7 - 2.3 GHz and Ku-band 15.25-17.25 GHz. During re-entry into the earth's atmosphere there was a period when plasma formed around the shuttle and blocked all radio communication directly with the earth. From Figure 2.1, estimate the minimum electron density of the plasma formed around the shuttle during the communication black-out[2]. [10^{12} cm^{-3}.]

[2] The problem of spacecraft blackout was solved in 1988 by NASA with the launch of the Tracking and Data Relay Satellite System which allowed communication from the 'topside' (away from the earth) via a relay from orbiting satellites

5.6

The ionosphere surrounding the earth comprises plasma with electron density varying with the height above the earth's surface. Letter symbols have been given to different ionospheric regions at different heights: the D-region at 50 - 100 km with electron density 10^3 cm^{-3}, the E region at 100 km with electron density 10^5 cm^{-3} and the F_2 region at 150 - 200 km with the maximum electron density $\approx 10^6$ cm^{-3}. An ionosonde is a radar system first employed in the 1920s which transmits radar pulses vertically and records the time taken to receive a reflected signal from the ionosphere. Determine the range of frequencies needed to obtain a reflection from the D, E and F_2 regions and the approximate expected time delay between sending the radar pulse and receiving the reflection. [D-region 0.5 MHz, 0.3 - 0.6 ms; E-region 3 MHz, 0.6 ms; F_2 region 10 MHz, 1 - 1.2 ms.]

5.7

The group velocity of radar pulses from an ionosonde in the ionsophere (see Additional Exercise 5.6) affects the time for a pulse to be returned. Consider reflections propagating through 10 km of approximately uniform plasma material at an electron density $n_e = (1/2)n_{crit}$, where n_{crit} is the critical density. Calculate the extra pulse delay produced by changes to the group velocity from c by the radar pulse propagating up and back through such a 10 km length of plasma. [1.6×10^{-5} s.]

5.8

Consider a satellite orbiting in an approximately circular orbit at a speed v relative to the earth transmitting a signal at frequency ν_s. At a particular time, the satellite is at an angle θ to the vertical above a radio receiver on earth. The total electron content of the ionosphere $N_T = \int n_e dz$ is the integral of the electron density through the ionosphere along a vertical path. Allowing for the Doppler effect of the satellite motion on the radio frequency in the earth's frame of reference, show that the group velocity delay Δt due to the ionosphere of the received radio signal relative to the signal received when the satellite is vertical varies as

$$\Delta t \approx -\frac{e^2 v N_T}{2\pi\epsilon_0 m_0 c^2 \nu_s} |\tan\theta|$$

where positive velocity v is towards the radio receiver.

5.9

The first man-made satellite known as Sputnik 1 was launced into orbit on 4 October 1956 by the USSR. The Sputnik demonstrated a velocity of 29 000 km/hour relative to the earth and emitted radio 'beeps' at 20 MHz of 0.3 s duration with a gap of 0.3 s between beeps. Western listerners to Sputnik thought some telemetry information was conveyed as sometimes the beep duration and gap changed. Assuming a height above earth of 500 km when close to vertical above a radio receiver and a total electron content of 5×10^{17} m^{-2}, use the result of Additional Exercise 5.8 to estimate the change of the duration between successive beeps due to changes of the radio wave group velocity through the ionosphere. [0.005 s.]

5.10

Satellite lines are found at slightly higher wavelengths to hydrogen- and helium-like resonance lines from ions of charge, say, Z_i. In Section 12.6, the satellite lines are shown to vary in intensity approximately proportionally to $n_i n_e/T^{3/2}$, where n_i is the density of ions of charge Z_i, n_e is the electron density and T is the electron temperature . However, resonance line intensities vary approximately as $n_i n_e/T^{1/2}$. Solar flare plasmas have temperatures sufficient to excite hydrogen- and helium-like resonance and satellite lines up to iron enabling this variation in intensity with temperature to be used as a diagnostic of the solar flare electron temperature. Assume that a solar flare has a cylindrical cross-section, that measurements of spectral line intensities are integrated over the cross-section, that $n_i \propto n_e$ and that the electron density and electron temperature variations with radius r are such that

$$n_e(r) = n_e(0)\exp(-(r/\Delta r_n)^2),$$

$$T(r) = T_0 \exp(-(r/\Delta r_T)^2)$$

where Δr_n and Δr_T are measures of the spatial width of the solar flare density and temperature. Show that the ratio $R(T_0)$ of the intensity of the satellite lines to the resonance line from a solar flare with a peak temperature T_0 has a dependence

$$R(T_0) = R_u(T_{ref})\frac{T_{ref}}{T_0}\,\frac{2(\Delta r_T/\Delta r_n)^2 - 1/2}{2(\Delta r_T/\Delta r_n)^2 - 3/2},$$

where $R_u(T_{ref})$ is the intensity ratio assuming a uniform plasma at some reference temperature T_{ref}. [Assuming a uniform solar flare plasma of temperature T_0 gives $R(T_0) \propto 1/T_0$, whereas a non-uniform solar flare plasma of peak temperature T_0 with $\Delta r_T = \Delta r_n$ has $R(T_0) \propto 3/T_0$.]

5.11

The intensity I of radiation for an optically thin plasma is obtained by integrating the emission coefficient ϵ along a line-of-sight. For a plasma with circular symmetry, show that the intensity viewing along a line-of-sight at a minimum distance y from the centre of the axis of circular symmetry is related to the emission coefficient $\epsilon(r)$ variation with radius r by

$$I(y) = 2\int_y^\infty \frac{\epsilon(r) r dr}{\sqrt{y^2 - r^2}}.$$

5.12

The equation for the intensity $I(y)$ given in the Additional Exercise (5.11) can be inverted to explicity give the emission coefficient $\epsilon(r)$ as a function of radius r using the Abel inversion formula. We have that

$$\epsilon(r) = -\frac{1}{\pi}\int_r^\infty \frac{dI(y)}{dy}\frac{dy}{\sqrt{y^2 - r^2}}.$$

Consider the intensity $I(y)$ of light from the solar corona which decreases approximately linearly with distance y from the photosphere of the Sun when observed from the earth. We have approximately that

$$I(y) = \frac{I(R_S)(R-y)}{R - R_S}$$

for $R_S < y < R$, and $I(y) = 0$ for $y \geq R$. Here R_S is the radius of the solar photosphere. Show that the emission coefficient $\epsilon(r)$ for the light from the solar corona varies as

$$\epsilon(r) = \frac{I(R_S)}{\pi(R - R_S)} \ln\left[\frac{1 + \sqrt{1 - (r/R)^2}}{r/R}\right].$$

6 Tokamaks

6.1

A spectrometer used in a tokamak has a line of sight passing close to the 'inboard' vacuum wall with a minimum major radius R denoted by R_{in}. Plasma rotation without shear often occurs such that the bulk velocity v of the plasma varies with angle θ to the centre of the torus such that $v = R\,(d\theta/dt)$. Show that the Doppler shift $\delta\omega$ of line emission of frequency ω_0 in the rest frame of reference due to plasma rotation at any radius R is given by

$$\delta\omega = \frac{R_{in}\,\omega_0}{c}\frac{d\theta}{dt}.$$

6.2

The $n = 2{-}1$ transitions in He-like nickel along with Li-like satellite lines have been recorded for the JET tokamak. The dielectronic satellite line intensities relative to the resonance line increases by a factor of two for some discharges. What is the change in plasma conditions most likely to cause such a change? [The central electron temperature has dropped by a factor two.]

6.3

The spectral width of high-Z impurity line emission from the centre of tokamak plasmas is used to deduce central ion temperatures T. The Lorentzian profile due to natural broadening often needs to be taken into account. Show that the natural broadening full width at half maximum is equal to or greater than one tenth of the Doppler broadened full width at half maximum for H-like $n = 2 - 1$ emission if

$$k_B T \leq 10^2 \frac{4c^2 m_p}{9\ln 2}\frac{\hbar^2 A_{21}^2 Z^5}{R_d^2} = 5.5 \times 10^{-5} Z^5 \ \ \mathrm{eV}$$

where A_{21} is the transition probability for the hydrogen $n = 2 - 1$ (Lyman α) transition, R_d is the Rydberg energy and m_p is the mass of a proton.

7 Laser-produced plasmas

7.1

In laser-produced plasmas, inverse bremsstrahlung is often the major absorption mechanism for the laser light. The electron $n_e(z)$ and ion $n_i(z)$ density profiles in a laser-produced plasma typically decrease exponentially with distance z from the critical density n_{crit} along the target normal towards the laser, such that

$$n_e(z) = n_{crit} \exp(-z/L)$$

$$n_i(z) = n_{crit}/Z_{av} \exp(-z/L)$$

where L is the density profile scalelength and Z_{av} is the average ion charge in the plasma. The electron temperature $k_B T$ is to a first approximation constant with distance $z \geq 0$ from the critical density. Use the expression obtained in Exercise 5.8 to show that the optical depth $\tau_{abs} = \int K_{ff}\, dz$ for inverse bremsstrahlung of the laser light incident normally in a laser-produced plasma is approximately given by

$$\tau_{abs} \approx 1.35 \times 10^{-2} \frac{Z_{av} L}{\lambda^2 (k_B T)^{3/2}}$$

where the plasma scalength L is measured in microns, the temperature $k_B T$ is measured in eV and λ is the laser wavelength measured in microns.

7.2

Laser light is often incident at an angle of incidence θ_0 to the normal of a solid target. The laser light penetrates to a turning point of electron density $n_e = n_{crit} \cos^2 \theta_0$ (see Section 2.4.4). Show that the optical depth $\tau_{abs}(\theta_0)$ for inverse bremsstrahlung of the laser light incident at angle θ_0 in a laser-produced plasma is given by

$$\tau_{abs}(\theta_0) \approx 1.35 \times 10^{-2} \frac{Z_{av} L \cos^3 \theta_0}{\lambda^2 (k_B T)^{3/2}}$$

using the notation and conditions of the Additional Exercise 7.1.

7.3

Use Figure 2.6 to show that the angle of incidence θ_0 for maximum resonance absorption at the critical density in a laser-produced plasma is given by

$$\sin\theta_0 \approx 0.38 \left(\frac{\lambda}{L}\right)^{1/3}$$

assuming the laser wavelength λ and the profile scalelength L are measured in the same units.

7.4

Any focusing lens produces a range of angles of incidence θ in the near-field (away from the focus) dependent on simple geometry. Laser-produced plasma focusing geometries are often defined by the f-number f_{no} of the lens which is the ratio of the focal length to the diameter of the lens. Show that in the near-field, light incident normally to a planar target after passing through a lens of f-number f_{no} will have a maximum angle of incidence θ_{max} for the light rays given by

$$\sin\theta_{max} = \frac{1}{2f_{no}}.$$

7.5

In short scalelength laser-plasmas, resonance and inverse bremsstrahlung absorption as discussed in previous Additional Exercises becomes small. Another absortion process known as vacuum heating[3] where electrons oscillate across the solid target/vacuum boundary starts to dominate. Use the results of Additional Exercises 7.3 and 7.4 to show that vacuum heating for normal incident light is likely to dominate when the plasma scalelength L is such that

$$L < \left(\frac{0.38}{2f_{no}}\right)^3 \lambda.$$

[3]The concept of 'vacuum heating' was first proposed by Brunel and is sometimes known as Brunel heating, see F Brunel 1987 Phys. Rev. Lett. 59, 52. 'Not-so-resonant, resonant absorption'.

7.6

A technique known as VISAR (Velocity Interferometer System for Any Reflector) uses time-resolved interferometry of light reflected from a surface to measure the velocity of the surface. VISAR is commonly employed to measure the velocity of shock waves propagating in plasma as a shock wave exhibits a highly reflecting steep increase in density and pressure at the front of the wave. (a) If a VISAR using laser light of $\lambda = 0.53\mu$m normally incident onto a shock front detects 2 fringe shifts in 10 ps using a Mach-Zehnder interferometer, determined the shock velocity. (b) Assume that there is a spatially and temporally varying electron density $n_e(z,t)$ a distance z ahead of the shock such that

$$n_e(z,t) = n_s \exp\left(-\frac{z}{v_p t}\right).$$

where n_s is the electron density immediately ahead of the shock front and v_p is a velocity comparable to the shock velocity v_s. Show that for n_s much less than the critical density n_{crit} for the probing light, that the plasma ahead of the shock front produces a phase shift $\Delta\phi$ in the VISAR additional to the phase change due to the changing position of the shock front given by

$$\Delta\phi = \frac{2\pi n_s (v_p - v_s)\, t}{\lambda n_{crit}}.$$

[(a) 5×10^4 ms^{-1}.]

7.7

Parallel rays of a probe laser are incident into a laser-produced plasma parallel to the plane target surface and traverse a distance Δz through the plasma. A lens of numerical aperture N_A focussed at the centre of the target collects refracted laser light and images this onto a detector. Light incident at distances r from the target surface such that $r < r_{min}$ is refracted at angles $\theta > \theta_{max}$, where $\theta_{max} = N_A$ and appears black in the image. The electron density $n_e(r)$ in the laser-produced plasma decreases with distance r from the solid target electron density n_s and is uniform along Δz, such that

$$n_e(r) = n_s \exp(-r/L)$$

where L is a density scalelength. Show that

$$\frac{\exp(-r_{min}/L)}{L} \approx \frac{2n_{crit}}{n_s}\frac{N_A}{\Delta z},$$

where n_{crit} is the critical electron density for the probe laser wavelength.

7.8

Ignoring relativistic effects, show that the electron 'quiver' velocity v_q in the electric field of an electromagnetic wave is such that the value of $(v_q/c)^2$ averaged over a cycle of the wave has a value $a_0^2/2$, where a_0 is the reduced vector potential (see section 2.4.1).

7.9

The ponderomotive force F on an electron arises from the spatial gradient of the ponderomotive potential $<U>$. Consider a pulse of electromagnetic radiation incident on a stationary electron. Show that the ponderomotive force due to the rising edge of a light pulse causes a change of momentum dp/dt on the electron in the direction of light propagation such that $dp/dt = (1/c)d(<U>)/dt$ and that hence during the light pulse the momentum of the electron in the direction of propagation is equal to $<U>/c$. [In section 2.4.1 the microscopic motion of an electron in an electromagnetic wave is shown to produce a cycle-averaged drift momentum given by $<U>/c$.]

8 Spectroscopy

8.1

Consider two quantum states, a lower state 1 and and an upper state 2 with energy separation of $\hbar\omega_{21}$. From the relationship between the absorption oscillator strength f_{12} and radiative transition probability A_{21} between the two states (Equation 4.29), show that the frequency integrated emission coefficient ϵ_{21} for radiative transitions between states 2 and 1 is given by

$$\epsilon_{21} = \frac{2\hbar\omega_{21}^3}{c} r_e \left[g_1 f_{12} \frac{N_2}{g_2} \right]$$

where r_e is the classical electron radius (see Equation 3.4), g_1 is the degeneracy of state 1 and N_2 is the population density of state 2. [For closely spaced excited energy levels with populations $N_2 \propto g_2$, the intensity of emitted radiation is proportional to $g_1 f_{12}$. The value of the oscillator strength times the lower state degeneracy is often found in tabulations of spectral lines as it gives a good guide to the relative intensity of a spectral line.]

8.2

Consider two excited quantum states 2 and 3 separated by a small energy difference ΔE within the same ionisation stage. Show that the ratio of the emission coefficients at frequencies ω_{21} and ω_{31} from these excited states to the ground state 1 can usually be approximated by

$$\frac{\epsilon_{31}}{\epsilon_{21}} \approx \left(\frac{\omega_{31}}{\omega_{21}}\right)^3 \frac{g_1 f_{13}}{g_1 f_{12}}$$

to an accuracy of 10%, provided the electron temperature T is such that

$$k_B T > 10\,\Delta E.$$

8.3

The Balmer α ($n = 3-2$) and Balmer β ($n = 4-2$) lines of hydrogen have respective transition probabilities of 4.4101×10^7 s^{-1} and 8.4193×10^6 s^{-1}. Assuming LTE between the upper quantum states for these transitions, show that the electron temperature T is related to the ratio $R = \epsilon_{32}/\epsilon_{42}$ of the emission coefficients for the two lines by

$$\frac{k_B T}{R_d} = \frac{0.04861}{-0.7808 + \ln R}$$

where $R_d = 13.6$ eV is the Rydberg energy.

8.4

Differentiate the expression for the temperature in Additional Exercise 8.3 to show that the fractional error $\Delta T/T$ in a temperature measured using the Balmer line emission coefficient ratio R is such that

$$\frac{\Delta T}{T} = \left(\frac{k_B T}{0.04861 R_d}\right) \frac{\Delta R}{R}$$

where $\Delta R/R$ is the fractional error in the Balmer line emission coefficient ratio.

8.5

Considering the discussion in Chapter 5, show that both free-free and free-bound continuum emission can be used to determine a plasma temperature T by determing the slope s of a plot of the natural logarithm of the continuum emission coefficient as a function of photon energy $\hbar\omega$ with

$$k_B T = -\frac{1}{s}.$$

Show that the error ΔT in the temperature measurement is related to the error Δs in the measurement of the slope such that

$$\frac{\Delta T}{T} = \frac{\Delta s}{s}.$$

8.6

An optically thin expanding plasma contains ions with a velocity v along a line of sight varying linearly as $v = K_v z$ (with K_v constant with z) and a density profile $N(z)$ of the upper quantum state of ions emitting a spectral line such that

$$N(z) = N(0)\, \exp\left(-4\sqrt{\ln 2}\left(\frac{z}{\Delta z}\right)^2\right).$$

Assuming an infinitely narrow spectral line profile centred on the frequency ω_0 in the frame of the emitting ion, show that the full-width at half maximum $\Delta\omega$ of the line profile seen by a stationary observer viewing along the line of sight z is given by

$$\Delta\omega = \frac{\omega_0\, K_v\, \Delta z}{c}.$$

8.7

The Doppler broadening of spectral lines due to thermal motion of ions is used to measure ion temperatures. Bulk plasma motion as explored in Additional Exercise 8.6 can invalidate such measurements. Show that the line broadening due to a linear velocity gradient K_v and number distribution of ions as given in Additional Exercise 8.6 is equal to the Doppler broadening associated with an ion temperature T when

$$K_v \, \Delta z = 1.665 \sqrt{\frac{k_B T}{M}}$$

where M is the mass of the emitting ions.

8.8

Consider a stationary plasma affected by the opacity broadening of spectral lines. Use Figure 10.1 to estimate the maximum line centre optical depth τ_0 causing a less than 10% change to the spectral width of a spectral line. [0.3 assuming a Lorentzian profile.]

8.9

Plasmas are often 'backlit' by a broad spectrum of radiation which enables a measurement of the opacity of absorption lines. As absorption lines can be spectrally narrow, the instrument resolution $\Delta\omega_{Res}$ of a spectrometer recording the absorption feature is often larger than the spectral width $\Delta\omega_s$ of an absorption line. Assuming that the instrument resolution and absorption line profile can be both characterised by Milne profiles (see Section 10.3), show that the apparent optical depth τ_A measured for an absorption line of spectral width $\Delta\omega_s$ and peak optical depth τ_s recorded with an instrument resolution $\Delta\omega_{Res}$ is given by

$$\tau_A = -\ln\left[1 - \frac{\Delta\omega_s}{\Delta\omega_{Res}}\left(1 - e^{-\tau_s}\right)\right]$$

assuming $\Delta\omega_{Res} \geq \Delta\omega_s$. [This expression illustrates that the measurement τ_A can be significantly less than the actual τ_s. A saturation effect can occur in absorption spectroscopy where the measured optical depth does not increase with increasing actual optical depth: saturation here occurs when τ_s is large, so that the maximum apparent optical depth $\tau_A = -\ln[1 - \Delta\omega_s/\Delta\omega_{Res}]$.]

9 High density plasmas

9.1

The rate coefficients of collisional excitation, collisional ionisation and bound free photo-ionisation are modified at high density from the rates at low density due to free electron degeneracy effects by factors of respectively (see equations 13.34, 13.41 and 13.45):

$$R_{pq} = \frac{(2/n_e)(2\pi m_0 k_B T/h^2)^{3/2}}{1 - \exp(-\Delta E/k_B T)} \ln \left[\frac{1 + \exp(\mu/k_B T)}{1 + \exp((\mu - \Delta E)/k_B T)} \right],$$

$$R_{ion} = (2/n_e)(2\pi m_0 k_B T/h^2)^{3/2} \frac{J_{ion}(E_{ion})}{\exp(-E_{ion}/k_B T},$$

$$R^*_{ff} = (2/n_e)(2\pi m_0 k_B T/h^2)^{3/2} \exp\left(\frac{\hbar\omega}{k_B T}\right) I_{int}.$$

In order to express these factors in terms of the chemical potential μ (and ionisation energy ΔE_{ion}), show that the following substitution can be used:

$$(2/n_e)(2\pi m_0 k_B T/h^2)^{3/2} = \frac{\sqrt{\pi}}{2} \frac{1}{I_{1/2}(\mu/k_B T)}$$

where $I_{1/2}(\mu/k_B T)$ is the Fermi-Dirac integral of order 1/2 (see equation 13.6).

10 Errata

10.1

On p39, the number of photons in the volume of an atom should be

$$n_p = 2 \times 10^{-20} I.$$

where the intensity I is measured in $\mathrm{Wm^{-2}}$. This means that intensities $I > 10^{16}$ $\mathrm{Wcm^{-2}}$ are required to have at least two photons in the atomic volume.

10.2

On p116, the solid angle of radiation propagating at an angle θ to the normal to a plane subtends an angle $2\pi \sin \theta d\theta$ without the τ term as printed. Equation 6.5, 6.5 and the un-numbered integral at the top of p116 should appear with integrands of respectively $S(\tau) \exp(-\tau) d\tau du$, $S(\tau) u \exp(-\tau) d\tau du$ and $S(\tau) \exp(-\tau) 2\pi \sin \theta d\theta d\tau$.

10.3

On p267, equation 13.21 should have $Z - Z_{av}$ on the left hand side rather than just Z_{av}.