Automated Reasoning for Probabilistic Sequential Programs with Theorem Proving

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Overview

Background

- RoboChart[1]: DSL for robotics (state machines: reactive+time+probability), unification of semantics (Unifying Theories of Programming or UTP)

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- RoboChart[1]: DSL for robotics (state machines: reactive+time+probability), unification of semantics (Unifying Theories of Programming or UTP)
- Recent work[2]: probabilistic semantics to RoboChart (He et al.'s relational model[3]): sequential+probability

[1] https://robostar.cs.york.ac.uk/notations/
[2] Woodcock et al.: Probabilistic semantics for RoboChart - A weakest completion approach. UTP 2019
[3] He et al.: Deriving probabilistic semantics via the 'weakest completion'. ICFEM 2004

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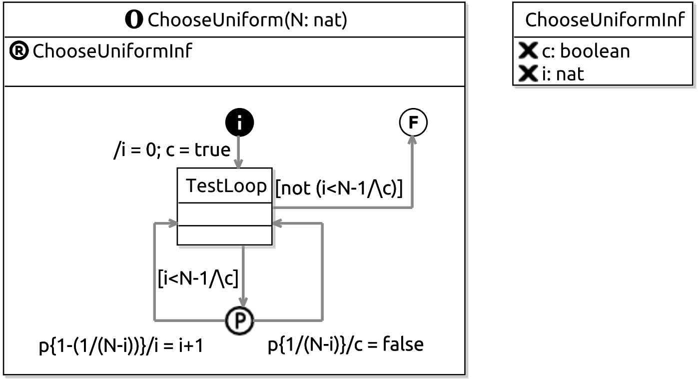
Our contributions

- A formalisation of the proof that embedding sequential composition is a homomorphism,
- A mechanisation of probabilistic designs in Isabelle/UTP for automated reasoning,
- With mechanisation, more interesting details are disclosed.
  - PMFs are convex-closed,
  - Probabilistic choice is not idempotent in general,
  - Embedding sequential composition is a homomorphism only for finite state space.

Outline

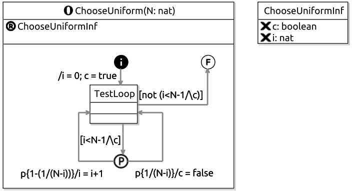
A RoboChart algorithm

Goal: a randomisation algorithm (the same probability $1/N$ to choose i from $[0, N-1]$)



A RoboChart algorithm

Question: does this model correctly implement the randomisation algorithm for any N?



Analysis by PRISM on a Linux server:

- $N = 100$: model construction (4s) + checking (0.002s);
- $N = 10,000$: 8s + 0.004s;
- $N = 100,000$: 830s + 0.011s;
- $N = 1,000,000$: not finished after several hours;
- $N = 1, \ldots\ldots\ldots\ldots$: ?

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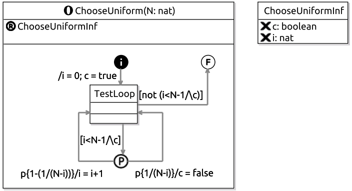

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Our solution: theorem proving

Nondeterministic probabilistic sequential programming language

pGCL[1]

$$P \quad ::= \quad \bot \mid I\!I \mid x := e \mid P \lhd b \rhd Q \mid P \sqcap Q \mid P \oplus_r Q \mid P \, ; \, Q \mid \mu X \bullet P(X)$$

[1] McIver,A.,Morgan,C.: Introduction to pGCL: Its logic and its model. Springer (January 2005)

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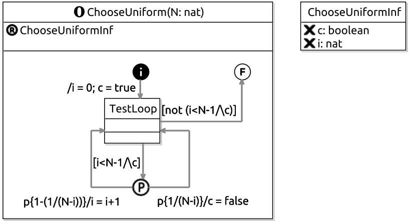
Randomisation algorithm in *pGCL*



$$\begin{aligned} &ChooseUniform(N) \triangleq \\ &\quad i := 0 \,;\ c := true \,; \\ &\quad \mu X \bullet \left(\begin{array}{l} ((c := false) \oplus_{1/(N-i)} (i := i+1)) \,;\ X \\ \lhd (i < (N-1) \wedge c) \rhd I\!I \end{array} \right) \end{aligned}$$

[1] McIver,A.,Morgan,C.: Introduction to pGCL: Its logic and its model. Springer (January 2005)

Outline

Relational semantics [He et al.] of sequential probabilistic programs

Embedding

$\mathcal{K}(D) \triangleq D/\rho$ $\quad$ $D \triangleq (p \vdash_n R)$ $\quad$ Embeddeding

Relational semantics of sequential probabilistic programs

Embedding

$$Y/K \triangleq \neg(\neg Y;\ K^{-})$$

Weakest prespecification

$$\mathcal{K}(D) \triangleq D/\rho \qquad D \triangleq (p \vdash_n R)$$

Embeddeding

Relational semantics of sequential probabilistic programs

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$$Y/K \triangleq \neg\, (\neg\, Y;\ K^{-})$$ Weakest prespecification

$$\rho \triangleq \left(true \vdash prob(s') > 0\right) \qquad prob : PROB\, (\triangleq S \to [0,1])$$ Forgetful function

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$\mathcal{K}(D) \triangleq D/\rho$	$D \triangleq (p \vdash_n R)$	Embeddeding

$$
\begin{tikzcd}
S \arrow[r, "P"] \arrow[dr, "D"'] & PROB \arrow[d, "\rho"] \\
 & S
\end{tikzcd}
$$

Relational semantics of sequential probabilistic programs

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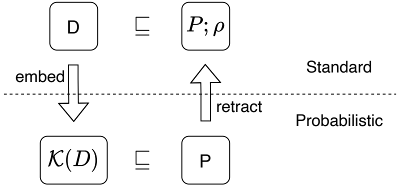

$$\mathcal{K}(D);\ \rho = D$$ Retraction

$$D \sqsubseteq (P;\ \rho) \Leftrightarrow (D/\rho) \sqsubseteq P$$

Relational semantics (embedding: a homomorphism)

Homomorphism on the structure of standard programs

$$\mathcal{K}(\bot) = \bot \qquad \mathcal{K}(x := e) = \left(true \vdash prob'(\mathbf{v}[e/x]) = 1\right)$$

$$\mathcal{K}(I\!I) = \left(\textbf{\textit{true}} \vdash prob'(\mathbf{v}) = 1\right) \qquad \mathcal{K}(P \lhd b \rhd Q) = \mathcal{K}(P) \lhd b \rhd \mathcal{K}(Q)$$

$$(P \oplus_r Q) = \ldots \qquad \mu X \bullet P(X) = \bigsqcap\{X \mid X \sqsupseteq P(X)\}$$

$$\mathcal{K}(P \sqcap Q) = \left(\bigsqcap r \in [0..1] \bullet \mathcal{K}(P) \oplus_r \mathcal{K}(Q)\right) \qquad (\sqsubseteq \mathcal{K}(P) \oplus_r \mathcal{K}(Q)) \qquad \text{Nondeterminism}^{*}$$

$$\mathcal{K}(P;\ Q) = \mathcal{K}(P);\uparrow \mathcal{K}(Q)$$

Relational semantics (previous [Woodcock et al.], a new contribution)

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$$\mathcal{K}(P;\ Q) = \mathcal{K}(P);\uparrow \mathcal{K}(Q) \qquad (P \sqsubseteq Q \Rightarrow (\uparrow P) \sqsubseteq (\uparrow Q))$$ Sequential composition+

Outline

Probabilistic state space and probabilistic choice

Probabilistic state space

- $prob :: [\alpha]pmf$ (Isabelle measure-based pmf).
- Probabilistic designs:

$$\mathcal{K}\,(p \vdash R(S,S)) = \left(p \vdash \left(\Sigma i \in S \mid (R\ \textbf{\textit{wp}}\ (\mathbf{v} = i)) \bullet prob'(i)\right) = 1\right)$$

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Probabilistic choice

- $[S]pmf$ is convex-closed in terms of distribution combination operator $+_r$;
- $+_r$ is idempotent: $p +_r p = p$;
- $\oplus_r$ is not idempotent: $P \oplus_r P = P$ only if $prob'$ in $P(s, prob')$ is convex-closed.
  - the distribution of a deterministic probabilistic program (singleton);
  - the distributions of embedding nondeterministic choice.

Sequential composition

Kleisli lifting

$$\uparrow (q \vdash R) \triangleq \left(\begin{array}{l} (\Sigma i \in [\![q]\!] \bullet prob\,(i) = 1) \vdash \\ \exists\, Q \bullet \left(\begin{array}{l} (\forall\, ss \bullet prob'(ss) = \mathbf{\Sigma} t \bullet prob(t) * (Q(t))(ss)) \wedge \\ \left(\forall\, s \bullet \left(\begin{array}{l} \neg\ (prob(\mathbf{v}') > \mathbf{0} \wedge \mathbf{v}' = s) \,; \\ (\neg\ R \,;\ (\forall\, t \bullet prob(t) = (Q(s))(t))) \end{array} \right) \right) \end{array} \right) \end{array} \right)$$

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Lifting

$$\uparrow (\mathcal{K}\,(\mathit{II})) = (\textbf{\textit{true}} \vdash prob' = prob)$$

$$P \sqsubseteq Q \Rightarrow \uparrow P \sqsubseteq \uparrow Q$$

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Lifting

$$\uparrow(\mathcal{K}\,(\mathit{I\!I})) = \left(\textbf{\textit{true}} \vdash prob' = prob\right)$$

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Sequential composition

$$P\,;_p\,Q \triangleq P;\ \uparrow Q$$

$$P\,;_p\,\mathcal{K}\,(\mathit{I\!I}) = P = \mathcal{K}\,(\mathit{I\!I})\ ;_p\,P \qquad \text{(left/right unit)}$$

$$\mathcal{K}(P;\ Q) = \mathcal{K}(P)\,;_p\,\mathcal{K}(Q) \qquad \text{Only if } S \text{ is finite}$$

Recursion

Theorem (Refinement introduction)

We assume

- R *is a well-founded relation:* $\mathbf{wf}\, R$;
- F *is monotonic:* $\forall\, P\ Q \bullet [\![P \sqsubseteq Q;\ P \textit{ is } \mathbf{N};\ Q \textit{ is } \mathbf{N}]\!] \Rightarrow F(P) \sqsubseteq F(Q)$;
- F *is a* $\mathbf{N}$*-healthy function:* $F \in \mathbf{N} \to \mathbf{N}$;
- *Induct step:* $\forall\, st \bullet ((p \wedge e = st) \vdash Q) \sqsubseteq F\,((p \wedge (e, st) \in R) \vdash Q)$;

then

$$(p \vdash Q) \sqsubseteq \mu\ F$$

Outline

Example 1

Probabilistic choice[1]

$$P1 \triangleq \left(\mathcal{K}\,(x := 0) \oplus_{1/3} \mathcal{K}\,(x := 1)\right)$$

$$P2 \triangleq \left(\mathcal{K}\,(x := x + 2) \oplus_{1/2} \mathcal{K}\,(x := x + 3)\right)$$

$$P3 \triangleq \left(\mathcal{K}\,(x := x + 4) \oplus_{1/4} \mathcal{K}\,(x := x + 5)\right)$$

$$P1 \,;_p\, (P2 \lhd x = 0 \rhd P3) = \left(\textit{\textbf{true}} \vdash \left(\begin{array}{l} prob'\,(\mathbf{v}[2/x]) = 1/6 \land prob'\,(\mathbf{v}[3/x]) = 1/6 \land \\ prob'\,(\mathbf{v}[5/x]) = 1/6 \land prob'\,(\mathbf{v}[6/x]) = 1/2 \end{array}\right)\right)$$

[1] Hehner, E.C.R.: Probabilistic predicative programming. MPC2004

Example 2

Probabilistic choice and nondeterministic choice[1]

$$P \triangleq (\mathcal{K}\,(x := 0) \sqcap \mathcal{K}\,(x := 1))$$

$$Q \triangleq \left(\mathcal{K}\,(y := 0) \oplus_{1/2} \mathcal{K}\,(y := 1)\right)$$

$$P ;_p Q = \left(\textbf{\textit{true}} \vdash \left(\begin{array}{c} (prob'\,(\mathbf{v}[0, 0/x, y]) = 1/2 \wedge prob'\,(\mathbf{v}[0, 1/x, y]) = 1/2) \vee \\ \\ \end{array} \right)\right)$$

$$Q ;_p P = \left(\textbf{\textit{true}} \vdash \left(\begin{array}{c} \\ \\ \\ \end{array} \right)\right)$$

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Example 3: the randomisation algorithm in RoboChart

Assume $N \geq 1$,

$$\left(\textbf{\textit{true}} \vdash \left(\left(\begin{array}{l} c \wedge i < (N-1) \Rightarrow \\ \quad \left(\left(\begin{array}{l} \forall j < (N-i-1) \bullet \\ \quad prob' \, (\mathbf{v}[j+i, \mathit{false}/i, c]) = 1/(N-i) \end{array} \right) \wedge \right. \\ \quad\quad prob' \, (\mathbf{v}[N-1, \mathit{true}/i, c] = 1/(N-i))) \end{array} \right) \wedge \right. \right. \\ \left. \left. (\neg \, (c \wedge i < (N-1)) \Rightarrow prob'(\mathbf{v}) = 1) \right) \right)$$

$$\sqsubseteq (\mu X \bullet \mathit{ChooseUniformBody}(N, X))$$

Choose in Theorem (refinement introduction): $e = N - i - (0 \lhd c \rhd 1)$ and $R = \{(x, y).x < y\}$

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$$\sqsubseteq \mathit{ChooseUniform}(N)$$

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- Analysed several examples;
- Future work: lift probabilistic designs to deal with reactive (instead of sequential) probabilistic systems.

Thank you!

https://robostar.cs.york.ac.uk/