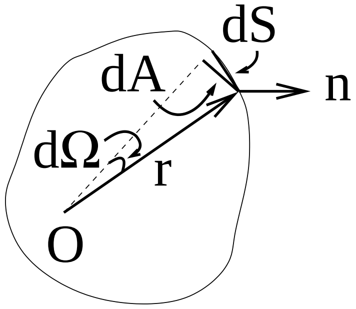
Summary Sheet 0
Mathematical Preliminaries

1 Solid Angle

Let S be a closed surface surrounding the origin; let dS be an element of area of the surface at a point $\mathbf{r}$; let $\mathbf{n}$ be a unit normal vector to dS, and let dA be the projection of dS onto a sphere of radius r centred on the origin as in the figure:



then the *solid angle* subtended by dS is

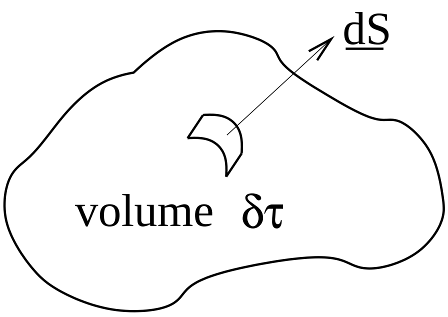
$$d\Omega = \frac{1}{r^2} dA$$

and the unit of solid angle is the *steradian*. Note that if we integrate over the complete surface, then we see that the total solid angle for any point, for any closed surface, is $\Omega = 4\pi$. Solid angle is a very useful concept in evaluating surface and volume integrals.

2 Div, Grad and Curl

2.1 Divergence

The divergence of a vector field $\mathbf{F}$, written as **div F** or $\nabla.\mathbf{F}$, represents the outwards flux of $\mathbf{F}$ per unit volume at the point $\mathbf{r}$:



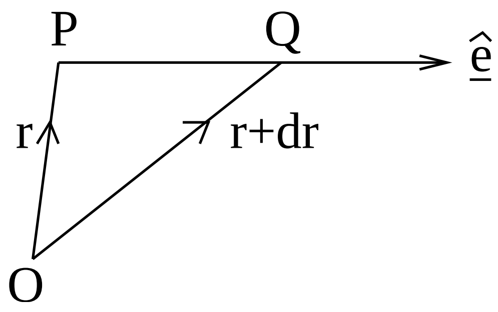
The outwards flux of $\mathbf{F}$ from the volume element $\delta\tau = \int_S \mathbf{F}.d\mathbf{S}$, where $d\mathbf{S}$ is directed out of the volume $\delta\tau$. Therefore,

$$\nabla.\mathbf{F} = \lim_{\delta\tau \to 0} \frac{\int_S \mathbf{F}.d\mathbf{S}}{\delta\tau}$$

In the definition of $\nabla.\mathbf{F}$, the limit of $\delta\tau \to 0$ is taken to define the outwards flux at a single point $\mathbf{r}$.

2.2 Gradient

The gradient of a scalar field ϕ, produces a vector field $\nabla\phi$, which acts in the direction which ϕ is most rapidly increasing. It's magnitude is given by the slope in that direction:

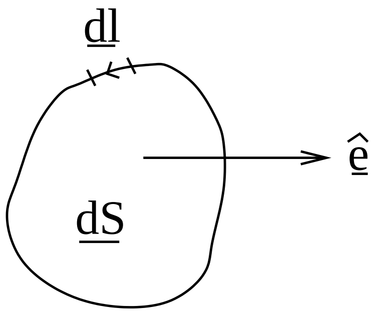


If $\widehat{\mathbf{e}}$ is a unit vector along a line between two points P and Q, with position vectors $\mathbf{r}$ and $\mathbf{r}+\delta\mathbf{r}$, the component of $\nabla\phi$ along PQ is

$$\widehat{\mathbf{e}}.\nabla\phi = \lim_{\delta\mathbf{r}\to 0}\frac{\phi(\mathbf{r}+\delta\mathbf{r})-\phi(\mathbf{r})}{\delta\mathbf{r}}$$

2.3 Curl

The curl of a vector field $\mathbf{F}$, $\nabla\times\mathbf{F}$, generates another vector field which is a measure of its vorticity:



If $\widehat{\mathbf{e}}$ is a unit vector perpendicular to a small surface area δS located at $\mathbf{r}$, $\nabla\times\mathbf{F}$ along $\widehat{\mathbf{e}}$ is defined by

$$\widehat{\mathbf{e}}.\nabla\times\mathbf{F} = \lim_{\delta S\to 0}\frac{\oint\mathbf{F}.d\mathbf{l}}{\delta S}$$

where the line integral is evaluated around the perimeter of δS in a right handed sense (right hand screw rule: $\widehat{\mathbf{e}}$ points in the direction a right handed screw goes when turned along the path $d\mathbf{l}$).

3 Identities

3.1 The Divergence Theorem

The total flux (= flow) of a vector field **F** out of a closed surface S equals the volume integral of $\nabla.\mathbf{F}$ over the volume enclosed by S:

$$\int_S \mathbf{F}.d\mathbf{S} = \int_V (\nabla.\mathbf{F}) d\tau$$

which follows from the definition of $\nabla.\mathbf{F}$.

3.2 Stokes Theorem

The line integral of a vector field **F** around a circuit l equals the flux of $\nabla \times \mathbf{F}$ through any surface S which spans l:

$$\oint_l \mathbf{F}.d\mathbf{l} = \int_S (\nabla \times \mathbf{F}).d\mathbf{S}$$

where the line integral is taken in a right-handed sense with respect to the direction of the field. This theorem follows from the definition of $\nabla \times \mathbf{F}$. (See Appendix B of Grant and Phillips for more details).

4 Div, Grad and Curl in Different Co-ordinate Systems

The form of div, grad and curl can be defined in various co-ordinate systems. The forms below are taken from the Physics departments booklet "Tables of constants and mathematical formulae" provided for use in physics examinations.

4.1 Cartesian Co-ordinates

Gradient

$$\nabla V = \widehat{\mathbf{x}}\frac{\partial V}{\partial x} + \widehat{\mathbf{y}}\frac{\partial V}{\partial y} + \widehat{\mathbf{z}}\frac{\partial V}{\partial z}$$

Divergence

$$\nabla.\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Curl

$$\nabla \times \mathbf{A} = \begin{vmatrix} \widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 \mathbf{A} = \widehat{\mathbf{x}}\nabla^2 A_x + \widehat{\mathbf{y}}\nabla^2 A_y + \widehat{\mathbf{z}}\nabla^2 A_z$$

4.2 Cylindrical Polar Co-ordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Gradient

$$\nabla V = \widehat{\mathbf{r}}\frac{\partial V}{\partial r} + \widehat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta} + \widehat{\mathbf{z}}\frac{\partial V}{\partial z}$$

Divergence

$$\nabla.\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \theta}A_\theta + \frac{\partial}{\partial z}A_z$$

Curl

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \widehat{\mathbf{r}} & r\widehat{\theta} & \widehat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

Laplacian

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}$$

4.3 Spherical Polar Co-ordinates

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

Gradient

$$\nabla V = \widehat{\mathbf{r}}\frac{\partial V}{\partial r} + \widehat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta} + \widehat{\phi}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

Divergence

$$\nabla.\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}A_\phi$$

Curl

$$\nabla \times \mathbf{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \widehat{\mathbf{r}} & r\widehat{\theta} & r\sin\theta\widehat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

Laplacian

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$