Summary Sheet 1
Electrostatics

Lecture 1

The force a point charge q_1 at $\mathbf{r}_1$ exerts on a point charge q_2 at $\mathbf{r}_2$ is defined by Coulomb's law as

$$\mathbf{F}_{21} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|(\mathbf{r}_2 - \mathbf{r}_1)|^3}$$

The force on a point charge q_1 due to a continuous distributions of charge with a volume charge density ρ

$$\mathbf{F}_1 = \frac{q_1}{4\pi\varepsilon_0} \int_V \rho(\mathbf{r}) \frac{(\mathbf{r}_1 - \mathbf{r})}{|(\mathbf{r}_1 - \mathbf{r})|^3} d\tau$$

For charges distributed over a surface or a line the volume charge density is replaced by a surface charge density $\sigma(\mathbf{r})$, or a line charge density $\lambda(\mathbf{r})$, and the integration is performed over a surface S, or a line l, in each case respectively.

Electrostatic forces are two-body forces so that the force experienced by a charge q_j due to a collection of other charges q_i is

$$\mathbf{F}_j = \frac{1}{4\pi\varepsilon_0} \Sigma_{i \neq j} \frac{q_i q_j (\mathbf{r}_j - \mathbf{r}_i)}{|(\mathbf{r}_j - \mathbf{r}_i)|^3}$$

This is the principle of superposition.

Electric field is defined in terms of the Coulomb force on a test charge q by

$$\mathbf{E}(\mathbf{r}) = \lim_{q \to 0} \frac{\mathbf{F}(\mathbf{r})}{q}$$

Lecture 2

The work done per unit charge in moving a charge q through an electric field from A to B is

$$\frac{W}{q} = -\int_A^B \mathbf{E}.d\mathbf{l}$$

The work done is independent of path, and depends only on the positions A and B. The electrostatic field is conservative, so that $\oint \mathbf{E}.d\mathbf{l} = 0$, and hence it is also free of vortices so that $\nabla \times \mathbf{E} = 0$.

The potential difference between A and B equals the work done per unit charge by an external force in moving from A to B. With the zero of potential taken at ∞, the potential

$$\phi(\mathbf{r}) = -\int_\infty^{\mathbf{r}} \mathbf{E}.d\mathbf{l}$$

This can be written in differential form so that $\mathbf{E} = -\nabla\phi$. Both the electrostatic field $\mathbf{E}$ and the potential ϕ obey the principle of superposition.

Lecture 3

The flux of $\mathbf{E}$ through a surface S is $\int_S \mathbf{E}.d\mathbf{S}$, where the flux is positive outwards through the surface S enclosing the volume V. Gauss's law states that this flux is related to the charge enclosed within this surface S so that

$$\int_S \mathbf{E}.d\mathbf{S} = \frac{\Sigma q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{r})d\tau$$

Gauss's law can be written in differential form as

$$\nabla.\mathbf{E} = \frac{\rho}{\varepsilon_0}$$

The electrostatic potential energy stored by bringing together a charge distribution is

$$\begin{aligned} U &= \frac{1}{2}\Sigma_{i\neq j} q_i \phi_j(\mathbf{r}_i) \\ &= \frac{1}{2}\int_V \rho(\mathbf{r}_i)\phi(\mathbf{r})d\tau \end{aligned}$$

Lecture 4

The electric potential of a charge distribution can be expressed in terms of a multipole expansion

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{k_0}{r} + \frac{k_1}{r^2} + \frac{k_2}{r^3} + \cdots\right]$$

where $k_0, k_1, k_2 \cdots$ are the monopole, dipole, and quadrupole moments etc.

A vector dipole moment is defined as

$$\mathbf{p} = \int_V \mathbf{r}\rho(\mathbf{r})d\tau$$

For the case of a point charge dipole, $\mathbf{p}$ is given by $\mathbf{p} = q\mathbf{a}$, where the dipole has point charges $\pm q$ separated by $\mathbf{a}$. The corresponding dipole potential can be written as

$$\phi(\mathbf{r}) = \frac{\mathbf{r}.\mathbf{p}}{4\pi\varepsilon_0 r^3}$$

A dipole in a uniform external electric field $\mathbf{E}$ experiences no net force, but experiences a torque $\Gamma = \mathbf{p}\times\mathbf{E}$, and possesses a potential energy $U = -\mathbf{p}.\mathbf{E}$. In a non-uniform field there is a net force, given by $\mathbf{F} = (\mathbf{p}.\nabla)\mathbf{E}$.

Lecture 5

Conductors contain "free electrons" which can move within the conductor. Net charges on conductors reside on their outer surfaces, with the volume of the conductor itself being an equipotential and having $\mathbf{E} = 0$. The $\mathbf{E}$ fields are created by the surface charges which act as sources/sinks of $\mathbf{E}$. The electric fields begin/end on the positive/negative surface charges just outside the volume of the conductor and are normal to the conductor surface.

Capacitance, C, measures the capacity of a conductor to hold charge at a certain potential. For an isolated conductor, $C = Q/V$, where Q is the charge on the conductor, and V is its potential. For a capacitor, C is still given by Q/V, where $\pm Q$ are the charges on each conductor making up the capacitor and V is the potential difference between them. Work is done in charging a capacitor, so potential energy U is stored in the capacitor given by

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2$$

This potential energy is stored in the electric field created between the capacitor plates and has an energy density $= \varepsilon_0 \mathbf{E}.\mathbf{E}/2$. In general, any electric field carries an energy density of this form, so that the total energy stored in an $\mathbf{E}$ field is

$$U = \frac{1}{2}\varepsilon_0 \int_V \mathbf{E}.\mathbf{E} d\tau$$

Lecture 6

The theory of electrostatics can be formulated in terms of boundary value problems in the form of Poisson's equation

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0},$$

or when there are no net charges, so that $\rho = 0$, in the form of Laplace's equation

$$\nabla^2 \phi = 0$$