Summary Sheet 2.
Magnetostatics and Electromagnetic Induction.

Lecture 7.

The current density $\mathbf{J}$ is defined to be the charge flowing normally through a surface $d\mathbf{S}$, so that $dI = \mathbf{J}.d\mathbf{S}$. Hence the total current flowing through a surface S is given by

$$I = \int_S \mathbf{J}.d\mathbf{S}$$

If a current flows into/out of a volume V, then the total charge contained within V must increase/decrease, since charge is conserved. This can be written as

$$\frac{d}{dt}\int_V \rho d\tau = -\int_S \mathbf{J}.d\mathbf{S},$$

which can be converted into differential form using the divergence theorem so that

$$\nabla.\mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

which is known as the continuity equation (or conservation of charge equation).

A metal conductor in an $\mathbf{E}$-field will have a current flowing such that

$$\mathbf{J} = \sigma \mathbf{E}$$

where σ is the conductivity. The conductivity is related to resistivity ρ by $\sigma = 1/\rho$. In a wire of length l and cross-sectional area A, the total resistance R in terms of ρ is given by $R = \rho l/A$.

Charges moving with a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ feel a Lorentz force $\mathbf{F}$ given by

$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. This equation defines the magnetic field $\mathbf{B}$, with $\mathbf{B}$ in Teslas (T) when $\mathbf{F}$, q and $\mathbf{v}$ have units of N, C, and m s^{-1} respectively. In the presence of both $\mathbf{E}$ and $\mathbf{B}$ fields, the total force felt by the particle is

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

When charges flow in a conductor to form a current they still experience the Lorentz force. An element of wire $d\mathbf{l}$ carrying a current I experiences a force

$$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$$

in a magnetic field $\mathbf{B}$.

Lecture 8.

A current element $Id\mathbf{l}$ produces a magnetic field $d\mathbf{B}$ at a point P at $\mathbf{r}$ given by

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}.$$

For an entire circuit, the total field at P is given by

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint \frac{\mathrm{d}\mathbf{l} \times \hat{\mathbf{r}}}{r^2}$$

which is known as the Biot-Savart law. For a distributed current this can be written as

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J} \times \hat{\mathbf{r}}}{r^2} \mathrm{d}\tau.$$

Since a current element $I_1 d\mathbf{l}_1$ will generate a field $d\mathbf{B}_1$, then a current carrying element $I_2 d\mathbf{l}_2$ in it's vicinity will experience a force

$$d\mathbf{F}_2 = \frac{\mu_0}{4\pi} \frac{I_1 I_2 \left(d\mathbf{l}_2 \times (\mathrm{d}\mathbf{l}_1 \times \mathbf{r})\right)}{r^3}.$$

Lecture 9.

Magnetic flux Φ is defined by

$$\Phi = \int_S \mathbf{B}.d\mathbf{S}$$

and has units of T m^2 (or Webers). The total magnetic flux crossing any closed surface S bounding a volume V is zero so that

$$\int_S \mathbf{B}.d\mathbf{S} = 0.$$

This is Gauss's law for $\mathbf{B}$. This can also be written in differential form using the divergence theorem so that

$$\nabla.\mathbf{B} = 0.$$

This is one of Maxwell's equations. Physically $\nabla.\mathbf{B} = 0$ means that magnetic field lines are continuous, and hence that there are no magnetic charges (i.e., the are no magnetic monopoles).

The line integral of $\mathbf{B}$ around a closed loop C is determined by the current flowing through the surface S bounded by C. This is Ampere's law which states that

$$\oint \mathbf{B}.d\mathbf{l} = \mu_0 \int_S \mathbf{J}.d\mathbf{S} = \mu_0 I.$$

This can be transformed into differential form using Stoke's theorem so that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

for steady (non-time varying) currents.

Lecture 10.

Magnetic vector potential $\mathbf{A}$ is defined in terms of $\mathbf{B}$ so that

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

For magnetostatics, a Poisson equation relating $\mathbf{A}$ and $\mathbf{J}$ can be derived so that

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

which has the formal solution

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r})}{r} \mathrm{d}\tau.$$

Lecture 11.

A closed current loop in a magnetic field will experience the Lorentz force $d\mathbf{F} = Id\mathbf{l}\times\mathbf{B}$ on each element of current in the loop. In a uniform field the loop experiences no net force, however it does experience a torque

$$\Gamma = Id\mathbf{S} \times \mathbf{B},$$

where d$\mathbf{S}$ represents the area of the loop. We can define the magnetic dipole moment of a loop carrying current I with area d$\mathbf{S}$ as

$$\mathbf{m} = I\mathbf{S} \ \left(= \int_S Id\mathbf{S}\right)$$

Hence the torque experienced by a magnetic dipole in a field $\mathbf{B}$ is $\Gamma = \mathbf{m}\times\mathbf{B}$, and the magnetic potential energy which can exist due to work done against this torque is

$$U_M = -\mathbf{m}.\mathbf{B}$$

A magnetic dipole moment $\mathbf{m}$ generates a magnetic field given by

$$B_r = \frac{\mu_0}{4\pi}\frac{2m\cos\theta}{r^3}, \ B_\theta = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^3}, \ B_\phi = 0.$$

This can be compared to the $\mathbf{E}$-field due to an electric dipole $E_r = 2p\cos\theta/(4\pi\varepsilon_0 r^3)$, $E_\theta = p\sin\theta/(4\pi\varepsilon_0 r^3)$, $E_\phi = 0$.

Since the current generating a magnetic dipole can be considered to represent the circular motion of (massed) charge carriers around a loop, magnetic dipoles have associated angular momentum $\mathbf{L}$. For electron charge carriers

$$\mathbf{m} = -\frac{e}{2m_e}\mathbf{L}.$$

As magnetic dipoles experience a torque in a field $\mathbf{B}$, there must be a change in their angular momentum in the presence of this torque (Torque = rate of change of angular momentum). This causes the magnetic dipoles to precess around the external field $\mathbf{B}$ with the Larmor (angular) frequency $\omega_L = mB/L$.

Lecture 12.

An induced electromotive force (e.m.f.) can be generated in a circuit by changes in the magnetic flux linking the circuit. Motional electromotance occurs when a circuit moves in a static $\mathbf{B}$-field, whilst transformer electromotance occurs when the $\mathbf{B}$ cutting a stationary circuit changes in time. In both cases, the resulting electromotance is given by the rate of change of magnetic flux cutting the circuit so that

$$V = \oint \mathbf{E}.d\mathbf{l} = -\frac{d\Phi}{dt}.$$

This is Faraday's law of electromagnetic induction. This induced e.m.f. may generate an induced current which will itself generate a magnetic field. The flux created by the induced currents is in a sense which opposes the original change of flux (Lenz's law).

Faraday's law

$$\oint \mathbf{E}.d\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B}.d\mathbf{S}$$

can be transformed into differential form using Stoke's theorem to give

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

This is one of Maxwell's equations. When $\mathbf{B}$ varies in time, $\mathbf{E}$ can be generated by both charges and magnetic field changes so that

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}.$$

If $\mathbf{B}$ is constant in time, then $\nabla \times \mathbf{E} = 0$ as expected for electrostatic fields, and $\mathbf{E}$ is given by $\mathbf{E} = -\nabla\phi$.

Lecture 13.

A current loop (1) carrying current I_1 generates a magnetic field $\mathbf{B}_1$ which may link an adjacent current loop (2). The flux linking (2) due to I_1 in (1) is

$$\Phi_{21} = M_{21} I_1$$

where M_{21} is the mutual inductance. A similar argument applies to the flux linking (1) due to a current I_2 in (2), so that $\Phi_{12} = M_{12} I_2$. In fact $M_{21} = M_{12} = M$, where M represents the mutual inductance of the two circuits. Changes in either current will generate changes in the flux cutting the other so that, for example,

$$V_2 = -\frac{d}{dt}\Phi_{21} = -M\frac{dI_1}{dt}.$$

M can be calculated using Neumann's formula

$$M_{12} = \frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2}\frac{d\mathbf{l}_1.d\mathbf{l}l_2}{r_{21}} = M_{21}.$$

A current loop also generates flux linking itself. The flux linkage is proportional to I, so

$$\Phi = LI$$

where L is the self-inductance of the loop. If the current in the loop changes, an e.m.f. V will be induced which acts to oppose the change in Φ

$$V = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}.$$

An inductor (a set of current loops such as a solenoid, for example) carrying a current I will have generated a magnetic field $\mathbf{B}$. Work is done to generate this field, and this energy is stored in the magnetic field itself. For an isolated inductor, the stored potential energy is

$$U_P = \frac{1}{2}LI^2.$$

If there are two circuits with self-inductances L_1 and L_2, and a mutual inductance M, the total potential energy stored is

$$U = \frac{1}{2} L_1 I_1^2 + M I_1 I_2 + \frac{1}{2} L_2 I_2^2.$$

The potential energy stored in any magnetic field (such as that created by an inductor) has an energy density $\mu_0 \mathbf{B}.\mathbf{B}/2$. Hence the total potential energy stored in a magnetic field is

$$U = \frac{1}{2\mu_0} \int_V \mathbf{B}.\mathbf{B} d\tau$$

(c.f., the electrostatic energy stored in an $\mathbf{E}$-field, $U = \frac{1}{2}\varepsilon_0 \int_V \mathbf{E}.\mathbf{E} d\tau$).