Summary Sheet 3
Dielectrics, Magnetic Materials, and Maxwell's Equations.

Lecture 14.

Dielectrics have no free charges, and all their electrons are bound to their atoms. In response to an externally applied **E**-field, the centroids of the positive and negative charge distributions in the atoms of the dielectric are displaced so that the material acquires an electric polarization. Electric polarization **P** represents the dielectric's electric dipole moment per unit volume, so that

$$\mathbf{P} = N\mathbf{p},$$

where N is the number of atoms per unit volume. As a result of this polarization, the dielectric can develop polarization surface charge densities σ_b given by

$$\sigma_b = \mathbf{P}.\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the unit normal to the dielectric surface. If the dielectric is non-uniform, then polarization can generate a finite net volume polarization charge density ρ_b given by

$$\rho_b = -\nabla.\mathbf{P}.$$

so if **P** is uniform, $\nabla.\mathbf{P} = 0$ and hence $\rho_P = 0$.

E-fields are generated by charges, so that in dielectrics **E** may be generated by both free (ρ_f) and bound (i.e., polarization, ρ_b) charge densities. Hence, in Gauss's law, ρ must be represented by $\rho = \rho_f + \rho_b$. If we define an electric displacement vector as

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P},$$

then Gauss's law for **D** can be in terms of ρ_f only. In this case it becomes

$$\begin{aligned} \nabla.\mathbf{D} &= \rho_f \\ \int_S \mathbf{D}.d\mathbf{S} &= \int_V \rho_f d\tau \end{aligned}$$

and so only free charges are sources of **D**.

Most dielectrics we will consider are Linear (the polarization is proportional to E), Isotropic (polarization independent of the direction of **E**), and Homogeneous (polarization independent of position) - these are called LIH dielectrics. We can define the electric susceptibility χ_E in terms of

$$\mathbf{P} = \chi_E\varepsilon_0\mathbf{E}$$

where χ_E is dimensionless and is independent of both position, and the orientation and magnitude of **E** in an LIH dielectric. Since $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$, then we can write

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\mathbf{E}(1 + \chi_E).$$

If we define a relative permittivity ε_r to be

$$\varepsilon_r = 1 + \chi_E$$

then

$$\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E}$$

with $\varepsilon_r \geq 1$, and it describes the total permittivity ε in units of ε_0 (i.e., $\varepsilon = \varepsilon_0 \varepsilon_r$).

If a capacitor is filled with dielectric of relative permittivity ε_r, the field between the plates is reduced by a factor ε_r, whilst the capacitance is increased by a factor of ε_r, as compared to their values in a vacuum.

Lecture 15.

At the interface between different media, boundary conditions for ϕ, $\mathbf{D}$ and $\mathbf{E}$ can be derived. ϕ must be continuous across the boundary, whilst the normal components of $\mathbf{D}$ at the interface are related by the free charge density on the boundary

$$D_{1\perp} - D_{2\perp} = \sigma_f.$$

This means that $D_\perp$, the normal component of $\mathbf{D}$, is continuous unless the interface carries free charge. Similarly, the tangential components of $\mathbf{E}$ are continuous across a boundary.

In the presence of dielectrics, free charges are sources of $\mathbf{D}$, and in this case a Poisson equation for ϕ can be derived

$$\nabla^2 \phi = -\frac{\rho_f}{\varepsilon_0 \varepsilon_r}$$

so if $\rho_f = 0$, we have Laplace's equation $\nabla^2 \phi = 0$.

When dielectrics are present, the electrostatic energy density stored in the fields increases by a factor of ε_r, so that the energy density can be written as

$$\frac{1}{2}\varepsilon_0 \varepsilon_r \mathbf{E}.\mathbf{E} = \frac{1}{2}\mathbf{D}.\mathbf{E}$$

and so the total stored potential energy in the fields in the presence of dielectrics is

$$U = \frac{1}{2}\int_V \mathbf{D}.\mathbf{E}\mathrm{d}\tau$$

Lecture 16.

In an external magnetic field $\mathbf{B}_0$, a magnetic material can acquire a macroscale magnetization. Magnetization is defined to be the magnetic dipole moment per unit volume, so that

$$\mathbf{M} = N\mathbf{m}$$

where $\mathbf{m}$ is the dipole moment of an atom, and there are N atoms per unit volume. Magnetization can generate additional magnetic fields so that the total magnetic field is

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_M$$

$\mathbf{M}$ can result from the orbital motions of electrons, or occur because of the intrinsic spins of electrons or nuclear particles. $\mathbf{B}$ can be increased or decreased depending on the type of magnetic material present.

Diamagnets: In the absence of $\mathbf{B}_0$, the net dipole moment of each atom in a diamagnetic material is zero. However, $\mathbf{B}_0$ can change the orbital motion of the electrons through the action of the Lorentz force, and the resulting changes in dipole moments reduce $\mathbf{B}$.

Paramagnets: In the absence of $\mathbf{B}_0$, each atom in a paramagnetic material has a non-zero net dipole moment, however they are randomly oriented so that macroscopically $\mathbf{M} = 0$. In the presence of $\mathbf{B}_0$, these tend to align parallel to $\mathbf{B}_0$, so paramagnetic materials increase $\mathbf{B}$.

Ferromagnets: Macroscale domains within ferromagnetic materials have their dipole moments oriented parallel to each other, however the domains themselves are randomly oriented in the absence of $\mathbf{B}_0$. When $\mathbf{B}_0$ is applied, the domains become aligned parallel to $\mathbf{B}_0$ which can produce a very large magnetization and consequent large increase in $\mathbf{B}$.

The current elements making up the dipole moments in a magnetized material can combine to produce a macroscale magnetization current on the materials surface. The surface magnetization current density $\mathbf{J}_s$ which is produced is given by

$$\mathbf{J}_s = \mathbf{M} \times \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the unit normal at the surface. If $\mathbf{M}$ is non-uniform, internal magnetization current densities $\mathbf{J}_M$ can be produced, given by

$$\mathbf{J}_M = \nabla \times \mathbf{M}.$$

In a medium which is both electrically conducting and magnetizable, we must take account of both the conduction ($\mathbf{J}_f$) and magnetization ($\mathbf{J}_M$) currents so that the total current density is given by

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_M.$$

If we define the magnetic $\mathbf{H}$-field as

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}_0 - \mathbf{M}$$

then we can write Ampères law in the presence of magnetized materials as

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathbf{J_f} \\ \oint \mathbf{H}.\mathrm{d}\mathbf{l} &= I_f \end{aligned}$$

where I_f represents the free current, such as that due to moving conduction electrons.

For isotropic, homogeneous, non-ferromagnetic materials, we can define the magnetic susceptibility χ_B as

$$\mathbf{M} = \chi_B \frac{\mathbf{B}}{\mu_0}$$

where χ_B is dimensionless and can be > 0 (paramagnetic) or < 0 (diamagnetic).

Since $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}_0 - \mathbf{M}$, we can define the relative permittivity of a magnetic material μ_r as

$$\mu_r = 1/(1 - \chi_B).$$

Hence

$$\begin{array}{rcccl} \mathbf{H} & = & \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} & = & \frac{1}{\mu_0}\mathbf{B}(1 - \chi_B) \\ & or & \mathbf{B} & = & \mu_r \mu_0 \mathbf{H} \end{array}$$

In the presence of magnetic materials $\mathbf{B}$ is increased by a factor of μ_r (note that μ_r can be greater or less than 1).

Lecture 17.

Since $\nabla.\mathbf{B} = 0$, taking the divergence of $\mathbf{H}$ gives

$$\nabla.\mathbf{H} = \frac{1}{\mu_0}\nabla.\mathbf{B} - \nabla.\mathbf{M} = -\nabla.\mathbf{M}$$

and so $\nabla.\mathbf{H}$ is non-zero when $\nabla.\mathbf{M}$ is non-zero. This will occur where $\mathbf{M}$ is discontinuous, such as at the edges of magnetic materials. This means that there can be sources and sinks of $\mathbf{H}$, and that lines of $\mathbf{H}$ do not have to be continuous.

Boundary conditions can be derived which govern the behaviour of $\mathbf{B}$ and $\mathbf{H}$ at interfaces. The normal component of $\mathbf{B}$ is continuous across an interface so that

$$B_{1\perp} = B_{2\perp},$$

where $B_\perp$ is the component of $\mathbf{B}$ normal to the interface. Similarly, the component of $\mathbf{H}$ parallel to the interface is also continuous so that

$$H_{1||} = H_{2||}$$

In the presence of magnetized material, the total potential energy density stored in the fields is given by $\frac{1}{2}\mathbf{B}.\mathbf{H}$ per unit volume. Hence the total potential energy stored is

$$U = \frac{1}{2}\int_V \mathbf{B}.\mathbf{H}\mathrm{d}\tau$$

Lecture 18.

Ampères law $\nabla \times \mathbf{B} = \mu_0 J$ can be shown to be incomplete for time varying currents. To satisfy conservation of charge, an additional term $\mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t$ called the displacement current must be added so that Ampères law becomes

$$\nabla \times \mathbf{B} = \mu_0 \left[\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right].$$

Electromagnetism can then be described by four field equations, known as Maxwell's equations, which *in vacuo* are

$$\begin{aligned}
\nabla . \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla . \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \left[\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right]
\end{aligned}$$

In matter, the macroscopic fields obey Maxwell's equations in the following form

$$\begin{aligned}
\nabla . \mathbf{D} &= \rho_f \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla . \mathbf{B} &= 0 \\
\nabla \times \mathbf{H} &= \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}
\end{aligned}$$

Maxwell's equations describe all electromagnetic phenomena, including electromagnetic waves. In free space, where $\rho = 0$ and $\mathbf{J} = 0$, the following wave equations for $\mathbf{E}$ and $\mathbf{B}$ can be derived

$$\begin{aligned}
\nabla^2 \mathbf{E} &= \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\
\nabla^2 B &= \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.
\end{aligned}$$

These equations describe waves which propagate with a speed $c = 1/\sqrt{\mu_0 \varepsilon_0}$. This is the speed of light (*in vacuo*), so that Maxwell's equations can describe light in the form of an electromagnetic wave.