UNIVERSITY OF YORK

BA, BSc and MMath Examinations 2002
MATHEMATICS
Complex Analysis

Time Allowed: $1\frac{1}{2}$ hours.

Answer <u>*three*</u> *questions.*
Standard calculators will be provided.

1. (a) Let $a, b, z \in \mathbb{C}$, $a \neq b$, $k \in (0, \infty)$ and suppose that

$$\left|\frac{z-a}{z-b}\right| = k.$$

Show that the locus of the point z is a circle or a straight line.

(b) Suppose $\lambda \in \mathbb{C}$ and $r \in (0, \infty)$, $r \neq |\lambda|$. Let

$$w(z) = \frac{r(z-\lambda)}{r^2 - \overline{\lambda} z}.$$

Prove that when $|w(z)| = 1$, $|z| = r$.

2. Let $f(z)$ be defined on an open set $A \subseteq \mathbb{C}$ say and let $a \in A$.

(a) Define the derivative $f'(a)$ of the complex function. Let $c \in \mathbb{C}$ and n be a non-negative integer. Show that when $f(z) = cz^n$,

$$f'(a) = n\,ca^{n-1},$$

stating clearly any assumptions made.

(b) Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at the point $z = (x, y) \in A$. Prove that the functions $u\colon A \to \mathbb{R}$, $v\colon A \to \mathbb{R}$ satisfy the Cauchy-Riemann equations.

(c) Explain what it means to say that f is a *regular* function on A. Find a regular function $f : \mathbb{C} \to \mathbb{C}$ with real part $u\colon \mathbb{C} \to \mathbb{R}$ given by

$$u(x, y) = 2(x + 1)y.$$

3. Define a *path* in the complex plane. Let f be a complex valued function defined in a region of the complex plane.

(a) Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise continuously differentiable path and let f be continuous. Define the path integral

$$\int_\gamma f(z)dz \tag{1}$$

in terms of the Riemann integral (which need not be defined but those properties used should be stated). Now suppose $f(z) = F'(z)$ for some continuous function F. Prove that

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

and deduce that if γ is a closed path, the integral vanishes (you may quote any relevant theorems in your proof). State but do not prove Cauchy's theorem.

(b) Evaluate the integral (1) when (i) γ: $[0, 2\pi] \to \mathbb{C}$: $t \mapsto e^{it}$ and $f(z) = cz^n, n \in \mathbb{Z}$.

4. Let f be a complex valued function defined in a region A of the complex plane. Explain what is meant by the statement that f has a *simple pole* at a. Also explain what a *residue* of f is. State but do not prove Cauchy's residue theorem.

Use the theory of residues to evaluate the integral,

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta},$$

where $a > 1$, justifying your arguments where necessary. Hence or otherwise evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{(a + \cos\theta)^2}.$$

1. *Solution (a) bookwork, (b) unseen bookwork*

(a) Since
$$\left|\frac{z-a}{z-b}\right| = k,$$
we have $|z-a|^2 = k^2|z-b|^2$, *i.e.*,
$$(z-a)(\overline{z}-\overline{a}) = z\overline{z} - z\overline{a} - a\overline{z} + a\overline{a} = k^2(z\overline{z} - z\overline{b} - b\overline{z} + b\overline{b}),$$
i.e.,
$$(1-k^2)z\overline{z} - z(\overline{a} - k^2\overline{b}) - \overline{z}(a - k^2 b) + |a|^2 - k^2|b|^2 = 0.$$
When $k \neq 1$, this can be written as
$$z\overline{z} - z\overline{c} - \overline{z}c + |c|^2 - \rho^2 = |z-c|^2 - \rho^2 = 0,$$
which is the equation of a circle centred at $c = (a-k^2b)/(1-k^2)$ and with radius
$$\begin{aligned}
\rho^2 &= c\overline{c} - \frac{|a|^2 - k^2|b|^2}{1-k^2} \\
&= \left(\frac{a-k^2b}{1-k^2}\right)\left(\frac{\overline{a}-k^2\overline{b}}{1-k^2}\right) - \frac{|a|^2 - k^2|b|^2 - k^2|a|^2 + k^4|b|^2}{(1-k^2)^2} \\
&= \frac{|a|^2 - k^2a\overline{b} - k^2b\overline{a} + k^4b\overline{b} - |a|^2 + k^2|b|^2 + k^2|a|^2 - k^4|b|^2}{(1-k^2)^2} \\
&= \frac{k^2(a\overline{a} + b\overline{b} - a\overline{b} - \overline{a}b)}{(1-k^2)^2} = \frac{k^2((a-b)(\overline{a}-\overline{b})}{(1-k^2)^2} = \frac{k^2|a-b|^2}{(1-k^2)^2}
\end{aligned}$$

When $k = 1$, it is the equation of a straight line – the perpendicular bisector of the segment joining a to b.

Could also use geometry or observe that LHS is modulus of a Möbius transformation which preserves circles and straight lines since Möbius transformations decompose into compositions of translations, scalar multiplication and inversions, all of which preserve circles and straight lines. $\boxed{\text{15 Marks.}}$

(b) $r \in (0, \infty)$, $\lambda \in \mathbb{C}$, $|\lambda| \neq r$ and

$$w(z) = \frac{r(z-\lambda)}{r^2 - \overline{\lambda} z}.$$

When $\lambda = 0$, $w(z) = z/r$, so that when $|w(z)| = 1$, $|z| = r$. Otherwise write

$$w(z) = \frac{r(z-\lambda)}{r^2 - \overline{\lambda} z} = \frac{-r}{\overline{\lambda}} \frac{z-\lambda}{z - r^2/\overline{\lambda}}$$

Then $|w(z)| = 1$ implies

$$\left| \frac{z-\lambda}{z - r^2/\overline{\lambda}} \right| = 1.\frac{|\overline{\lambda}|}{r} = \frac{|\lambda|}{r} = k$$

and

$$a = \lambda, \ b = \frac{r^2}{\overline{\lambda}}.$$

Since $|\lambda| \neq r$, $k \neq 1$.

By the above, the locus of z is a circle with centre

$$\begin{aligned} c = \frac{a - k^2 b}{1 - k^2} &= \left(\lambda - \left(\frac{|\lambda|}{r} \right)^2 \frac{r^2}{\overline{\lambda}} \right) (1-k^2)^{-1} \\ &= \left(\lambda - \frac{\lambda\overline{\lambda}}{r^2} \frac{r^2}{\overline{\lambda}} \right) (1-k^2)^{-1} = 0. \end{aligned}$$

The radius is given by

$$\begin{aligned} \rho^2 = \frac{k^2 |a-b|^2}{(1-k^2)^2} &= \frac{k^2 |\lambda - r^4/\overline{\lambda}|^2}{(1-k^2)^2} \\ &= \frac{k^2 r^4 |\lambda\overline{\lambda} r^{-2} - 1|^2}{|\overline{\lambda}|^2 (1-k^2)^2} = \frac{k^2 r^4 |k^2 - 1|^2}{|\overline{\lambda}|^2 (1-k^2)^2} \\ &= \frac{k^2 r^4}{|\lambda|^2} = r^2, \end{aligned}$$

as required.

15 Marks.

Total: 30 Marks

2. *Solution (a) simple exercise, (b) bookwork, (c) unseen standard exercise*

(a) The derivative $f'(a) = \lim_{z\to a} \frac{f(z)-f(a)}{z-a}$, where $z \to a$ in any manner. When $f(z) = cz^n$,

$$\begin{aligned} f'(a) &= \lim_{z\to a} \frac{cz^n - ca^n}{z-a} \\ &= c \lim_{z\to a} \frac{(z-a)(z^{n-1} + z^{n-2}a + \cdots + za^{n-2} + a^{n-1})}{z-a} \\ &= c \lim_{z\to a} (z^{n-1} + z^{n-2}a + \ldots + za^{n-2} + a^{n-1}) = nca^{n-1}. \end{aligned}$$

$\boxed{\text{5 Marks.}}$

(b) The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since $f(z) = u(x,y) + iv(x,y)$ is differentiable in A, $f'(z)$ exists for each $z = (x,y) \in \mathbb{R}^2$, *i.e.*,

$$f'(z) = \lim_{z'\to z} \frac{f(z') - f(z)}{z' - z} = \lim_{\zeta\to 0} \frac{f(z+\zeta) - f(z)}{\zeta}$$

where $\zeta = z' - z$ and the limit of the difference quotient

$$(f(z+\zeta) - f(z))/\zeta$$

is independent of the way ζ approaches 0. Suppose $\zeta = (h,k) = h + ik \to 0$ through real values, (i.e., $k = 0$ and $h \to 0$). Then

$$\begin{aligned} \frac{f(z+\zeta) - f(z)}{\zeta} &= \frac{f(z+h) - f(z)}{h} \\ &= \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \end{aligned}$$

and

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{h\to 0} \frac{v(x+h,y) - v(x,y)}{h} \\ &= \frac{\partial u(x,y)}{\partial x} + i \frac{\partial v(x,y)}{\partial x} \end{aligned}$$

Next suppose $\zeta = (h,k) = h + ik \to 0$ through imaginary values (i.e., $h = 0$). Then

$$f'(z) = \lim_{k\to 0} \frac{u(x,y+k) + iv(x,y+k) - u(x,y) - iv(x,y)}{ik}$$
$$= \lim_{k\to 0} \frac{u(x,y+k) - u(x,y)}{ik} + i \lim_{k\to 0} \frac{v(x,y+k) - v(x,y)}{ik}$$
$$= \frac{\partial u(x,y)}{i\partial y} + \frac{\partial v(x,y)}{\partial y}.$$

Hence

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

and equating real and imaginary parts,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ or } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which are the Cauchy-Riemann equations. $\boxed{\text{15 Marks.}}$

(c) A regular (or holomorphic or analytic) function f is differentiable at each point in an open set. $\boxed{\text{2 Marks.}}$

$2(x+1)y = 2xy + 2 = u(x,y)$ and

$$\frac{\partial u}{\partial x} = 2y = \frac{\partial v}{\partial y} \Rightarrow v(x,y) = y^2 + C(x)$$

Therefore $v(x,y) = y^2 + C(x)$ and $\frac{\partial u}{\partial y} = 2x + 2 = -\frac{\partial v}{\partial x} \Rightarrow v(x,y) = -x^2 - 2x + D(y)$, whence

$$v(x,y) = y^2 + C(x) = -x^2 - 2x + D(y),$$

so that $C'(x) = -2x - 2$ and so $C(x) = -x^2 - 2x + C$, C a constant, and similarly, $D(y) = y^2 + D$, D a constant. On substituting, $v(x,y) = y^2 - x^2 - 2x + C = -x^2 + y^2 - 2x + D$, whence $C = D$ and $v(x,y) = -2x - x^2 + y^2 + C$. Thus

$$f(z) = u(x,y) + iv(x,y) = 2xy + 2y + i(y^2 - x^2 - 2x) + C$$
$$= -i(x^2 - y^2 - i2xy) - 2i(x+iy) = -i(x+iy)^2 - 2i(x+iy)$$
$$= \frac{1}{i}(z^2 + 2z) + C \qquad \boxed{\text{8 Marks.}}$$

$\boxed{\text{Total: 30 Marks}}$

3. *Solution: bookwork (a) bookwork (b) bookwork and exercise covered in lectures*

A *path*, usually denoted by γ, in the complex plane is a continuous function $\gamma\colon [a,b] \to \mathbb{C}\colon t \mapsto \gamma(t)$. For $\alpha, \beta \in \mathbb{C}$, the path γ goes from α to β if $\gamma(a) = \alpha, \gamma(b) = \beta$. The image $C = \gamma([a,b])$ of the path is also used to denote the path. $\boxed{\text{5 Marks.}}$

(a) A path γ is continuously differentiable if the derivative γ' is continuous. The path γ is piece-wise continuously differentiable if γ consists of a finite number of continuously differentiable subpaths $\gamma_1, \ldots, \gamma_k$. This means that γ' is cts except at finitely many points. Also $f(\gamma(t)) = (f \circ \gamma)(t)$ is continuous (composition of two continuous funtions). Hence the real and imaginary parts $U(t), V(t)$ say are also cts except at finitely many points, so that the (real) Riemann integrals $\int_a^b U(t)dt$, $\int_a^b V(t))dt$ exist. Hence the integral

$$\begin{aligned}\int_a^b f(\gamma(t))\,\gamma(t)\,dt &= \int_a^b (U(t) + iV(t))dt \\ &= \int_a^b U(t)dt + i\int_a^b V(t)dt,\end{aligned}$$

exists. This defines the path integral of f along γ, *i.e.*,

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_a^b (f \circ \gamma)(t))\,\gamma'(t)\,dt.$$

Since $f(z) = F'(z)$, where F is cts,

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b F'(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b (F \circ \gamma)'(t))\,dt = [F(\gamma(t))]_a^b \\ &= F(\gamma(b)) - F(\gamma(a)) = 0\end{aligned}$$

if $\gamma(b) = \gamma(a)$, *i.e.*, if γ is a closed path. $\boxed{\text{10 Marks.}}$

Cauchy's theorem: Let f be regular inside and on the simple closed curve $C = \gamma([a,b])$. Then

$$\int_C f(z)dz = \int_\gamma f(z)dz = 0.$$

$\boxed{\text{5 Marks.}}$

(b) $\gamma\colon [0, 2\pi] \to \mathbb{S}^1\colon t \mapsto e^{it}$ and $f(z) = cz^n, n \in \mathbb{Z}$.
$\mathbb{S}^1 = \gamma([0, 2\pi]) = \{e^{it}\colon t \in [0, 2\pi]\} = \{z \in \mathbb{C}\colon |z| = 1\}$

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\
&= \int_{\mathbb{S}^1} z^n dz = \int_0^{2\pi} c(e^{it})^n ie^{it}dt \\
&= ic\int_0^{2\pi} (e^{i(n+1)t})^n \, dt \\
&= ic\left[\frac{e^{i(n+1)t}}{n+1}\right]_0^{2\pi} = ic[1-1] = 0,
\end{aligned}$$

providing $n \neq -1$.
When $n = -1$,

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_\gamma \frac{c}{z}\, dz = \int_0^{2\pi} c\frac{1}{e^{it}}ie^{it}dt \\
&= ic\int_0^{2\pi} e^{-it+it}\, dt = ic\int_0^{2\pi} dt \\
&= 2\pi ic.
\end{aligned}$$

When n is a non-negative integer, then $c\int_{\mathbb{S}^1} z^n\, dz = 0$ by Cauchy's theorem but for $n \in \mathbb{Z}$, $n \neq -1$, the integral vanishes as a consequence of the above since

$$z^n = \frac{1}{n+1}(z^{n+1})'.$$

$\boxed{\text{10 Marks.}}$

$\boxed{\text{Total: 30 Marks}}$

4. *Solution : bookwork (a) unseen modification of simple standard application covered in lectures, (b) unseen standard exercise with a ending*

The complex valued function f has a pole of order n at $a \in A$ if f has an isolated singularity at a, i.e., if for some $c_n \neq 0$ and some function g regular at a, f can be written

$$f(z) = \frac{c_n}{(z-a)^n}\, g(z) \ (= \sum_{k=-n}^{\infty} c_k (z-a)^k).$$

When $n = 1$ the pole is simple. $\boxed{\text{5 Marks.}}$

(a) To evaluate the integral

$$I = I(a) = \int_0^{2\pi} \frac{d\theta}{a + \cos\theta},$$

where $a > 1$, put $z = e^{i\theta}$. Then

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + \frac{1}{z}),$$

$dz = ie^{i\theta} d\theta = iz d\theta$ and the integral

$$I = \frac{1}{i} \int_{|z|=1} \frac{dz}{z(a + \frac{1}{2}(z + \frac{1}{z}))} = -2i \int_{|z|=1} \frac{dz}{z^2 \ + 2az + 1}$$

Denominator of integrand

$$= z^2 + 2az + 1 = (z - a + \sqrt{a^2 - 1})(z - a - \sqrt{a^2 - 1})$$

The product of the two (real) roots is

$$(-a + \sqrt{a^2 - 1})(-a - \sqrt{a^2 - 1}) = 1$$

so only the root

$$\alpha = -a + \sqrt{a^2 - 1} \in D^2,$$

where D^2 is the unit disc $\{z : |z| \leq 1\}$, i.e., α is inside the circle $S^1 = \{z : |z| = 1\}$ and the conjugate root $\beta = -a - \sqrt{a^2 - 1}$ is outside D^2. Hence the integrand $f(z) = \frac{1}{z^2+2az+1}$ has one simple pole inside D^2 at $z = \alpha = -2 + \sqrt{a^2-1}$ with residue

$$\begin{aligned}
\operatorname{res}\{f;\alpha\} &= \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{(z-\alpha)}{z^2+4z+1} \\
&= \lim_{z\to\alpha}\frac{(z-\alpha)}{(z-\alpha)(z-\beta)} = \frac{1}{\alpha-\beta} \\
&= \frac{1}{\sqrt{a^2-1}}
\end{aligned}$$

But by the Residue Theorem

$$\int_{|z|=1}\frac{dz}{z^2+2az+1} = -2i\int_{|z|=1}\frac{dz}{(z-\alpha)(z-\beta)} = 2\pi i(-2i)\operatorname{res}\{f;\alpha\}$$

whence

$$I = I(a) = \frac{4\pi}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}$$

$\boxed{\text{15 Marks.}}$

To evaluate the integral

$$\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2},$$

differentiate $I(a)$ wrt a: then

$$I'(a) = -2\pi\frac{1}{2}(a^2-1)^{-3/2}2a = -2\pi a(a^2-1)^{-3/2}.$$

But by differentiating under the integral sign (OK since range finite and $a > 1$)

$$I'(a) = -\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2},$$

whence

$$\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}.$$

Alternatively could consider

$$J = \int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2}\int_{|z|=1}\frac{dz}{(z^2+2az+1)^2} = -2i\int_{|z|=1}\frac{dz}{(z-\alpha)^2(z-\beta)^2}$$

and the Residue Theorem corresponding to a double pole at α.

$\boxed{\text{10 Marks.}}$

$\boxed{\text{Total: 30 Marks}}$