UNIVERSITY OF YORK

BA, BSc and MMath Examinations 2003
MATHEMATICS
Complex Analysis

Time Allowed: $1\frac{1}{2}$ hours.

Answer <u>*three*</u> *questions.*
Standard calculators will be provided.

1. (a) Let n be a positive integer. Find the roots of

$$z^n + z^{n-1} + \cdots + z + 1.$$

(b) Let $n \geq 2$ and let α be a root of the polynomial

$$P(z) = 1 + z + 2z^2 + \cdots + nz^n.$$

Verify that $\alpha \neq 1$ and by considering the product $P(z)(z-1)$ or otherwise prove that

$$n\alpha^{n+1} = 1 + \alpha^2 + \cdots + \alpha^n.$$

Hence or otherwise show that $|\alpha| \not> 1$.

2. Let f be a complex valued function defined on an open set $A \subseteq \mathbb{C}$.

(a) Define the derivative $f'(a)$ of f at $a \in A$.

(b) Suppose $f(z) = u(x,y) + iv(x,y)$ is differentiable at the point $z = x+iy \in A$. Prove that the functions $u\colon A \to \mathbb{R}$, $v\colon A \to \mathbb{R}$ satisfy the Cauchy-Riemann equations.

(c) Explain what it means to say that f is a *holomorphic* function on A. Verify that the function

$$u(x,y) = e^{-y}\cos x$$

is harmonic and find a conjugate function $v(x,y)$. Express $u(x,y)+iv(x,y)$ as a holomorphic function $f(z)$ which satisfies $f(0) = 1$.

3. (a) Let $\sum_{n=0}^{\infty} c_n z^n$ be a power series. Show that the set

$$\{z \in \mathbb{C}\colon \sum_{n=0}^{\infty} |c_n z^n| \text{ converges}\}$$

is non-empty. Define the radius of convergence of the power series.

(b) Find the radius of convergence R for each of the following series.

$$(i) \quad \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n \qquad (ii) \sum_{n=0}^{\infty} \frac{1}{(2i)^n} z^n \qquad (iii) \sum_{n=1}^{\infty} \frac{z^{n-1}}{n^{3/2}} .$$

Discuss convergence in the case $|z| = R$ when R is finite.

(c) Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth path and let f be continuous. Define the path integral

$$\int_\gamma f(z)dz$$

in terms of the complex Riemann integral. State, but do not prove, Cauchy's theorem for functions holomorphic inside and on a contour (a closed simple piecewise smooth path).

4. (a) Prove the Fundamental Theorem of Algebra (you may assume Liouville's theorem that a bounded complex function which is holomorphic everywhere is constant).

(b) Let a be a positive real number. Use the theory of residues to evaluate the integral

$$\int_0^\pi \frac{d\theta}{a^2 + \sin^2\theta},$$

justifying your arguments where necessary.

1. *Solution (a) easy and familiar exercise, (b) unseen variation of a hard exercise*

(a) The polynomial equation

$$z^n + z^{n-1} + \cdots + z + 1 = 0$$

has n roots α_k, $k = 1, \ldots, n$, $\alpha_k \neq 1$. When $\alpha \neq 1$,

$$\alpha^n + \alpha^{n-1} + \cdots + \alpha + 1 = \frac{\alpha^{n+1} - 1}{\alpha - 1} = 0, \ \ i.e., \ \ \alpha^{n+1} = 1,$$

so that the n roots are

$$\alpha_k = e^{2\pi i \frac{k}{(n+1)}}, \ \ k = 1, \ldots, n.$$

$\boxed{\text{10 Marks.}}$

(b)

$$\begin{aligned} P(z)(z-1) &= z + z^2 + 2z^3 + \cdots + (n-1)z^n + nz^{n+1} - \\ &\quad - (1 + z + 2z^2 + \cdots + nz^n) \\ &= nz^{n+1} - \big(z^n + \cdots + z^2 + 1\big) = 0. \end{aligned}$$

Since $P(\alpha) = 0$,

$$P(\alpha)(\alpha - 1) = n\alpha^{n+1} - \big(\alpha^n + \cdots + \alpha^2 + 1\big) = 0,$$

Hence

$$n\alpha^{n+1} = \alpha^n + \cdots + \alpha^2 + 1.$$

Taking moduli:

$$\begin{aligned} |n\alpha^{n+1}| = n|\alpha|^{n+1} &= \big|\alpha^n + \cdots + \alpha^2 + 1\big| \\ &\leq |\alpha^n| + |\alpha^{n-1}| + \cdots + |\alpha^2| + |1| \\ &\leq |\alpha|^n + |\alpha|^{n-1} + \cdots + |\alpha|^2 + 1 \end{aligned}$$

by the triangle inequality and since $|zz'| = |z||z'|$.

Suppose $|\alpha| > 1$. Then $|\alpha|^m < |\alpha|^n$, $m < n$. Hence

$$\begin{aligned} n|\alpha|^{n+1} &\leq |\alpha|^n + |\alpha|^{n-1} + \cdots + |\alpha|^2 + 1 \\ &< |\alpha|^n + |\alpha|^n + \cdots + |\alpha|^n + |\alpha|^n = n|\alpha|^n. \end{aligned}$$

Thus

$$n|\alpha|^{n+1} < n|\alpha|^n,$$

i.e., $|\alpha| < 1$, a contradiction, whence $|\alpha| \leq 1$.

$\boxed{\text{20 Marks.}}$

$\boxed{\text{Total: 30 Marks}}$

2. *Solution (a), (b) bookwork, (c) unseen standard exercise*

(a) The derivative $f'(a) = \lim_{z\to a} \frac{f(z)-f(a)}{z-a}$, where $z \to a$ in any manner with $z \neq a$. $\boxed{\text{5 Marks.}}$

(b) The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since $f(z) = u(x, y)+iv(x, y)$ is differentiable in A, $f'(z)$ exists for each $z = (x, y) \in \mathbb{R}^2$, *i.e.*,

$$f'(z) = \lim_{z'\to z} \frac{f(z') - f(z)}{z' - z} = \lim_{\zeta\to 0} \frac{f(z + \zeta) - f(z)}{\zeta}$$

where $\zeta = z' - z$ and the limit of the difference quotient

$$(f(z + \zeta) - f(z))/\zeta$$

is independent of the way ζ approaches 0. Suppose $\zeta = (h, k) = h + ik \to 0$ through real values, (i.e., $k = 0$ and $h \to 0$). Then

$$\begin{aligned}\frac{f(z+\zeta)-f(z)}{\zeta} &= \frac{f(z+h)-f(z)}{h}\\ &= \frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h}\end{aligned}$$

and

$$\begin{aligned}f'(z) &= \lim_{h\to 0} \frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h}\\ &= \lim_{h\to 0} \frac{u(x+h,y)-u(x,y)}{h} + i\lim_{h\to 0}\frac{v(x+h,y)-v(x,y)}{h}\\ &= \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial x}\end{aligned}$$

Next suppose $\zeta = (h, k) = h + ik \to 0$ through imaginary values (i.e., $h = 0$). Then

$$\begin{aligned}f'(z) &= \lim_{k\to 0} \frac{u(x,y+k)+iv(x,y+k)-u(x,y)-iv(x,y)}{ik}\\ &= \lim_{k\to 0} \frac{u(x,y+k)-u(x,y)}{ik} + i\lim_{k\to 0}\frac{v(x,y+k)-v(x,y)}{ik}\\ &= \frac{\partial u(x,y)}{i\partial y} + \frac{\partial v(x,y)}{\partial y}.\end{aligned}$$

Hence

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

and equating real and imaginary parts,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ or } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which are the Cauchy-Riemann equations. 15 Marks.

(c) A regular (or holomorphic or analytic) function f is differentiable at each point in an open set.
Since $u(x,y) = e^{-y}\cos x$, $\partial u/\partial x = -e^{-y}\sin x$, $\partial u/\partial y = -e^{-y}\cos x$, whence

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial(-e^{-y}\sin x)}{\partial x} = -e^{y}\cos x, \ \frac{\partial^2 u}{\partial y^2} = \frac{\partial(-e^{-y}\cos x)}{\partial y} = e^{y}\cos x,$$

whence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -e^{-y}\cos x + e^{-y}\cos x = 0$$

and u is harmonic. To find the conjugate, consider

$$\frac{\partial u}{\partial x} = -e^{-y}\sin x = \frac{\partial v}{\partial y},$$

whence $v(x,y) = e^{-y}\sin x + C(x)$. Next

$$\frac{\partial u}{\partial y} = -e^{-y}\cos x = -\frac{\partial v}{\partial x},$$

whence $v(x,y) = e^{-y}\sin x + D(y) = e^{-y}\sin x + C(x)$, so that $C(x) = D(y) = C$, a constant independent of x, y. Thus

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) = e^{-y}\cos x + i(e^{-y}\sin x + C) \\ &= e^{-y}(\cos x + i\sin x) + iC = e^{-y}e^{ix} + iC = e^{i^2 y}e^{ix} + iC \\ &= e^{i(x+iy)} + iC = e^{iz} + iC = 1, \end{aligned}$$

when $z = 0$ if $C = 0$. The function $f(z) = e^{iz}$ is holomorphic for all $z \in \mathbb{C}$ (and so is entire and u and v are harmonic). **Or** Could observe after some trial and error that $f(z) = e^{iz}$

10 Marks.

Total: 30 Marks

3. *Solution: (a) slightly modified bookwork, (b) standard unseen examples (c) bookwork, unseen exercise*

(a) The set

$$\mathcal{C} := \{z \in \mathbb{C}\colon \sum_{n=0}^{\infty} |c_n z^n| \text{ converges}\}$$

is not empty since the power series $\sum_{n=0}^{\infty} |c_n z^n| = |c_0|$ when $z = 0$, whence $0 \in \mathcal{C}$. The radius of convergence is

$$R := \sup\{|z| : \sum_{n=0}^{\infty} |c_n z^n| \text{ converges}\},$$

where the set $\{|z| : \sum_{n=0}^{\infty} |c_n z^n| \text{ converges}\}$ is non-empty since it contains 0.

$\boxed{\text{7 Marks.}}$

(b) (i) Let $z \in \mathbb{C}$ and

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} z^n = \sum_{n=0}^{\infty} u_n$$

and use ratio test:

$$\frac{|u_{n+1}|}{|u_n|} = \left|\frac{2^{n+1} z^{n+1}}{(n+1)!}\right| \left|\frac{n!}{2^n z^n}\right| = \frac{2|z|}{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence series converges by limit form of ratio test for any $z \in \mathbb{C}$, *i.e.*, $R = \infty$. $\boxed{\text{5 Marks.}}$

(ii)

$$\sum_{n=0}^{\infty} \frac{1}{(2i)^n} z^n = \sum_{n=0}^{\infty} \left(\frac{z}{2i}\right)^n = \lim_{N\to\infty} \left(1 + \frac{z}{2i} + \cdots + \frac{z^N}{(2i)^N}\right)$$
$$= \lim_{N\to\infty} \frac{1 - (z/2i)^{N+1}}{1 - z/(2i)} = \frac{1}{1 - z/(2i)}$$

and so converges absolutely if $|z/(2i)| = |z|/2 < 1$, *i.e.*, if $|z| < 2$ (also ratio test). The series cannot converge when $|z| \geq 2$ since then the n-th term of the series satisfies

$$\left|\frac{z^n}{(2i)^n}\right| = \frac{|z|^n}{2^n} \geq \frac{2^n}{2^n} = 1 \not\to 0$$

as $n \to \infty$, a necessary condition for convergence. $\boxed{\text{5 Marks.}}$

(iii)

$$|\sum_{n=1}^{\infty} \frac{z^n}{n^{3/2}}| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{n^{3/2}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty$$

when $|z| \leq 1$, since

$$\sum_{n=1}^{\infty} n^{-\alpha} < \infty$$

for $\alpha > 1$. Hence by comparison test, series converges absolutely for $|z| \leq 1$, thus series converges on the unit circle $S^1(0,1) = \{z \colon |z| = 1\}$. For $|z| > 1$,

$$\left| \frac{z^n}{n^{3/2}} \right| = \frac{|z|^n}{n^{3/2}} \to \infty$$

as $n \to \infty$ and the series diverges for $|z| > 1$. Could use ratio test. $\boxed{\text{5 Marks.}}$

(c)

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt,$$

where the integral on the RHS is a complex Riemann integral of a complex function of a real variable t; well defined since the integrand is piece-wise continuous.

Cauchy's theorem: Let f be holomorphic inside and on the Jordan curve $C = \gamma([a,b])$. Then

$$\int_C f(z)dz = \int_\gamma f(z)dz = 0.$$

$\boxed{\text{8 Marks.}}$

$\boxed{\text{Total: 30 Marks}}$

4. *Solution : (a) bookwork (b) a standard unseen example*

(a) Fundamental Theorem of Algebra: A complex polynomial $P(z)$ of degree $n \geq 1$ has n roots, counted according to multiplicity.

Suppose $P(z) = a_n z^n + \cdots + a_1 z + a_0$, $a_n \neq 0$ has no roots. Then

$$1/P(z) = \frac{1}{a_n z^n + \cdots + a_1 z + a_0}$$

is defined and differentiable for all $z \in \mathbb{C}$ since

$$\frac{d}{dz}\frac{1}{P(z)} = \frac{-1}{P(z)^2}P'(z).$$

consider the holomorphic function $1/P(z)$. Moreover $1/P(z) \to 0$ as $|z| \to \infty$, since $a_n \neq 0$. Thus $1/P(z)$ is bounded, whence $P(z)$ is bounded and entire. Thus by Liouville's thm, $P(z)$ is constant. Thus $P(z) = P(0) = a_0$ and so if $\deg P \geq 1$, P has a root α say.

Complete proof by induction on degree. When $n = 1$,

$$P(z) = a_1 z + a_0, \text{ where } a_1 \neq 0,$$

has one root $\alpha = -a_0/a_1$. Now let $P(z)$ have degree $n \geq 2$ and assume the result for polynomials of degree $n-1$. Then by the above, $P(z)$ has a root α. Hence by the Remainder Theorem for polynomials,

$$P(z) = (z - \alpha)Q(z),$$

where $Q(z)$ is a polynomial of degree $n-1$ and so by the inductive hypothesis has P has exactly $n - 1 + 1 = n$ roots and result follows.

$\boxed{\text{10 Marks.}}$

(b) Change the $\sin^2\theta$ to $\cos 2\theta$ using $\sin^2\theta = (1 - \cos 2\theta)/2$:

$$I = \int_0^\pi \frac{d\theta}{a^2 + \sin^2\theta} = \int_0^\pi \frac{d\theta}{a^2 + (1 - \cos 2\theta)/2}$$
$$= \int_0^\pi \frac{2d\theta}{2a^2 + 1 - \cos 2\theta} = \int_0^{2\pi} \frac{d\varphi}{2a^2 + 1 - \cos\varphi}$$

where $\varphi = 2\theta$. Now put $z = e^\varphi$, so that $dz = ie^{i\varphi}d\varphi = izd\varphi$. Also

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} = \frac{1}{2}(z + \frac{1}{z}).$$

Then

$$I = \int_0^{2\pi} \frac{d\varphi}{2a^2+1-\cos\varphi} = \int_{S^1(0,1)} \frac{dz/iz}{2a^2+1-(z+1/z)/2}$$
$$= 2i \int_{S^1(0,1)} \frac{dz}{z^2-2(2a^2+1)z+1}$$

The denominator $z^2 - 2(2a^2+1)z + 1$ has 2 (real) roots:

$$2a^2+1 \pm \sqrt{(2a^2+1)^2-1} = 2a^2+1 \pm \sqrt{4a^4+4a^2+1-1}$$
$$= 2a^2+1 \pm 2a\sqrt{a^2+1},$$

whence the integrand has one simple pole at

$$\alpha = 2a^2+1-2a\sqrt{a^2+1} \in (-1,1) \text{ and so inside } S^1(0,1)$$

and the other β say outside. Let

$$f(z) = 1/(z^2-2(2a^2+1)z+1), i.e., \; f(z) = (z-\alpha)^{-1}(z-\beta)^{-1}.$$

The residue of the integrand f at α is

$$\mathrm{res}\{f;\alpha\} = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{z-\alpha}{(z-\alpha)(z-\beta)}$$
$$= \frac{1}{\alpha-\beta} = -\frac{1}{4a\sqrt{a^2+1}}.$$

Hence by Cauchy's Residue Theorem,

$$I = 2i\int_{S^1(0,1)} \frac{dz}{z^2-2(2a^2+1)z+1} = 2i\int_{S^1(0,1)} f(z)dz = 2i\,2\pi i\,\mathrm{res}\{f;\alpha\}$$
$$= -4\pi\left(-\frac{1}{4a\sqrt{a^2+1}}\right) = \frac{\pi}{a\sqrt{a^2+1}}.$$

$\boxed{\text{20 Marks.}}$

$\boxed{\text{Total: 30 Marks}}$