Algebra Examination Crib Sheet (2022/23)

Compiled by Oliver Dixon on 2nd June 2023

De Moivre's Theorem: For $w, z \in \mathbb{C}$ such that $w = \rho e^{i\varphi}$ and $z = re^{i\theta}$, $wz = \rho r e^{i(\varphi+\theta)}$. Alternatively, for $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, $\cos(nx) + i\sin(nx) = (\cos x + i\sin x)^n$.	**Complex Sine**: For $z \in \mathbb{C}$, $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$.	**Complex Cosine**: For $z \in \mathbb{C}$, $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$.
Roots of Unity (Computing): There are exactly n n^{th} roots of unity. In general, $\zeta_1^{(n)} = e^0 = 1$, $\zeta_2^{(n)} = e^{2\pi i/n}$, $\zeta_3^{(n)} = e^{4\pi i/n}$, $\ldots$, $\zeta_n^{(n)} = e^{2(n-1)\pi i/n}$.	**Roots of Unity (Properties)**: For $n \geq 2$, $\sum_{j=1}^{n} \zeta_j^{(n)} = 0$ and $\prod_{j=1}^{n} \zeta_j^{(n)} = (-1)^{n+1}$. All roots also lie on the unit circle, such that $\lvert\zeta_j^{(n)}\rvert = 1$ for $1 \leq j \leq n$.	**Euclid's Algorithm (1)**: Given $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, there exist q_1 and r_1 such that $a = q_1 b + r_1$ and $0 \leq r_1 < b$. If $r_1 = 0$, then $\gcd(a, b) = b$. Otherwise, $\ldots$
Euclid's Algorithm (2): $\ldots$ find q_2 and r_2 such that $b = q_2 r_1 + r_2$ and $0 \leq r_2 < r_1$. If $r_2 = 0$, set $\gcd(b, r_1) = \gcd(a, b) = r_1$. Otherwise, iterate on the remainders.	**Solving Congruence Equations (1)**: To solve $ax \equiv b \pmod{m}$, find $d = \gcd(a, m)$. If $d \nmid b$, there are no solutions. Otherwise, write $b = b'd$ and $d = sa + tm$.	**Solving Congruence Equations (2)**: A single solution is $x = sb'$; let $m' = m/d$. All numbers congruent to $sb' \pmod{m'}$ comprise the full solution set.
Chinese Remainder Theorem: If m_1 and m_2 are s.t. $\gcd(m_1, m_2) = 1$, then there is a unique $x \pmod{m_1 m_2}$ that satisfies $x \equiv c_1 \pmod{m_1}$ and $x \equiv c_2 \pmod{m_2}$.	**Binomial Theorem**: $(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}$	**Binomial Coefficient**: $\binom{n}{j} = \frac{n!}{j!(n-j)!}$
Cross Product: $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$	**STP**: $\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \cdot \left[\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right] = \det \begin{pmatrix} t_1 & t_2 & t_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$	**Parametric Line**: An equation of a line can be written as $x = p_1 + tv_1$, $y = p_2 + tv_2$, and $z = p_3 + tv_3$, for $t \in \mathbb{R}$. (p_1, p_2, p_3) are the coordinates of a point on the line, and (v_1, v_2, v_3) denotes the direction of the line.
Point-to-Line: If $\underline{v}$ is a vector parallel to the line, $\underline{p}$ is the position vector of a point on the line, and $\underline{r}$ is a point, then the minimal distance from R to the line is $\lvert(\underline{p} - \underline{r}) \times \underline{v}\rvert / \lvert\underline{v}\rvert$.	**Plane Equations**: The position vector of an arbitrary point X in the plane is $\underline{x} = \underline{p} + s\underline{u} + t\underline{v}$ for $s, t \in \mathbb{R}$, where $\underline{u}$ and $\underline{v}$ are parallel to the plane, and P is a plane point.	**Intersection of Line and Plane**: Write the line and plane in parametric form, equate each component, and solve for s, r, and t.
Normal Vector to a Plane: If $\underline{p}$ is the position vector of a known point, and $\underline{x}$ is the position vector of an arbitrary point, then $(\underline{x} - \underline{p}) \cdot \underline{n} = 0$, where $\underline{n}$ is the normal vector.	**Point-to-Plane**: The distance from a point A to a plane is $\lvert(\underline{p} - \underline{a}) \cdot \underline{n}\rvert / \lvert\underline{n}\rvert$, where $\underline{n}$ is the normal vector, and P is a known point in the plane.	**Matrix Multiplication**: If $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{n \times p}$, then the product AB is such that $(AB)_{ik} = \sum_{j=1}^{n} A_{ij} B_{jk}$. (For our purposes, the field $\mathbb{F}$ is the set of reals $\mathbb{R}$.)
Diagonal and Transpose Matrices: A square matrix $D \in \mathbb{F}^{n \times n}$ is *diagonal* if $D_{ij} = 0$ for all $i \neq j$. Diagonal matrices commute under multiplication. Also, for $B \in \mathbb{F}^{m \times n}$, $A^T_{ij} = A_{ji}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.	**Echelon Forms (1 and 2)**: $\begin{pmatrix} \star & \square & \square \\ 0 & \star & \square \\ 0 & 0 & \star \end{pmatrix} \begin{pmatrix} \star & \square & \square \\ 0 & \star & \square \\ 0 & 0 & 0 \end{pmatrix}$	**Echelon Forms (3 and 4)**: $\begin{pmatrix} \star & \square & \square \\ 0 & 0 & \star \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \star & \square & \square \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Reduced EFs: $\begin{pmatrix} \star & 0 & 0 \\ 0 & \star & 0 \\ 0 & 0 & \star \end{pmatrix} \begin{pmatrix} \star & 0 & \square \\ 0 & \star & \square \\ 0 & 0 & \star \end{pmatrix} \begin{pmatrix} \star & \square & 0 \\ 0 & 0 & \star \\ 0 & 0 & 0 \end{pmatrix}$	**Systems of Linear Equations**: Construct the *augmented matrix*, use row operations to transform it into an Echelon form, reinterpret as a system of linear equations, and solve for each component in terms of the parameter.	**Inverse (2x2)**: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
Determinant: In general, for a matrix A, $\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$. For an upper triangular matrix A^E with diagonal $\mu_0, \ldots, \mu_n$, $\det\left(A^E\right) = \prod_{i=1}^{n} \mu_i$.	**Computing an Eigensystem (1)**: For the *eigenvalues* of a matrix A, solve $\det(A - \lambda I_n) = 0$ for λ (the eigenvalues), where I_n is the identity matrix in $\mathbb{F}^{n \times n}$.	**Computing an Eigensystem (2)**: For each eigenvalue λ_0, the corresponding *eigenvector* $\underline{v}_0/C_0$ is such that $(A - \lambda_0 I_n)\underline{v}_0 = \underline{0}$, where $C_0 \in \mathbb{F}$ is a nonzero constant.
Eigendiagonalisation (1): If $A \in \mathbb{F}^{n \times n}$ has linearly independent eigenvectors $A\underline{x}_1 = \lambda_1 \underline{x}_1, \ldots, A\underline{x}_n = \lambda_n \underline{x}_n$, then A is similar to the diagonal matrix Λ with $\lambda_0, \ldots, \lambda_n$.	**Eigendiagonalisation (2)**: If the columns of a matrix M are given by the eigenvectors of A, then $AM = M\Lambda$, and $\Lambda = M^{-1}AM$.	**Eigendiagonalisation (3)**: If we have $A = M\Lambda M^{-1}$ with matrices M invertible and Λ diagonal, then $A^n = (M\Lambda M^{-1})(M\Lambda M^{-1})\ldots(M\Lambda M^{-1}) = M\Lambda^n M^{-1}$.

Note: Unless specified otherwise, an arbitrary point X (uppercase Latin character) in n-space has position vector $\underline{x} \in \mathbb{F}^n$, where $\underline{x} = (x_1, x_2, \ldots, x_n)$.

https://github.com/oliverdixon/exam-crib-sheets/tree/master/Algebra

Norm of Matrix: For a matrix $A \in \mathbb{R}^{n\times n}$, its Euclidean norm is defined as $\lVert A \rVert = \max_{\lVert \underline{v} \rVert \le 1} \lVert A\underline{v} \rVert$, where $\lVert \underline{v} \rVert$ is the Euclidean norm of $\underline{v}$ in $\mathbb{R}^n$.	**Exponential of a Matrix**: For some $A \in \mathbb{R}^{n\times n}$, $\exp A = \sum_{k=0}^{\infty} A^k/k!$. This sum is convergent, such that $\lVert \exp A \rVert = \exp \lVert A \rVert$. If $\dot{\underline{x}} = A\underline{x}$, then the general solution is given by $\underline{x}(t) = \exp(tA)\underline{v}$.	**CODEs**: $\ldots \underline{x}(t) = \mathrm{e}^{tA}\underline{v} = M \begin{pmatrix} \mathrm{e}^{\lambda_1 t} & 0 & \ldots \\ 0 & \mathrm{e}^{\lambda_2 t} & \ldots \\ \ldots & \ldots & \mathrm{e}^{\lambda_n t} \end{pmatrix} M^{-1}\underline{v}$
Real Symmetric Matrices: Let $A \in \mathbb{R}^{n\times n}$ with $A^T = A$. Then, the eigenvalues of A are real, and distinct eigenvectors are orthogonal. A is *symmetric* if $A^T = A$.	**Orthogonal Matrices**: A square matrix $M \in \mathbb{F}^{n\times n}$ satisfying $M^T M = I_n$ is an *orthogonal* matrix. The columns of such a matrix are mutually orthonormal vectors.	**Row Reduction Methods**: When finding the Echelon form, we can multiply rows by non-zero scalar constants, take the sum of two rows, and interchange rows.
Coupled ODEs (1): Let A be the matrix of coefficients given in the coupled linear system of ODEs. M and Λ are such that $M^{-1}AM = \Lambda$: M is the composed eigenvector matrix and Λ is the diagonal matrix of eigenvalues.	**Coupled ODEs (2)**: Note that $(x', y') = M^{-1}(x, y)$. Then, $\mathrm{d}(x', y')/\mathrm{d}t = \Lambda(x', y')$. Expand $\Lambda(x', y')$ and solve (component-wise) for the first-order ODEs $x'(t)$ and $y'(t)$. [Recall that $u'(x) = cu(x) \implies u(x) = A\mathrm{e}^{cx}$.]	**The Remainder Theorem**: Let p and q be an n- and m-degree complex polynomials respectively, with $m \le n$. There exists polynomials s and r of degrees $n - m$ and $k < m$ s.t. $p(x) = q(x)s(x) + r(x)$.
Cofactors/Minors: For each pair ij, the *cofactor* c_{ij} of $A \in \mathbb{F}^{n\times n}$ is s.t. $c_{ij} = (-1)^{i+j}\det(A_{ij})$, where A_{ij} is the *minor* of A (obtained by removing row i and column j).	**Adjoint and Inverse (3x3)**: The *adjoint* matrix of A is $\operatorname{adj} A = C^T$, where $C = [c_{ij}]$. For a matrix A, the *inverse* is $A^{-1} = \operatorname{adj} A/\det A$.	**Linear Independence**: Vectors $\underline{v}_1, \ldots, \underline{v}_m \in \mathbb{F}^n$ are *linearly independent* if and only if $\alpha_1\underline{v}_1, \ldots, \alpha_m\underline{v}_m = \underline{0}$ is solved by $\alpha_1 = \ldots = \alpha_m = 0$, for $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$.

An amusing collection of mathematical poetry ...
... some of which involves Linear-Algebraic themes:

`https://people.math.harvard.edu/~knill/poetry/`