Calculus Examination Crib Sheet (2022/23) Compiled by Oliver Dixon on 25th May 2023

L'Hôpital's Rule: If f and g are differentiable functions at x_0, $f(x_0) = g(x_0) = 0$, and $g'(x_0) \neq 0$, then $\lim_{x \to x_0} f(x)/g(x) = \lim_{x \to x_0} f'(x)/g'(x)$.	**The IVT**: Suppose $a < b$ and f is continuous on $[a, b]$. Then, for every y such that $\min(f(a), f(b)) < y < \max(f(a), f(b))$, there exist $x_0 \in (a, b)$ s.t. $f(x_0) = y$.	**The Chain Rule**: If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x, and $(f \circ g)'(x) = f'(g(x))g'(x)$.
The IFT: If $f : I \to \mathbb{R}$ is continuous and strictly monotonic, then $f^{-1} : J \to I$ is also continuous, where $J = f(I)$ and $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.	**The MVT**: If $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b), then there exist $x_0 \in (a, b)$ such that $f'(x_0) = [f(b) - f(a)]/(b - a)$.	**Classifying CPs**: If $f : [a, b] \to \mathbb{R}$, f', and f'' are sensibly defined, and $x_0 \in (a, b)$ s.t. $f'(x_0) = 0$, then $f''(x_0) > 0$ means local min., and $f''(x_0) < 0$ means local max.
Taylor's Theorem (1): If $f \in C^{N+1}(I)$ and $x \in I$, then $f(x) = \sum_{n=0}^{N} \left[f^{(n)}(x_0)(x - x_0)^n\right]/n! + 1/N! \int_{x_0}^{x} (x - t)^N f^{(N+1)}(t)\, \mathrm{d}t$.	**Taylor's Theorem (2)**: The terms under the summation are the *Taylor polynomial* of f at x_0, of order N. The integral term is known as the *error in integral form*.	**Taylor's Theorem (3)**: The *Lagrange form* of the error is $R_N(x) = \left[(x - x_0)^{N+1} f^{(N+1)(c)}\right]/(N + 1)!$, for some c between x_0 and x.
Diff. Eq. (1): If $u'(x) = cu(x)$, where $c \in \mathbb{R} \setminus \{0\}$ and A is an arbitrary constant, then $u(x) = A\mathrm{e}^{cx}$.	**Diff. Eq. (2)**: If $u''(x) = -c^2 u(x)$, then $A\cos(cx) + B\sin(cx)$, where A and B are arbitrary constants.	**Diff. Eq. (3)**: If $u''(x) = c^2 u(x)$, then $u(x) = A\mathrm{e}^{cx} + B\mathrm{e}^{-cx} = C\cosh(cx) + D\sinh(cx)$, for arb. constants C, D.
Simple Diff. Eqs.: A *simple differential equation* has the form $y'(x) = f(x)$, and has solutions $y = \int f(x)\, \mathrm{d}x + C$, for some arbitrary constant C.	**Separable Diff. Eqs.**: A *separable differential equation* has the form $y'(x) = f(x)/g(y)$. It has solutions $G(y) = F(x) + C$, where $F' = f$ and $G' = g$.	**Integrating Factors (1)**: A first-order ODE is *linear* if it has the form $a(x)y'(x) + b(x)y + c(x) = 0$. In *standard form*, this is $y'(x) + P(x)y + Q(x) = 0 \ldots$
Integrating Factors (2): ... This can be solved to give $y = -\left[\int Q(x)F(x)\, \mathrm{d}x + C\right]/F(x)$, where $F(x) = \exp \int P(x)\, \mathrm{d}x$ is the *integrating factor*.	**Derivative of Arc Sine**: $\dfrac{\mathrm{d}}{\mathrm{d}x} \arcsin(x) = \dfrac{1}{\sqrt{1 - x^2}}$	**Derivative of Arc Cosine**: $\dfrac{\mathrm{d}}{\mathrm{d}x} \arccos(x) = \dfrac{-1}{\sqrt{1 - x^2}}$
Derivative of Arc Tangent: $\dfrac{\mathrm{d}}{\mathrm{d}x} \arctan(x) = \dfrac{1}{1 + x^2}$	**Radian Measure (1)**: If $(x, y) \in \mathbb{R}^2$ with $(x, y) \neq (0, 0)$, then there is a unique solution to $x = r\cos\theta$ and $y = r\sin\theta$ for $\theta \in (-\pi, \pi]$ and $r > 0$.	**Radian Measure (2)**: If $x > 0$, then $\theta = \arctan(y/x)$. If $x = 0$, $\theta = \mathrm{sgn}(y)\pi/2$. If $x < 0$, then $\theta = \arctan(y/x) + \pi$ if $y \geq 0$, or $\theta = \arctan(y/x) - \pi$ otherwise.
Complex Circular Trigonometric Functions: For $z \in \mathbb{C}$, $\sin(z) = (\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})/(2\mathrm{i})$, and $\cos(z) = (\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})/2$. Therefore, $\tan(z) = \mathrm{i}(\mathrm{e}^{-\mathrm{i}z} - \mathrm{e}^{\mathrm{i}z})/(\mathrm{e}^{-\mathrm{i}z} + \mathrm{e}^{\mathrm{i}z})$.	**Complex Hyperbolic Trigonometric Functions**: For $z \in \mathbb{C}$, $\sinh(z) = (\mathrm{e}^{z} - \mathrm{e}^{-z})/2$, $\cosh(z) = (\mathrm{e}^{z} + \mathrm{e}^{-z})/2$, and $\tanh(z) = \sinh(z)/\cosh(z)$.	**Trigonometric Identities (Hyperbolic Form)**: For x and y, $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$, and $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$.
2nd-Order ODE (MD.1): Consider $R(y'', y', x) = 0$. To solve for $y(x)$, we define a new dependent variable as the derivative of the old dependent variable.	**... (MD.2)**: We then solve the resulting first-order ODE, and integrate the solution. This works in cases of *missing dependent variables*.	**2nd-Order ODE (MI.1)**: Consider $R(y'', y', y) = 0$. To solve this *autonomous ODE*, we first define a new independent variable as the old dependent variable.
... (MI.2): Define a dependent variable as the derivative of the old dependent variable. Rewrite the expression in terms of these new variables, and solve.	**... (MI.3)**: Rewrite the solution in terms of the original variables, and solve the resulting first-order differential equation.	**2nd-Order ODE (HC.1)**: Consider a *homogeneous linear ODE* in $y(x)$ with constant coeffs. Take an ansatz of $\mathrm{e}^{\lambda x}$, substitute this into the auxiliary equation, and solve.
... (HC.2): If $\lambda \in \mathbb{R}$ is a root of the aux. eq., then $\mathrm{e}^{\lambda x}$ is a solution to the ODE. If $\alpha \pm \mathrm{i}\beta \in \mathbb{C}$ are roots, then $\mathrm{e}^{\alpha x}\cos\beta x$ and $\mathrm{e}^{\alpha x}\sin\beta x$ are ODE solutions.	**... (HC.3)**: If λ is an m-times repeated root of the aux. eq., with $m \leq n$, then multiplying these solutions by powers of x, up to x^{m-1}, gives more solutions.	**... (HC.4)**: The general solution of the ODE is an arbitrary linear combination of these real and complex solutions.
2nd-Order ODE (IL.1): For an *inhomogeneous linear ODE*, first find the general solution of the corresponding homogeneous equation.	**... (IL.2)**: Find one solution of the inhomogeneous ODE, and sum it with the solution to the homogeneous ODE for the general solution.	**2nd-Order ODE (IC.1)**: For an *inhomogeneous linear ODE with constant coeffs.*, take an ansatz which is of the same type as the RHS, with undetermined coeffs.
... (IC.2): If this ansatz "overlaps" with the general solution of the homogeneous ODE, multiply that part of the guess by x.	**... (IC.3)**: Insert this into the ODE and determine the coeffs. Substitute these values into the ansatz, and take the sum of the particular and general solution.	**2nd-Order ODE (CP.1)**: For a system of coupled ODEs in $x(t)$ and $y(t)$, find $\ddot{x}$ and $\dot{y}$. Use $\dot{y}$ to eliminate $\dot{y}$, and use $\dot{x}$ to eliminate y in the ODE.

... (CP.2): Find the general solution of the resulting ODE for $x(t)$, and compute $\dot{x}(t)$. Use $\dot{x}$ to write y in x and $\dot{x}$, and compute $y(t)$ from the general solution of $x(t)$.	**Basic FS**: The *Fourier Series* for $f\colon[-\pi,\pi]\to\mathbb{R}$ is $S(x)=a_0/2+\sum_{n=1}^\infty(a_n\cos nx+b_n\sin nx)$, with $a_n=\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x/\pi$ and $b_n=\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x/\pi$.	**Tangent Lines to Curves**: If $\underline{r}_0$ is lying on the level curve $f(x,y)=c$, then $\nabla f(\underline{r_0})\cdot(\underline{r}-\underline{r}_0)$. In three variables, this also applies to three-space planes.
Periodic Extensions: If $f\colon[-\pi,\pi)\to\mathbb{R}$, then its *periodic extension* $\tilde{f}\colon\mathbb{R}\to\mathbb{R}$ is defined by $\tilde{f}(x+2\pi k)\coloneqq f(x)$ for $k\in\mathbb{Z}$ and $-\pi\le x<\pi$.	**FCT (1)**: If $f\colon[-\pi,\pi]\to\mathbb{R}$ is a PWCD† function, and $\tilde f\colon\mathbb{R}\to\mathbb{R}$ is its 2π-PE, then at $x\in\mathbb{R}$, the FS of f converges to $\lim_{N\to\infty}S_N(x)=S(x)=[\tilde f(x^+)+\tilde f(x^-)]/2$.	**FCT (2)**: If $\tilde f$ is continuous at x, then $S(x)=\tilde f(x)$. [† *Piecewise continuously differentiable function*]
PT: If $f\colon[-\pi,\pi]\to\mathbb{R}$ is a PWCD with Fourier coefficients a_0, a_n, and b_n for $n\in\mathbb{N}$, then $\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x/\pi\equiv a_0^2/2+\sum_{n=1}^\infty(a_n^2+b_n^2)$.	**Half-Range Series**: For $f\colon[0,\pi]\to\mathbb{R}$, $S_c(x)=a_0/2+\sum_{n=1}^\infty a_n\cos nx$, and $S_s(x)=\sum_{n=1}^\infty b_n\sin nx$, where $a_n=2\int_0^\pi f(x)\cos nx\,\mathrm{d}x/\pi$ and $b_n=2\int_0^\pi f(x)\sin nx\,\mathrm{d}x/\pi$.	**Complex Exponential Series**: For complex-valued coefficients $c_n\in\mathbb{C}$, $S(x)=\sum_{n=-\infty}^\infty c_n\mathrm{e}^{inx}$, where $c_n=\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx}\,\mathrm{d}x/(2\pi)$ and $\overline{c_n}=c_{-n}$ with $n\in\mathbb{N}\cup\{0\}$.
FS on Other Intervals (1): For a function over $[-L/2,L/2]$, $S(x)=a_0/2+\sum_{n=1}^\infty[a_n\cos(2n\pi x/L)+b_n\sin(2n\pi x/L)]$, where ...	**FS on Other Intervals (2)**: ... the cosine coefficients are $a_n=2\int_{-L/2}^{L/2}f(x)\cos(2n\pi x/L)\,\mathrm{d}x/L$, and the sine coefficients are $b_n=2\int_{-L/2}^{L/2} f(x)\sin(2n\pi x/L)\,\mathrm{d}x/L$.	**Clairaut's Theorem**: If $f(x,y)$ and f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing (a,b), and they are all cont. at (a,b), then $f_{xy}(a,b)=f_{yx}(a,b)$.
GCL: If $f(x,y)$ is a CDF, and $\underline{r}(t)=[x(t),y(t)]$ is a pair of diff. functions, then $F'(t)=x'(t)f_x[x(t),y(t)]+y'(t)f_y[x(t),y(t)]$, where $F(t)=f[\underline{r}(t)]$.	**Gradient**: For some $f(x,y,z)$, $\nabla f\coloneqq\partial f/\partial x\,\underline{i}+\partial f/\partial y\,\underline{j}+\partial f/\partial z\,\underline{k}$. If $\underline{r}_0$ is a point, and $\underline{u}$ is a unit vector then the DD of f is $D_{\underline{u}}f(\underline{r}_0)\coloneqq\lim_{h\to0}[f(\underline{r}_0+h\underline{u})-f(\underline{r}_0)]/h$.	**Level Surface**: A *level set* of a three-variable function $F(x,y,z)$ is defined to be $\{(x,y,z)\in\mathbb{R}^3\mid F(x,y,z)=c\}$, for some constant $c\in\mathbb{R}$.
Tangent Vector: Let $\mathcal{S}$ be the level surface of $F\colon\mathbb{R}^3\to\mathbb{R}$ passing through $\underline{r}_0$. If $\underline{u}$ is a vector tangent to $\mathcal{S}$ at $\underline{r}_0$, then $0=\underline{u}\cdot\nabla F(\underline{r}_0)$.	**Imp. Diff.**: If $y(x)$ is defined implicitly by $f(x,y)=c$ then $\mathrm{d}y/\mathrm{d}x=-f_x(x,y)/f_y(x,y)$. For $F(x,y,z)=c$, then $(\partial z/\partial x)_y=-F_x/F_z$ and $(\partial z/\partial y)_x=-F_y/F_z$.	**Laplacian**: The *Laplacian* of $f(x,y)$ is $\Delta f\equiv\nabla^2 f=\partial^2 f/\partial x^2+\partial^2 f/\partial y^2$. If $F(r,\theta)=f(r\cos\theta,r\sin\theta)$, then $\Delta f\equiv\nabla^2 f=\partial^2F/\partial r^2+(\partial F/\partial r)/r+(\partial^2F/\partial\theta^2)/r^2$.
Types of Regions: *Type-One* Region: $R_1=\{(x,y)\in\mathbb{R}^2\mid a\le x\le b,\ \varphi_1(x)\le y\le\varphi_2(x)\}$; *Type-Two* Region: $R_2=\{(x,y)\in\mathbb{R}^2\mid c\le y\le d,\ \psi_1(y)\le x\le\psi_2(y)\}$.	**Fubini's Theorem**: Let $f\colon R\to\mathbb{R}$ be cont., with $R\subset\mathbb{R}^2$. If R is T1, then $\iint_R f\,\mathrm{d}A=\int_a^b\int_{\varphi_1(x)}^{\varphi_2(x)}f(x,y)\,\mathrm{d}y\,\mathrm{d}x$. If R is T2, then $\iint_R f\,\mathrm{d}A=\int_c^d\int_{\psi_1(y)}^{\psi_2(y)}f(x,y)\,\mathrm{d}x\,\mathrm{d}y$.	**Centroid**: The *centroid* of $R\subset\mathbb{R}^2$ is the point $(\overline{x},\overline{y})$ such that $\overline{x}=\iint_R x\,\mathrm{d}A/\mathcal{A}(R)$ and $\overline{y}=\iint_R y\,\mathrm{d}A/\mathcal{A}(R)$, where $\mathcal{A}(R)$ is the area of R.
The Jacobian: $J\coloneqq\det\begin{pmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{pmatrix}$	**Double Integral Transform (1)**: If $x(u,v)$ and $y(u,v)$ are CDFs, $f(x,y)$ is cont., $R\subset\mathbb{R}^2$, and $\mathcal{S}$ is in the (u,v)-plane that maps one-to-one with R, then ...	**Double Integral Transform (2)**: ... change of variables can be achieved with the double integral result:: $\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\iint_{\mathcal{S}}f[x(u,v),y(u,v)]\,\lvert J(u,v)\rvert\,\mathrm{d}u\,\mathrm{d}v$.
Local Extrema: A function $f\colon\mathbb{R}^2\to\mathbb{R}$ has a *local minimum* at (x_0,y_0) if $f(x_0,y_0)\le f(x,y)$ for all (x,y) in some disc centered at (x_0,y_0).	**Stationary and Saddle Points**: If $\nabla f(x_0,y_0)=0$, then (x_0,y_0) is a *stationary point*. If (x_0,y_0) is a stationary point, but not an extremum, then it is a *saddle point*.	**Hessian**: $\lvert H(x,y)\rvert\coloneqq\left\lvert\begin{pmatrix}f_{xx}(x,y)&f_{xy}(x,y)\\f_{yx}(x,y)&f_{yy}(x,y)\end{pmatrix}\right\rvert\eqqcolon\Delta$
Classifying Stationary Points (1): Let $f(x,y)$ be a cont. twice-diff. function, and suppose that it has a stationary point at (x_0,y_0) with discriminant $\Delta=\Delta(x_0,y_0)$	**Classifying Stationary Points (2)**: If $\Delta>0$, then $f_{xx}(x_0,y_0)>0$: local min. at (x_0,y_0), and $f_{xx}(x_0,y_0)<0$: local max. Alternatively, $\Delta<0$ implies a saddle point.	**CS Points**: A point $(x_0,y_0)\in\mathcal{C}$ is a *constrained stationary point* of f if $D_{\underline{u}}f(x_0,y_0)=\underline{u}\cdot\nabla f(x_0,y_0)=0$ holds for all vectors $\underline{u}$ tangent to $\mathcal{C}$ at (x_0,y_0).
Rewriting the Derivative: $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}=\dfrac{\mathrm{d}u}{\mathrm{d}x}=u\dfrac{\mathrm{d}u}{\mathrm{d}y}$	**Vector Fields (Condition)**: If a CD vector field $\underline{f}=(u,v)$ is a gradient, i.e. $\underline{f}=\nabla\varphi$ for some SF $\varphi(x,y)$, then its components satisfy $\partial v/\partial x=\partial u/\partial y$.	**Vector Fields (Identification)**: We want to find an SF φ s.t. $\partial\varphi/\partial x=u$. Integrate up, and subsitute to find a closed form for the constant of integration $g(y)$.
Arc Length: If γ is s.t. $\gamma\colon[a,b]\to\mathbb{R}^2$, where $\gamma(t)=(x(t),y(t))$ and x,y are continuous differentiable, $L(\gamma)=\int_a^b\sqrt{x'(t)^2+y'(t)^2}\mathrm{d}t$.	**FTC (1)**: If $f\colon[a,b]\to\mathbb{C}$ is a continuous function, then $\mathrm{d}/\mathrm{d}x\left[\int_a^x f(y)\mathrm{d}y\right]=f(y)$, where $F\colon[a,b]\to\mathbb{R}$ is such that $F(x)=\int_a^x f$.	**FTC (2)**: If f is differentiable on $[a,b]$ and f' is continuous on $[a,b]$, then $\int_a^b f'=f(b)-f(a)$.