Metric Spaces

Consolidated Lecture Notes

Collated and Typeset by Oliver Dixon

Based on the MAT00051I Lecture Series

University of York

Semester I, 2023/24

Compilation Date	19th December 2023
Author Contact	od641@york.ac.uk
Sources Link	https://github.com/oliverdixon/l5-notes/tree/master/metric-spaces
Latest Commit Hash	5af8285603a077399bdac641c79dc223aa8de0b8 (master)
York Web Link	https://www-users.york.ac.uk/~od641/l5-notes/metric-spaces.pdf
Document Licence	All rights reserved (draft copy)

Structural Information

This document consolidates the content delivered during lectures #1 through #17 of the undergraduate *Metric Spaces* module, presented throughout Semester 1 of the 2023/24 academic year at the University of York. Each lecture introduction is annotated with references to the relevant Panopto lecture recording, accessible exclusively to UoY-authenticated users, and references to the germane sections in [SV06] and [OSe07], denoted by the icons , , and respectively. Comments and corrections are warmly welcomed, and should be directed to the Author via the Author Contact as printed on the title page.

Lecture Index

Draft Copy Only: To-Do List

In no particular order:

- In progress: lecture XIV;
- Fix page boundaries/trailing under-filled vertical boxes;
- Add Shirali and S&V book references;
- Improve the length and prose style of lecture-introduction paragraphs;

DRAFT/UNFINISHED

- Lectures:
  - XV;
  - XVI;
  - XVII: *end of examinable content.*

Lecture 1: Introduction to Metric Spaces

Lecture One introduces the concept of a *metric* as a generalisation of the notion of distance between two points in a set. Three *canonical metrics* on $\mathbb{R}^N$ are presented; these are then generalised further, and a short proof verifies the compliance of the generalised Euclidean metric with the relevant axioms.

26th September 2023
TODO
TODO

Definition 1.1 Metric Space

Suppose that X is a set, and $d\colon X \times X \to [0, \infty) \subset \mathbb{R}$. Then, d is a *metric* on X if and only if the following properties hold for $a, b, c \in X$:

M1) *Positivity.* $d(a, b) \geq 0$;

M2) *Equality.* $d(a, b) = 0 \iff a = b$;

M3) *Symmetry.* $d(a, b) = d(b, a)$;

M4) *Triangularity.* $d(a, b) \leq d(a, c) + d(b, c)$.

The tuple (X, d) is a *metric space.*

Definition 1.2 Canonical Metrics on $\mathbb{R}^N$

We can consider three metrics on $\mathbb{R}^N$: d_1, d_2, and d_∞, each of which have a domain of $\mathbb{R}^N \times \mathbb{R}^N$ and a codomain of $[0, \infty)$:

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{N} |\mathbf{x}_i - \mathbf{y}_i| \tag{1.1}$$

$$d_2(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{N} (\mathbf{x}_i - \mathbf{y}_i)^2\right]^{1/2} \tag{1.2}$$

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq N} |\mathbf{x}_i - \mathbf{y}_i| \tag{1.3}$$

Unless otherwise stated, $\mathbb{R}^N$ is endowed with d_2 Euclidean metric. This is consistent with our current understanding of the real line, which uses the absolute value $|x - y|$ to denote distance between $x, y \in \mathbb{R}$.

Example 1.1 Unit Circles in the Three Canonical Spaces

Using the definitions of the canonical metrics equation 1.1, equation 1.2, and equation 1.3 (and a very loose understanding of a *circle*) we can draw the unit "circles" generated in $\mathbb{R}^2$ under each of these metrics. For instance, figure 1.3 shows the boundary of the set S^2_∞, where

$$S^2_\infty = \left\{(x, y) \in \mathbb{R}^2 \colon d_\infty(x, y) \leq 1\right\}. \tag{1.4}$$

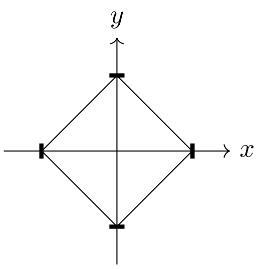

Figure 1.1: Unit Circle in d_1

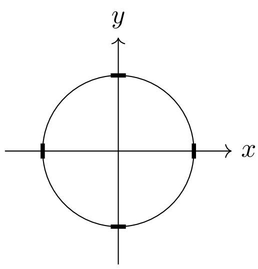

Figure 1.2: Unit Circle in d_2

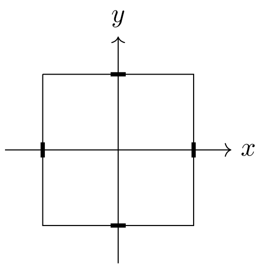

Figure 1.3: Unit Circle in d_∞

Theorem 1.1 The Generalised Metric is a Metric

Consider the d_p metric, where $d_p \colon \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ is a generalisation of the d_2 Euclidean metric for $p \in \mathbb{N}$:

$$(\mathbf{x}, \mathbf{y}) \mapsto \left[\sum_{i=1}^{N} |\mathbf{x}_i - \mathbf{y}_i|^p\right]^{1/p} \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^N. \tag{1.5}$$

Then, $(\mathbb{R}^N, d_p)$ is a metric space.

Proof. To show that d_p is a metric on $\mathbb{R}^N$, we must verify that d_p is in compliance with the constraints enumerated in definition 1.1. The positivity, equality, and symmetry axioms are easy to show, so we will focus on the triangularity property here, proving it by demonstrating a reduction to *Minkowski's Theorem.*

Let $\mathbf{a}_k$ and $\mathbf{b}_k$ be such that

$$d_p(\mathbf{x}, \mathbf{z}) =: \left[\sum_{k=1}^{N} |\mathbf{a}_k|^p\right]^{1/p} \tag{1.6}$$

$$d_p(\mathbf{y}, \mathbf{z}) =: \left[\sum_{k=1}^{N} |\mathbf{b}_k|^p\right]^{1/p}. \tag{1.7}$$

Then, note that $d_p(\mathbf{x}, \mathbf{y})$ (as defined in equation 1.5) can be written in terms of $\mathbf{a}_k$ and $\mathbf{b}_k$, since

$\mathbf{a}_k = \mathbf{x}_k - \mathbf{z}_k$ and $\mathbf{b}_k = \mathbf{y}_k - \mathbf{z}_k$ for $k = 1, \ldots, N$:

$$d_p(\mathbf{x}, \mathbf{y}) = \left[\sum_{k=1}^{N} |\mathbf{a}_k + \mathbf{b}_k|^p\right]^{1/p}. \tag{1.8}$$

The triangle inequality, as stated in axiom M4, requires that

$$\left[\sum_{k=1}^{N} |\mathbf{a}_k + \mathbf{b}_k|^p\right]^{1/p} \leq \left[\sum_{k=1}^{N} |\mathbf{a}_k|^p\right]^{1/p} + \left[\sum_{k=1}^{N} |\mathbf{b}_k|^p\right]^{1/p}. \tag{1.9}$$

This inequality is equivalent to the well-known Minkowski's Theorem; thus, d_p satisfies the triangle inequality over $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^N$. □

Lecture 2: Real Analysis Prerequisites

Lecture Two introduces the concept of the *supremum* and *infimum* as properties of any subset of the reals. The sets of *bounded* and *continuous* functions are introduced as $B([0,1])$ and $C([0,1])$ respectively, and we prove that the "sup-metric" d_∞ forms a metric on $B([0,1])$.

28th September 2023

TODO

TODO

Definition 2.1 Supremum and Infimum

If $S \subset \mathbb{R}$ is a set, then $\sup S$ is defined to be the *least upper bound* of S. This is defined to be the smallest $b \in \mathbb{R}$ such that $x \leq b$ for all $x \in S$. The infimum of S, $\inf S$, is defined analogously as the *greatest lower bound*.

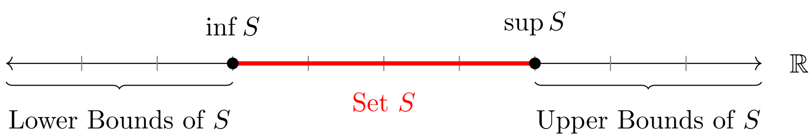

Figure 2.1: $S \subset \mathbb{R}$ and its bounding points on the real line

Definition 2.2 The ℓ^∞ Set of Bounded Sequences

Consider $\mathbb{R}^\mathbb{N}$: the set of all sequences of reals. We cannot work with this entire space, since many real sequences are unbounded, and the d_1 and d_2 canonical metrics give rise to non-finite sums. Therefore, we consider the set ℓ^∞ as the *set of all bounded real sequences*:

$$X \in \ell^\infty \iff \exists M > 0 \text{ such that } |X_n| \leq M \text{ for all } n \in \mathbb{N}. \tag{2.1}$$

Then, the infinity metric is defined in terms of the supremum, since a sequence with infinite terms mightn't possess a maximum:

$$d_\infty(X, Y) = \sup\{|X_i - Y_i| : i \in \mathbb{N}\} \text{ for } X, Y \in \ell^\infty. \tag{2.2}$$

Definition 2.3 The Set of Bounded Functions

$B([0,1])$ is the *set of all bounded functions* f such that $f\colon [0,1] \to \mathbb{R}$.

Definition 2.4 The Set of Continuous Functions

$C([0,1])$ is the *set of all continuous functions* f such that $f\colon [0,1] \to \mathbb{R}$.

Theorem 2.1 The set of bounded functions over $[0,1]$ with the sup-metric forms a metric space

Consider the d_∞ metric on $B([0,1])$ defined in terms of the supremum, such that the upper bound needn't lie in the set:

$$d_\infty\colon B([0,1]) \times B([0,1]) \to [0,\infty) \text{ such that } (f,g) \mapsto \sup\{|f(t) - g(t)| : t \in [0,1]\}. \tag{2.3}$$

Then, $(B([0,1]), d_\infty)$ is a metric space.

Proof. We must verify that $d_\infty : B([0,1]) \times B([0,1]) \to [0,\infty)$ satisfies the metric axioms described in definition 1.1 for all $f, g, h \in B([0,1])$.

- Since $f - g$ is a bounded function, there exists an $M \geq 0$ for which $f(t) - g(t) \leq M$ for all $t \in [0,1]$. Thus, $\sup\{|f(t) - g(t)| : t \in [0,1]\} \geq 0$, and $d_\infty(f,g) \geq 0$ for all $f, g \in B([0,1])$.
- $\boxed{\Longrightarrow}$ If $f = g$, then $|f(t) - g(t)| = 0$ for all $t \in [0,1]$, so $d_\infty(f,g) = \sup\{0, 0, \ldots\} = 0$.

  $\boxed{\Longleftarrow}$ Furthermore, if $d_\infty(f,g) = 0$, then we know that $\sup\{|f(t) - g(t)| : t \in [0,1]\} = 0$. We know that $|f(t) - g(t)| \geq 0$, so $|f(t) - g(t)| = 0$ follows immediately, from which we can conclude that $f(t) = g(t)$ for all $t \in [0,1]$, hence $f = g$.

  Thus, $d_\infty(f,g) = 0 \iff f = g$.
- By the symmetry of the standard metric on $\mathbb{R}$, the symmetry of d_∞ on $B([0,1])$ follows immediately:

$$d_\infty(f,g) = \sup\{|f(t) - g(t)| : t \in [0,1]\} \tag{2.4}$$
$$= \sup\{|g(t) - f(t)| : t \in [0,1]\} \tag{2.5}$$
$$= d_\infty(g,f). \tag{2.6}$$

- By the triangularity property of the standard metric on $\mathbb{R}$,

$$\begin{aligned} d_\infty(f,g) &= \sup\{|f(t)-g(t)| : t \in [0,1]\} && (2.7) \\ &= \sup\{|f(t)-h(t)+h(t)-g(t)| : t \in [0,1]\} && (2.8) \\ &\leq \sup\{|f(t)-h(t)| : t \in [0,1]\} + \sup\{|h(t)-g(t)| : t \in [0,1]\} && (2.9) \\ &= d_\infty(f,h) + d_\infty(h,g), && (2.10) \end{aligned}$$

  hence d_∞ possesses the property of triangularity on $B([0,1])$.

Thus, $(B([0,1]), d_\infty)$ is a metric space. □

Lecture 3: Norms, Subspaces, and Isometric Maps

Lecture Three opens with a counterexample to challenge a common misconception. It continues to introduce the concept of *norms* as generalisations of the absolute value function, *metric subspaces*, and *isometric maps*, complemented by a simple example.

3rd October 2023
TODO S'cd
TODO S&V

Example 3.1 Binary Strings

Despite the examples seen thus far, metric spaces needn't support an associated arithmetic or algebraic structure. For instance, let $\Sigma = \{0,1\}^{\mathbb{N}}$ be the set of binary strings (sequences of zeroes and ones). If $\sigma = (a_n)_{n=1}^\infty \in \Sigma$ and $\sigma' = (b_n)_{n=1}^\infty \in \Sigma$, consider the distance function $d\colon \Sigma \times \Sigma \to [0,\infty)$ such that

$$(\sigma, \sigma') \mapsto \begin{cases} 0 & \text{if } a_n = b_n \quad \forall n \in \mathbb{N}; \\ 1/\min\{n \colon a_n \neq b_n\} & \text{otherwise.} \end{cases} \tag{3.1}$$

Despite there being no inherent algebraic structure or ordering on Σ, d is a metric that is inversely proportional with the earliest point at which two given binary strings diverge, thus inducing a very natural— although unconventional—notion of *distance*.

This example was adapted from Ben Green's University of Oxford Metric Spaces and Complex Analysis *Michaelmas 2021 course notes.*

Definition 3.1 Norm

A *norm* is an abstraction of the absolute value. Suppose that V is a normable vector space. Then $||\cdot||\colon V \to \mathbb{R}$ is such that, for all $\mathbf{x}, \mathbf{y} \in V$, N1) $||\mathbf{x}|| \geq 0$;

N2) $||\mathbf{x}|| = 0 \iff \mathbf{x} = \mathbf{0}$;

N3) $||\lambda\mathbf{x}|| = |\lambda|||\mathbf{x}||$ for all $\lambda \in \mathbb{R}$;

N4) $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

V equipped with $||\cdot||$ is a *normed space*. Note that any norm can give rise to a metric; such a metric is sometimes called the *metric induced by the norm*.

Definition 3.2 Metric Subspace

Suppose that (X, d) and (Y, e) are metric spaces. We say that X is a *metric subspace* of Y if $X \subseteq Y$ and d is a restriction of e to $X \times X$.

Definition 3.3 Isometric Map

Suppose that (X, d) and (Y, e) are metric spaces, and that $\phi \colon X \to Y$ is surjective. Then ϕ is called an *isometric map* if and only if $e(\phi(a), \phi(b)) = d(a, b)$ for all $a, b \in X$. This will later be used to define the most rigid definition of "sameness" for metric spaces.

Example 3.2 Complex Isometry

Each $(a, b) \in \mathbb{R}$ is associated with a unique $z = a + ib \in \mathbb{C}$ such that $\Re(z) = a$ and $\Im(z) = b$. Notice that $(a, b) \mapsto a + ib$ is a bijective map from $\mathbb{R}^2$ to $\mathbb{C}$, and hence qualifies as an isometry.

Lecture 4: Introduction to the Topology of Metric Spaces

Lecture Four begins an investigation into the topology of a metric space. In particular, we introduce *open and closed balls*, *interior points*, *boundary points*, and *exterior points*, and define *open* and *closed* sets in terms of these concepts. We take an example metric space and rigorously compute its interior, boundary, and exterior, and finally show (by example) that considering objects as either subsets or subspaces can vastly alter their topological properties.

5th October 2023

TODO S'cd

TODO S&V

Definition 4.1 Open and Closed Balls

Suppose that (X, d) is a metric space, and that $x_0 \in X$. For every $\epsilon > 0$, we define the *open ball centered at x_0 with radius ϵ* to be the set $B(x_0, \epsilon) = \{x \in X : d(x, x_0) < \epsilon\}$. Analogously, the *closed ball centered at x_0 with radius ϵ* is defined to be the set $\overline{B}(x_0, \epsilon) = \{x \in X : d(x, x_0) \leq \epsilon\}$.

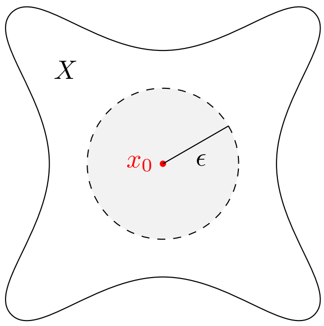

Figure 4.1: The open ball $B(x_0, \epsilon) \subset X$

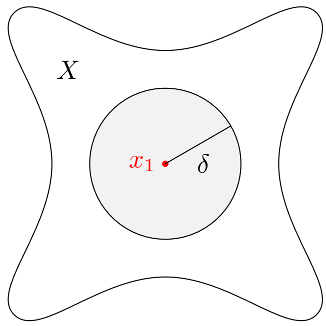

Figure 4.2: The closed ball $\overline{B}(x_1, \delta) \subset X$

Definition 4.2 Interior Points

Let $A \subset X$. An *interior point* $y \in X$ of A is an element for which $B(y, \epsilon) \subset A$ for some $\epsilon > 0$. That is, there is an open ball centred at y with radius ϵ that is completely contained within A. The set of all such points is denoted as A^o, and is called *the interior of* A.

Definition 4.3 Boundary Points

The element $y \in X$ is a *boundary point* of A if and only if for any $\epsilon > 0$, $B(y, \epsilon) \cap A \neq \emptyset$ and $B(y, \epsilon) \cap A^c \neq \emptyset$. That is, any open ball centred at y always intersects with A and its complement A^c. The set of all such points is denoted as ∂A, and is called *the boundary of* A.

Definition 4.4 Exterior Points

The element $y \in X$ is an *exterior point* of A if and only if for some $\epsilon > 0$, $B(y, \epsilon) \subset A^c$. That is, there exists an open ball centred at y which intersects only with the complement of A; this can also be interpreted as an interior point of the complement. The set of all such points is denoted as A^e, and is called *the exterior of* A.

Example 4.1 Illustration of Interior, Boundary, and Exterior Points

Consider an ambient space X, and the shaded subset $A \subseteq X$. We can illustrate examples of points from the interior, boundary, and exterior of A.

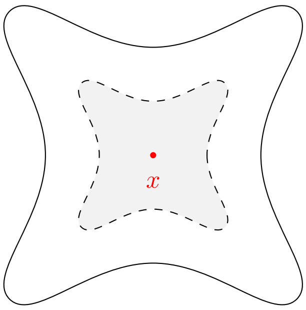

Figure 4.3: An interior point $x \in A^o$

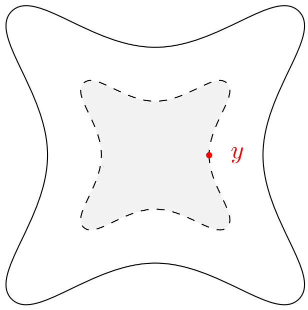

Figure 4.4: A boundary point $y \in \partial A$

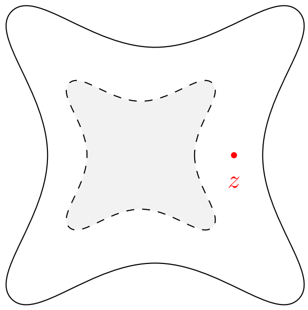

Figure 4.5: An exterior point $z \in A^e$

Note that the interior, boundary, and exterior are mutually disjoint and can be placed under the disjoint union operation to form the entire ambient space. This fact is henceforth denoted by "$A^o \coprod \partial A \coprod A^e = X$" for $A \subseteq X$.

Example 4.2 Finding the interior, boundary, and exterior of a set

Consider $(\mathbb{R}, d)$ with $A = (0,1] \subset \mathbb{R}$. Intuitively, we can conjecture that $A^o = (0,1)$, $\partial A = \{0,1\}$, and $A^e = \mathbb{R} \setminus (0,1) = (-\infty, 0) \cup (1, \infty)$, however these claims must be proven rigorously by (a) showing that the conjectured points do belong to the relevant set, and (b) showing that the conjectured points are the only elements to belong to the relevant set.

- First consider the interior. Take $x \in (0,1)$. By definition 4.2, we want to show that there is an $\epsilon > 0$ such that $B(x,\epsilon) \subset (0,1] = A$. Set $\epsilon_1 := x$, and $\epsilon_2 := 1 - x$. Given that $0 < x < 1$, we can take an $\epsilon := \min\{\epsilon_1, \epsilon_2\}$. Then, since $\epsilon/2 < \epsilon_1, \epsilon_2$,

$$B(x, \epsilon/2) = \{y \in \mathbb{R} \colon d(x,y) < \epsilon/2\} \subset A. \tag{4.1}$$

  This proves that $(0,1) \subseteq A^o$.

  We now need to eliminate the remaining candidates in $\mathbb{R} \setminus (0,1)$ from having possible membership in A^o. The points $x < 0$ and $x > 1$ can be discarded immediately, since any open ball centred at these points could never lie totally within A, due to their positive radii ϵ. Finally, we need to show that $\{0,1\} \not\subset A^o$. Without loss of generality, pick $x = 1$, and take an $\epsilon > 0$ to consider the open ball $B(x,\epsilon)$. Any such ball would contain a point that is strictly greater than 1, and hence would contain points outside of A. Thus, $x \notin A^o$, and $A^o = (0,1)$ as claimed.

- Now consider the boundary, as described in definition 4.3. We claim that $\{0,1\} = \partial A$, and first demonstrate that $\{0,1\} \subseteq \partial A$. Without loss of generality, we show that $0 \in \partial A$. Let $\epsilon > 0$, and consider $B(0,\epsilon) = (-\epsilon,\epsilon)$. Clearly, since $\epsilon > 0$, $(-\epsilon,\epsilon) \cap A \neq \emptyset$ and $(-\epsilon,\epsilon) \cap A^c \neq \emptyset$; thus, 0 is a boundary point.

Now, we show that there are no other boundary points of A in $\mathbb{R}$. We know that $\mathbb{R} = A^o \coprod \partial A \coprod A^e$, hence $A^o \cup \partial A = \emptyset$, thus $(0,1) \not\subset \partial A$. Without loss of generality for $x < 0$, consider points $x > 1$. Therefore, there exists an $\epsilon > 0$ such that $x = 1 + \epsilon$. Considering $B(x,\epsilon/2)$, we can see that $B(x,\epsilon/2) \subset A^c$, which implies that $B(x,\epsilon/2) \cap A = \emptyset$. Thus, $\{0,1\} = \partial A$.

- Finally, consider the exterior, as described in definition 4.4. Recall that the entire space can be expressed as a disjoint union, e.g. $\mathbb{R} = A^o \coprod A^e \coprod \partial A$. Hence,

$$A^e = \mathbb{R} \setminus (A^o \cup \partial A) \tag{4.2}$$
$$= \mathbb{R} \setminus [0,1] \tag{4.3}$$
$$= (-\infty,0) \cup (1,\infty), \tag{4.4}$$

as conjectured.

Definition 4.5 Open, Closed, and Clopen Sets

Let (X,d) be a metric space. A subset A of X is *open* if and only if $A \cap \partial A = \emptyset$. A subset F of X is *closed* if and only if $\partial F \subseteq F$. Note that a set can be both open and closed: typical examples are the empty set and the entire space; these sets are called *clopen*.

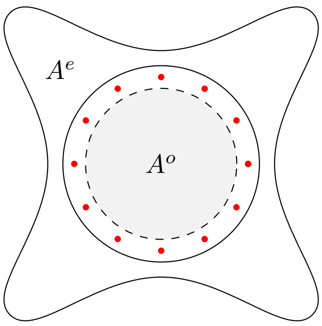

Figure 4.6: An open set A does not contain its boundary points.

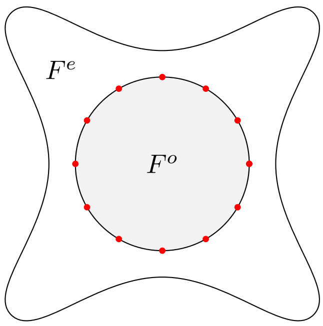

Figure 4.7: A closed set F contains its boundary points.

Example 4.3 Subset vs. Subspace

Consider $(\mathbb{R},d)$ with $A = (0,1) \cup (1,2)$. If A is considered as a *subset* if $\mathbb{R}$, then $\partial A = \{0,1,2\}$. Hence, $A \cap \partial A = \emptyset$, and A is open by definition 4.5. Further, A is not closed, since $\partial A \not\subseteq A$.

If A is considered as a *subspace* of $\mathbb{R}$, then $\partial A = \emptyset$, since $\{0, 1, 2\} \not\subseteq A$, and we cannot consider the points outside of the subspace when determining its topology. Hence, A is closed. But $A \cap \partial A = \emptyset$, since $\partial A = \emptyset$; thus A is also open, and is ultimately clopen when interpreted as a subspace.

Lecture 5: Results on the Topology of Metric Spaces

Lecture Five proves multiple important theorems: a set is open if and only if its complement is closed; a set is open if and only if we can place an open ball around every point and stay inside of the set; and an open set can be expressed as a union of open balls. We also introduce the notion of *a topology* T_d as the collection of all open subsets, and prove theorems related to closure under the familiar union and intersection set operations.

10th October 2023

TODO S'cd

TODO S&V

Theorem 5.1 A subset is open if and only if its complement is closed

Consider a set $A \subseteq X$. Then, A is open if and only if A^c is closed.

Proof. This can be proven by unravelling the definitions of open and closed sets (definition 4.5) and boundary points (definition 4.3).

$\boxed{\implies}$ First, suppose that A is open. If $A = \emptyset$, then $A^c = X$. The entire space is known to be clopen, and thus closed. If $A \neq \emptyset$, then $A \cap \partial A = \emptyset$, and $\partial A \subseteq A^c$. We can now see a useful equality by using the fact that $(A^c)^c = A$:

$$\partial (A^c) = \{y \in X : \forall \epsilon > 0\, B(y, \epsilon) \cap A^c \neq \emptyset \wedge B(x, \epsilon) \cap (A^c)^c \neq \emptyset\} \tag{5.1}$$

$$= \{y \in X : \forall \epsilon > 0\, B(y, \epsilon) \cap A^c \neq \emptyset \wedge B(x, \epsilon) \cap A \neq \emptyset\} \tag{5.2}$$

$$= \partial A. \tag{5.3}$$

Since $\partial A \subseteq A^c$, and $\partial A = \partial (A^c)$, we know that $\partial (A^c) \subseteq A^c$. Hence, A^c is closed.

$\boxed{\impliedby}$ Next, assume that A^c is closed, hence $\partial (A^c) \subseteq A^c$. Given that $\partial A = \partial (A^c)$, $\partial A \subseteq A^c$. Since $A^c \cap A = \emptyset$, we know that $\partial A \cap A = \emptyset$, and thus A is open. □

Definition 5.1 The topology of a metric space

The *topology* of a metric space (X, d) is denoted as T_d, and is defined to be *the collection of all open subsets of* X. Note that since $T_d \subseteq \mathcal{P}(X)$, for any set X, $\emptyset, X \in T_d$, so $T_d \neq \emptyset$.

Theorem 5.2 Equivalence between openness and the existence of open balls

Let $A \subseteq X$ be open. Then, every point of A is an interior point of A. Equivalently, A is open if and only if there is an open ball around every point in A that resides within A:

$$\forall x \in A \, \exists \epsilon > 0 \text{ such that } B(x, \epsilon) \subseteq A. \tag{5.4}$$

Proof. $\boxed{\Longrightarrow}$ First assume that A is open. By definition, $A \cap \partial A = \emptyset$. If $A = \emptyset$, then there exists no points to select, and the universal quantifier cannot select any points for x. If $A \neq \emptyset$, then there must be at least one $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A$ or $B(x, \epsilon) \subseteq A^c$ for any $x \in A$. But, since $x \in A$, it is not possible that $B(x, \epsilon)$ is entirely contained within A^c, since $x \in B(x, \epsilon)$. Hence, $B(x, \epsilon) \subseteq A^c$, as required.
$\boxed{\Longleftarrow}$ Next, suppose that $\forall x \in A \, \exists \epsilon > 0$ such that $B(x, \epsilon) \subseteq A$. Take any $x \in A$. Immediately, we can see that $x \notin \partial A$, since $B(x, \epsilon) \cap A^c = \emptyset$, because $B(x, \epsilon) \subseteq A$. Hence, A is open. $\square$

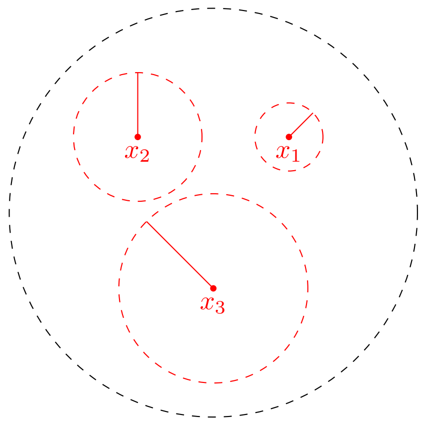

Figure 5.1: Every point supports an open ball

Theorem 5.3 The open ball is open

For any $x \in X$ and any $\epsilon > 0$, $B(x, \epsilon) \in T_d$. Since T_d is defined to be a collection of open sets, this statement is equivalent to the claim that "*the open ball is open*".

Proof. Take an $x \in X$ and construct the open ball $B(x, \epsilon)$, for a fixed $\epsilon > 0$. Take $y \in B(x, \epsilon)$ and let $\Delta := d(x, y)$. If $x = y$, then $B(x, \epsilon) = B(y, \epsilon) \subseteq B(x, \epsilon)$, there is nothing to do; therefore, we assume that $x \neq y$, thus $\Delta > 0$ and $0 < \Delta < \epsilon$. Let $\epsilon' := \min\{\Delta, \epsilon - \Delta\}$ and consider $B(y, \epsilon'/2)$.

To show the openness of the open ball, it is sufficient to show that $B(y, \epsilon'/2) \subseteq B(x, \epsilon)$. By the triangle inequality on d, for any $z \in B(y, \epsilon'/2)$,

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) && (5.5) \\ &\leq \Delta + \epsilon'/2 && (5.6) \\ &= \Delta + \min\{\Delta, \epsilon - \Delta\}/2 && (5.7) \\ &\leq \Delta + (\epsilon - \Delta)/2 && (5.8) \\ &< \Delta + \epsilon - \Delta && (5.9) \\ &= \epsilon && (5.10) \end{aligned}$$

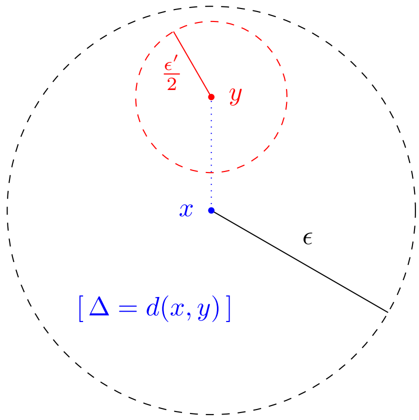

Figure 5.2: Careful construction of $B(y, \epsilon'/2)$

Thus, $z \in B(x, \epsilon)$ for all $z \in B(y, \epsilon'/2)$. Therefore, $B(y, \epsilon'/2) \subseteq B(x, \epsilon)$, and open balls are indeed open. □

Theorem 5.4 Elements of the topology are unions of open balls

If $A \in T_d$ and $A \neq \emptyset$, then A is a union of open balls.

Proof. Suppose that $A \neq \emptyset$ and is open. Take any $x \in A$, and we know by theorem 5.2 that there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A$. Then, we claim that

$$A = \bigcup_{x \in A} B(x, \epsilon(x)) \iff \underbrace{\left[A \subseteq \bigcup_{x \in A} B(x, \epsilon(x)) \wedge A \supseteq \bigcup_{x \in A} B(x, \epsilon(x)) \right]}_{\text{true by the principle of double-inclusion}}. \quad (5.11)$$

The leftmost conjunctive on the right-hand-side is clearly true: by placing an open ball of strictly positive radius around every point in A, the entire set will be covered, since $x \in B(x, \epsilon)$ for any x and $\epsilon > 0$. The rightmost conjunctive is also true, since each individual ball is wholly contained within A, and taking the union of all such interior balls cause any elements to "escape" the set in which they reside. Hence, A is the union of the open balls centered about every point in A. □

Theorem 5.5 Any union of open sets is open

Take *any* (finite, countably infinite, or uncountable) collection of open sets $\Lambda \subseteq T_d$. Then, for any Λ,

$$\bigcup_{\Omega \in \Lambda} \Omega \in T_d \text{ is open.} \quad (5.12)$$

Proof. Take $x \in \bigcup_{\Omega \in \Lambda} \Omega$. Thus, there exists an open $\Omega(x)$ such that $x \in \Omega(x)$. By definition 4.5, there exists an $\epsilon > 0$ such that $B(x,\epsilon) \subseteq \Omega(x)$. By transitivity, $B(x,\epsilon) \subseteq \bigcup_{\Omega\in\Lambda} \Omega$, and every union of open sets is open by theorem 5.4. □

Example 5.1 Non-finite open sets are not closed under intersection

Take $(\mathbb{R}, d)$ and consider $I_n := (-1/n, 1/n)$. Under infinite intersection, we calculate the singleton:

$$\bigcap_{n=1}^{\infty} I_n = \{0\}. \tag{5.13}$$

It is known that all singletons are not open—since any $B(x,\epsilon)$ with $\epsilon > 0$ would exceed the bounds of $\{x\}$—despite each individual I_n being open. Thus, the union property shown in theorem 5.5 does not apply with such generality to intersections.

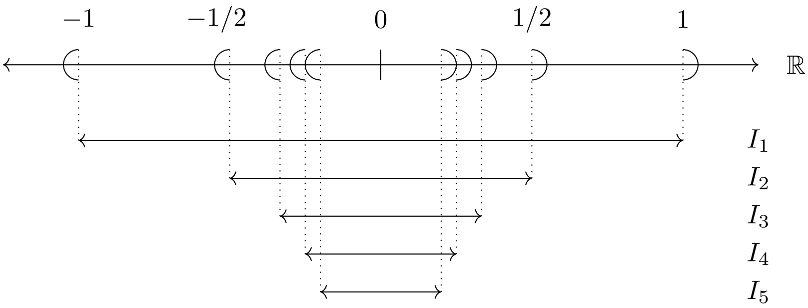

Figure 5.3: Nested intersections I_1 through I_5

Theorem 5.6 Any finite intersection of open sets is closed

Take *any finite* collection of open sets $\Omega_1, \ldots, \Omega_N \in T_d$. Then,

$$\bigcap_{i=1}^{N} \Omega_i \in T_d \text{ is open.} \tag{5.14}$$

Proof. If $\bigcap_{i=1}^{N} \Omega_i = \emptyset$, then there is nothing further to show, since $\emptyset$ is known to be open and thus a member of all topologies T_d. We now assume that the intersection is non-empty, from which we take an element x. Thus, there exists an $\epsilon_i > 0$ for which $B(x,\epsilon_i) \subseteq \Omega_i$. Let $\epsilon := \min\{\epsilon_1, \ldots, \epsilon_N\}$; then, $B(x,\epsilon) \subseteq B(x,\epsilon_i) \subseteq \Omega_i$ for any choice of $i = 1, \ldots, N$. Therefore,

$$B(x,\epsilon) \in \bigcap_{i=1}^{N} \Omega_i, \tag{5.15}$$

since $B(x,\epsilon)$ is a member of *every* Ω_i. Hence we can place an open ball with positive radius around any point and stay within the intersection; it is therefore open. □

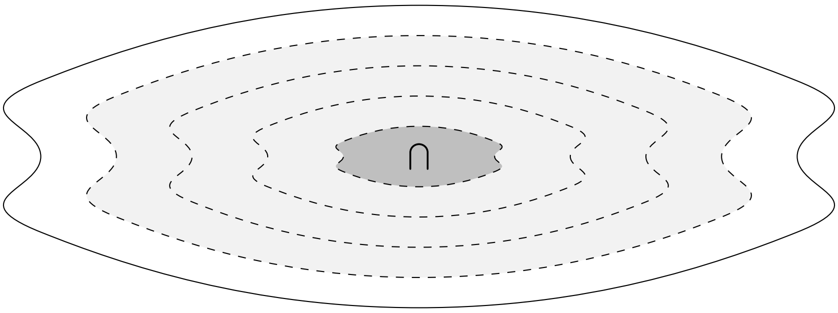

Figure 5.4: We consider the intersection of open sets using the smallest open ball $B(x, \epsilon)$.

Theorem 5.7 Summary of T_d Properties

Let (X, d) be a metric space, and let T_d be the topology induced by d. Then,

T1) $\emptyset, X \in T_d$;

T2) For any collection of open sets $\Lambda \subseteq T_d$, $\bigcup_{\Omega \in \Lambda} \Omega \in T_d$ is open (theorem 5.5);

T3) For any finite collection of open sets $\Omega_1, \ldots, \Omega_N \in T_d$, $\bigcap_{i=1}^{N} \Omega_i \in T_d$ is open (theorem 5.6).

Lecture 6: Closed Sets & Topological Equivalence

Lecture Six continues to cover the topology induced by a metric by deriving the corresponding properties of closed sets. We introduce the concept of *topological equivalence* as weaker method of determining "sameness" between metric spaces. We finally define the *closure* of a set, and prove that the closure is closed.

12th October 2023

TODO

TODO

Theorem 6.1 Summary of Closed Set Properties

We can easily derive a dual of theorem 5.7 for arbitrary and finite collections of *closed* sets.

F1) For any collection of closed sets $\mathcal{F}$,

$$\text{For all } F \in \mathcal{F},\ \underbrace{F^c \in T_d}_{\text{Theorem 5.1}} \implies \underbrace{\bigcup_{F \in \mathcal{F}} (F^c) \in T_d}_{\text{Axiom T2}} \implies \bigcap_{F \in \mathcal{F}} F \text{ is closed,} \qquad (6.1)$$

since $\left(\bigcap_{F \in \mathcal{F}} F\right)^c = \bigcup_{F \in \mathcal{F}} (F^c) \in T_d$ by De Morgan's laws.

F2) Similarly, for any finite collection of closed sets $F_1, \ldots, F_N$,

$$\left(\bigcup_{i=1}^{N} F_i\right)^c = \bigcap_{i=1}^{N} F_i^c \in T_d \implies \bigcup_{i=1}^{N} F_i \text{ is closed.} \tag{6.2}$$

Definition 6.1 Topological Equivalence

Let d and d^* be metrics on a set X. Then, (X, d) and (X, d^*) are *(topologically) equivalent* if and only if $T_d = T_{d^*}$; that is, d and d^* induce the same topologies.

Theorem 6.2 Determining Topological Equivalence

Let X be a set and let d and d^* be metrics on X. Then, $T_d = T_{d^*}$ if and only if there exists a scalar $\lambda > 0$ for which

$$\frac{1}{\lambda} d(x, y) \leq d^*(x, y) \leq \lambda d(x, y) \tag{6.3}$$

for all $x, y \in X$.

Example 6.1 $\mathbb{R}^2$ is topologically equivalent under the d_1, d_2, and d_∞ metrics

Recall the unit circles S_1^2 (figure 1.1), S_2^2 (figure 1.2), and S_∞^2 (figure 1.3) from example 1.1. We claim (and give an information demonstration to show) that $T_{d_1} = T_{d_2} = T_{d_\infty}$; that is, d_1, d_2, and d_∞ are topologically equivalent by definition 6.1.

We first consider the set $S_1^2 =: \Omega$ in the space $\left(\mathbb{R}^2, d_2\right)$. Take a point $x_0 \in \Omega$ and construct the open ball $B(x_0, \epsilon/2) \subseteq S_1^2$; such a ball can be created by considering an $\epsilon := \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, where ϵ_i for $i = 1, 2, 3, 4$ are the perpendicular distances from x_0 to the boundary of Ω by the d_2 metric. Then, $\Omega \in T_{d_2}$.

Next, consider $S_2^2 =: \Lambda$ in the space $\left(\mathbb{R}^2, d_1\right)$. Constructing open balls in d_1 around the points in Λ is easier still, since we only need to consider 'diamonds' which lie entirely within the Euclidean d_2 unit circle. By covering the entire set, we can conclude that $\Lambda \in T_{d_1}$.

Intuitively, we can see that the metrics d_1 and d_2 induce the same topologies; analogous arguments apply for establishing the equivalences to d_∞.

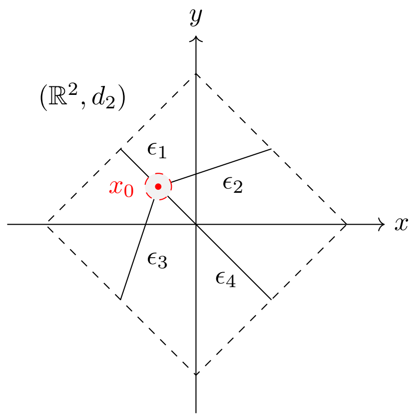

Figure 6.1: Considering the d_1-defined unit circle Ω as an open set in the $(\mathbb{R}^2, d_2)$ space.

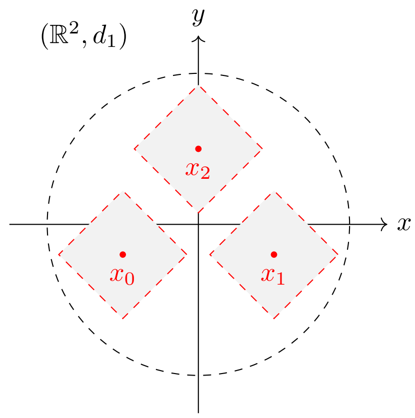

Figure 6.2: Considering the d_2-defined unit circle Λ as an open set in the $(\mathbb{R}^2, d_1)$ space.

Definition 6.2 Closure

Let (X, d) be a metric space and $A \subseteq X$. Then the *closure* of A, denoted by $\overline{A}$, is defined to be $\overline{A} = A \cup \partial A$.

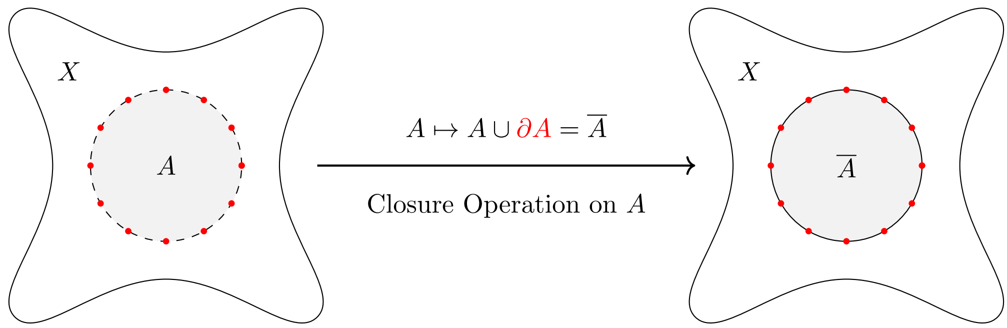

Figure 6.3: Encapsulating the boundary of an open set A is the most 'efficient' way of generating a closed set $\overline{A}$ with the same interior A^o.

Theorem 6.3 The closure is closed

For an open set $A \subseteq X$, the closure $\overline{A}$ is closed.

Proof. By theorem 5.1, $\overline{A}$ is closed if and only if $\left(\overline{A}\right)^c$ is open. If $\left(\overline{A}\right)^c = \emptyset$, then we are done since the empty set is known to be open; thus we assume that $\left(\overline{A}\right)^c \neq \emptyset$. To prove the openness of this non-empty set, we consider an arbitrary point $x \in \left(\overline{A}\right)^c$ and construct an $\epsilon > 0$ such that

$B(x, \epsilon) \subseteq \left(\overline{A}\right)^c$.

Since $\overline{A} = A \cup \partial A$, $x \notin A$ and $x \notin \partial A$. We can use rules from first-order logic combined with De Morgan's laws to negate the definition of a boundary point shown in definition 4.3:

$$x \in (\partial A)^c = \{y \in X : \forall \epsilon > 0\, B(y, \epsilon) \cap A \neq \emptyset \wedge B(y, \epsilon) \cap A^c \neq \emptyset\}^c \tag{6.4}$$

$$= \{y \in X : \forall \epsilon > 0\, B(y, \epsilon) \cap A = \emptyset \vee B(y, \epsilon) \cap A^c = \emptyset\} \tag{6.5}$$

$$= \{y \in X : \forall \epsilon > 0\, B(y, \epsilon) \subseteq A^c \vee B(y, \epsilon) \subseteq A\} \tag{6.6}$$

Since $x \notin A$, $B(x, \epsilon) \subseteq A^c$ is the only possibility. We also require that $x \notin \partial A$, so we need to show that there is an $\epsilon > 0$ such that $B(x, \epsilon) \cap \partial A = \emptyset$. By way of contradiction, suppose that there exists a $y \in B(x, \epsilon)$ such that $y \in \partial A$. By definition definition 4.3, for all $\delta > 0$,

$$B(y, \delta) \cap A \neq \emptyset \text{ and } B(y, \delta) \cap A^c \neq \emptyset. \tag{6.7}$$

If $d(x, y) =: \epsilon^* < \epsilon$, and $\hat{\epsilon} := \min\{\epsilon^*, \epsilon - \epsilon^*\}$, then $B(y, \hat{\epsilon}/2) \subseteq B(x, \epsilon)$. But $B(y, \hat{\epsilon}) \cap A \neq \emptyset$. This is a contradiction: no such y can exist, so $\left(\overline{A}\right)^c$ is open, and $\overline{A}$ is closed. □

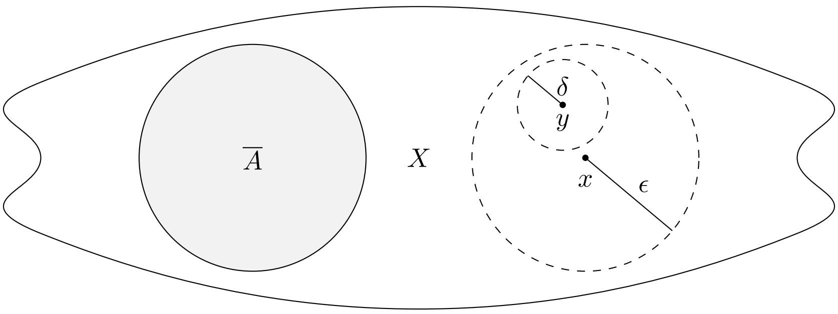

Figure 6.4: The open ball $B(x, \epsilon)$ lies entirely within the complement of the closure of A, within which $B(y, \delta)$ is nested.

Lecture 7: Limit Points and their Consequences

Lecture Seven demonstrates that a space is closed if and only if it is equal to its own closure, and moves to introduce the topic of limit/accumulation/cluster points whilst proving some useful related properties.

No Recording
TODO S'cd
TODO S&V

Theorem 7.1 Relationships between a closed set and its closure

If $A \subseteq X$ is closed, then $A = \overline{A}$; if $A = \overline{A}$, then A is closed. That is, A is closed if and only if $A = \overline{A}$.

Proof. $\boxed{\Longrightarrow}$ If $A = \overline{A}$, then $\overline{A}$ is closed by theorem 6.3. Thus A is closed.

$\boxed{\Longleftarrow}$ If A is closed, then $\overline{A} = A \cup \partial A$ by definition 6.2. By definition 4.5, $\overline{A} = \partial A \cup \overline{A} = A$. □

Definition 7.1 Limit/Accumulation/Cluster Points

Let (X, d) be a metric space, and $A \subseteq X$. A *limit point* $y \in X$ of A is an element of X for which

$$[B(y, \epsilon) \setminus \{y\}] \cap A \neq \emptyset \text{ for all } \epsilon > 0. \tag{7.1}$$

The *derived set* of A is denoted A', and is defined to be the set of all limit points of A. Equivalently (in the context of metric spaces, but not the more general topological spaces), we can say that $y \in X$ is a limit point of A if, for any $\epsilon > 0$, the intersection $B(y, \epsilon) \cap A$ contains infinitely many points of A. Limit points are sometimes called *accumulation points* or *cluster points*.

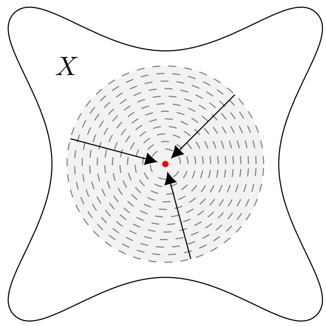

Figure 7.1: As the open balls contract *ad infinitum*, we can still find non-centroid points of A.

Theorem 7.2 The closure of a set is the union of the base set and its derived set

For a set $A \subseteq X$, $\overline{A} = A \cup A'$.

Proof. If A is closed, then the result is immediate: $\overline{A} = A$ (by theorem 7.1), so $A' \subseteq A = \overline{A}$. Thus, we assume that A is open. If $A' = \emptyset$, then $A' \subseteq \overline{A}$, since A has no limit points; therefore, we also assume that A is non-empty. We will show the equality $\overline{A} = A \cup A'$ for an open non-empty set A through the principle of double-inclusion.

$\boxed{\Longrightarrow}$ Without loss of generality, suppose that y is such that $y \in \partial A$ and $y \notin A$. For any choice of $\epsilon > 0$, we know that $B(y, \epsilon) \cap A \neq \emptyset$. Since $y \notin A$, there exists a $y' \neq y \in A, B(y, \epsilon)$. This is consistent with definition 7.1. Thus, y is a limit point, so $A' \subseteq \overline{A}$ and $A \cup A' \subseteq \overline{A}$.

$\boxed{\Longleftarrow}$ Given that A is not closed, there exists a $y \in A'$ such that $y \notin A$. Take $\epsilon > 0$ and note that $B(y, \epsilon) \cap A \neq \emptyset$ and $B(y, \epsilon) \cap A^c \neq \emptyset$. Thus, by definition 4.3, y is a boundary point. We now unravel the definitions:

$$
\begin{aligned}
A \text{ is closed} &\iff \partial A \subseteq A && \text{[Definition 4.5]} && (7.2)\\
&\iff A^c \text{ is open} && \text{[Theorem 5.1]} && (7.3)\\
&\iff A^c \cap \partial\left(A^c\right) = \emptyset && \text{[Definition 4.5]} && (7.4)\\
&\iff A^c \cap \partial A = \emptyset && \text{[Equations 5.1 to 5.3]} && (7.5)\\
&\iff \forall x \in A^c,\ \exists \epsilon > 0 \text{ s.t. } B(x,\epsilon) \subseteq A^c && \text{[Equation 7.3]} && (7.6)\\
&\iff A = \overline{A} && && (7.7)\\
&\iff A \supseteq A' && && (7.8)
\end{aligned}
$$

Hence, $\overline{A} = A \cup \partial A = A \cup A'$ for any $A \subseteq X$. □

Definition 7.2 Topological Interpretation of Closure

Let (X, d) be a metric space, and $A \subseteq X$. Then,

$$\overline{A} = \bigcap_{F \in \mathcal{F}} F, \tag{7.9}$$

where $\mathcal{F}$ is the collection of all open supersets of A. This means that $\overline{A}$ is the smallest closed superset of A.

Definition 7.3 Density

Let (X, d) be a metric space and $A \subseteq X$. Then A is said to be *dense* in X if and only if $\overline{A} = X$. Alternatively—but equivalently—for any $x \in X$ and any $\epsilon > 0$, there exists an $a \in A$ such that $d(x, a) < \epsilon$.

Lecture 8: Sequences and Convergence

Lecture Eight gives the definition of a *sequence* in a metric space, and motivates the study of various properties thereof, such as *convergence*. Some examples are provided, with familiar concepts from Real Analysis being recast into the more abstract setting of metric and topological spaces.

19th October 2023
TODO
TODO

Definition 8.1 Sequence

A *sequence* in a metric space (X, d) is an element of $X^{\mathbb{N}}$, where any element $x \in X^{\mathbb{N}}$ can be written as a countable set $x = \{x_1, \ldots, x_n, \ldots\}$, such that $x_i \in X$ for all $i \in \mathbb{N}$. Such a sequence is denoted

$$(x_n)_{n\in\mathbb{N}} \text{ or } (x_n)_{n=1}^{\infty}. \tag{8.1}$$

Definition 8.2 Convergence of Sequences (by the Metric)

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in (X, d). Then, $(x_n)_{n\in\mathbb{N}}$ *converges* to $x \in X$ if and only if

$$\text{For all } \epsilon > 0, \text{ there exists an } N = N(\epsilon) \text{ such that } d(x_n, x) < \epsilon \text{ for all } n > N. \tag{8.2}$$

If this is the case, we write $x_n \to x$ (as $n \to \infty$), and the x is *the* (unique) limit of $(x_n)_{n\in\mathbb{N}}$. If no such x exists, then $(x_n)_{n\in\mathbb{N}}$ is *divergent*.

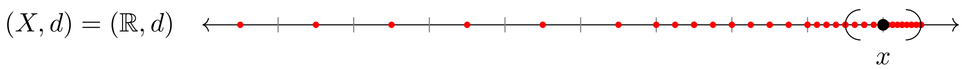

Figure 8.1: A sequence in $(\mathbb{R}, d)$, where d is the standard metric, converging to an $x \in \mathbb{R}$. The *zone of convergence* $(x - \epsilon, x + \epsilon)$ is denoted with parentheses about x.

Definition 8.3 Convergence of Sequences (by the Open Balls)

An equivalent notion of convergence can be expressed in terms of open balls, such that the tail of the sequence exists in the open ball centered at the limit. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in (X, d). Then, $(x_n)_{n\in\mathbb{N}}$ *converges* to $x \in X$ if and only if

$$\forall \epsilon > 0\, \exists N > 0 \text{ such that } x_n \in B(x, \epsilon) \text{ for all } n > N. \tag{8.3}$$

That is, for any given $\epsilon > 0$, we must find an index N for the sequence $(x_n)_{n\in\mathbb{N}}$ after which all points x_i $(i > N)$ exist within the open ball $B(x, \epsilon)$. If this holds, then the sequence (x_n) converges to the centre of the open ball x.

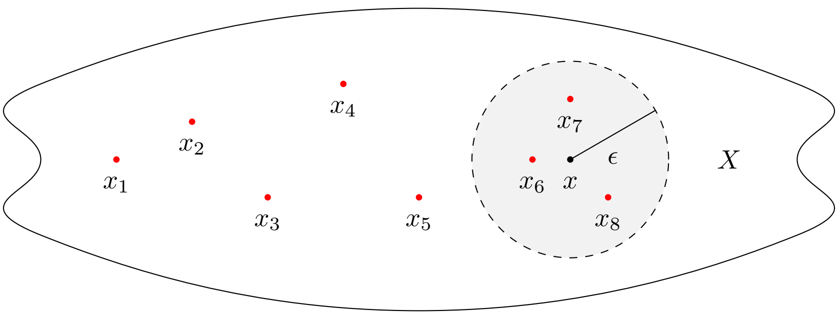

Figure 8.2: A sequence converging within the bounds of an arbitrarily sized open ball centred at the limit. In this example, $N = 5$, since all sequence points $x_6, x_7, x_8, \ldots$ belong to the pictured open ball.

Definition 8.4 Convergence of Sequences (by the convergence of a real sequence)

We can also use results from Real Analysis to consider the real sequence generated by the distances between each point in the space and the proposed limit x. For a sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d),

$$\lim_{n\to\infty} x_n = x \iff \lim_{n\to\infty} d(x_n, x) \to 0. \tag{8.4}$$

Theorem 8.1 Convergences of Real-Dimensional Vector Spaces

In $\mathbb{R}^k$ with any d_1, d_2, or d_∞ metrics, *overall convergence is equivalent to simultaneous component-wise convergence*. That is, for $\mathbf{x}_n \in \mathbb{R}^k$ with $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathbf{x}_n \to \mathbf{x} \iff \lim_{n\to\infty} x_i^{(n)} \to x_i \text{ for each } i \in \{1, \ldots, k\}. \tag{8.5}$$

Note that $\mathbf{x} = (x_1, \ldots, x_k)$ denotes a general vector/candidate limit, and $\mathbf{x}_n = \left(x_1^{(n)}, \ldots, x_k^{(n)}\right)$ denotes a k-dimensional vector in the sequence at index n. For this demonstration, we will consider the specific case of $\left(\mathbb{R}^k, d_\infty\right)$.

Proof. $\boxed{\Longrightarrow}$ Suppose that $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$. We want to show that, for each $i \in \{1, \ldots, k\}$, the components converge such that $x_i^{(n)} \to x_i$ as $n \to \infty$. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$, there exists an $N(\epsilon) > 0$ such that $d_\infty(\mathbf{x}_n, \mathbf{x}) < \epsilon$ for all $n > N$. By recalling the mapping definition of d_∞ from definition 1.2, we can derive that

$$\max_{1\le i\le k} \left|x_i^{(n)} - x_i\right| < \epsilon \text{ for all } n > N. \tag{8.6}$$

By the nature of the maximum function, we can generalise this inequality to all choices of $i \in \{1, \ldots, k\}$:

$$\left|x_i^{(n)} - x_i\right| < \epsilon \text{ for all } n > N \text{ for all } i \in \{1, \ldots, k\}. \tag{8.7}$$

We can immediately arrive at the desired limit for the real sequence: $\lim_{n\to\infty} x_i^{(n)} = x_i$ for all $i \in \{1, \ldots, k\}$.

$\boxed{\Longleftarrow}$ Now suppose that $\lim_{n\to\infty} x_i^{(n)} = x_i$ for all $i \in \{1, \ldots, k\}$, and let $\epsilon > 0$ be given. Then, for any suitable choice of i, there exists an $N_i(\epsilon) > 0$ such that $\left|x_i^{(n)} - x_i\right| < \epsilon$ for all $n > N_i$.

$$\begin{array}{c|c}
\begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \vdots \\ \vdots \\ \vdots \end{array} &
\begin{array}{cccccc}
x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & \dots & x_k^{(1)} \\
x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & \dots & x_k^{(2)} \\
x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & \dots & x_k^{(3)} \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{array} \\
 & \left\{ \begin{array}{ccccc}
N_1 & \downarrow & N_3 & \downarrow & N_k \\
\vdots & N_2 & \vdots & N_j & \vdots
\end{array} \right\}
\end{array}$$

Zones of Convergence

Take $N := \max\{N_1, \dots, N_k\}$, such that N is a sufficiently large index to induce convergence on all component sequences. That is,

$$\max_{1\le i\le k} \left|x_i^{(n)} - x_i\right| < \epsilon \text{ for all } n > N. \tag{8.8}$$

Thus, $\lim_{n\to\infty} x_i^n = x_i$ for all $i \in \{1, \dots, k\}$, as desired. □

Example 8.1 Convergence of Continuous Functions

Consider the sequence $(f_n)_{n=1}^{\infty}$ where $f_n \in C([0,1])$ and $t \mapsto t^n$. We claim the following:

1. In d_2, $(f_n)_{n=1}^{\infty}$ converges to zero;
2. In d_∞, $(f_n)_{n=1}^{\infty}$ diverges.

First consider the sequence under the d_2 metric. In general, we know that $x_n \to x$ as $n \to \infty$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. Therefore, we compute $d_2(f_n, 0)$:

$$d_2(f_n, 0) = \left[\int_0^1 |f_n(t)|^2 \, \mathrm{d}t\right]^{1/2} \tag{8.9}$$
$$= \sqrt{\frac{1}{2n+1}} \tag{8.10}$$

Given that $\lim_{n\to\infty} \sqrt{1/(2n+1)} = 0$, we can conclude that $f_n \to 0$ as $n \to \infty$. Next considering the sequence under the d_∞ metric, recall that, for all $n \in \mathbb{N}$,

$$d_\infty(f_n, 0) = \sup\{|f_n(t)| : t \in [0,1]\} = 1. \tag{8.11}$$

Since 1 does not approach 0, $f_n \not\to 0$ as $n \to \infty$. Therefore, we can see, by example, that changing the metric on a set can vastly alter its convergence properties.

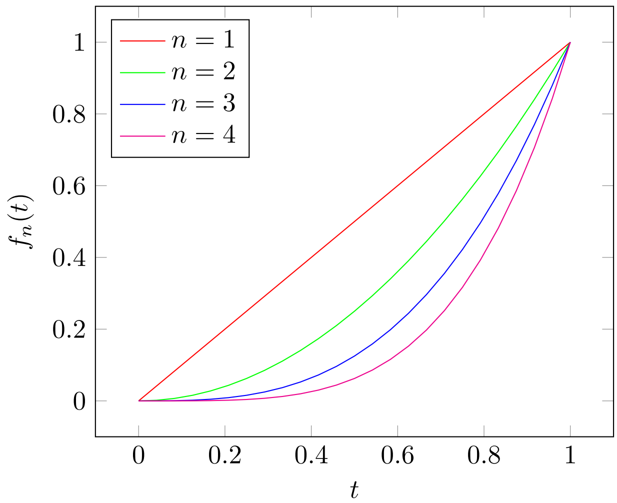

Figure 8.3: The graphs of $f_n(t)$ over $[0, 1]$ for $n = 1, 2, 3, 4$.

Theorem 8.2 Relationship between Mutual Convergence and Topological Equivalence

Suppose that (X, d) and (X, d^*) are topologically equivalent. Then,

$$\underbrace{(x_n)_{n=1}^{\infty} \to x}_{\text{Convergence in }(X,d)} \quad \Longleftrightarrow \quad \underbrace{(x_n)_{n=1}^{\infty} \to x.}_{\text{Convergence in }(X,d^*)} \tag{8.12}$$

Proof. Due to the topological equivalence of (X, d) and (X, d^*), and by theorem 6.2, there exists a scalar $\lambda > 0$ such that

$$\frac{1}{\lambda} d^*(x, y) \leq d(x, y) \leq \lambda d^*(x, y) \text{ for all } x, y \in X. \tag{8.13}$$

$\boxed{\Longrightarrow}$ Suppose that $(x_n)_{n=1}^{\infty} \to x$ as $n \to \infty$. Let $\epsilon > 0$ be given, and set $\epsilon^* := \epsilon/\lambda$. By definition 8.2, there exists an $N > 0$ such that $d(x_n, x) < \epsilon^*$ for all $n > N$. Since (X, d) and (X, d^*) are topologically equivalent,

$$\frac{1}{\lambda} d^*(x_n, x) \leq d(x_n, x) < \epsilon^* = \frac{\epsilon}{\lambda}. \tag{8.14}$$

Since $\lambda > 0$, we can multiply through by λ:

$$d^*(x_n, x) \leq \lambda d(x_n, x) < \epsilon. \tag{8.15}$$

Thus, $d^*(x_n, x) < \epsilon$ for all $n > N$, so $(x_n)_{n=1}^{\infty} \to x$ as $n \to \infty$ in (X, d^*).

$\boxed{\Longleftarrow}$ A near-identical argument applies: derive convergence in (X, d) by assuming convergence in (X, d^*). $\square$

Lecture 9: Cauchy Sequences & More on Convergence

Lecture Nine proves the uniqueness of limits, and begins to formalise the Cauchy Condition while introducing its various surprising relationships to 'traditional' convergence. We are also brought to consider the concept of pointwise convergence over a sequence of functions.

24th October 2023
TODO S'cd
TODO S&V

Theorem 9.1 Uniqueness of Limits of Convergent Sequences

Let (X, d) be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a convergent sequence in (X, d) such that $x_n \to x$, and $x_n \to y$, as $n \to \infty$. Then, $x = y$.

Proof. Assume, by way of contradiction, that $x \neq y$. Thus, $d(x, y) =: \epsilon < 0$. Set $\delta := \epsilon/2$. Given that $x_n \to x$ as $n \to \infty$ there exists an $N(\delta) > 0$ such that $d(x_n, x) < \delta = \epsilon/2$ for all $n > N$, by definition 8.2. Due to the purported simultaneous convergence to y, there also exists an $\widetilde{N}(\delta) > 0$ such that $d(x_n, y) < \delta = \epsilon/2$ for all $n > \widetilde{N}$.

Thus,

$$d(x_n, x) < \delta = \frac{\epsilon}{2} \text{ and } d(x_n, y) < \delta = \frac{\epsilon}{2} \tag{9.1}$$

for all $n > M$, where $M := \max\left\{N, \widetilde{N}\right\}$. By the triangle inequality,

$$\begin{aligned} d(x, y) &\leq d(x, x_n) + d(x_n, y) &(9.2)\\ &< \delta + \delta &(9.3)\\ &= \epsilon \text{ for all } n > M. &(9.4) \end{aligned}$$

Given that ϵ was defined to be equal to $d(x, y)$, this is a contradiction. Therefore, $x = y$, and the limits are unique. □

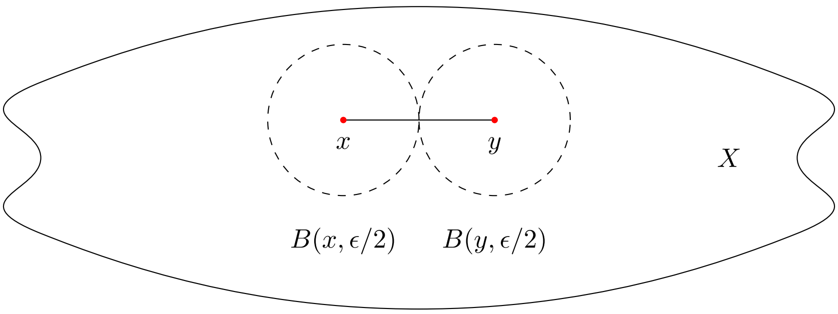

Figure 9.1: Given the required sequence convergences, we cannot have distinct limits.

Example 9.1 Abstract Infinite Series

We are often concerned with expressions of the form

$$S_\infty = \sum_{n=1}^{\infty} x_n, \tag{9.5}$$

where S_∞ is defined to be the limit of the sequence generated by the partial sums

$$S_k = \sum_{n=1}^{k} x_n \tag{9.6}$$

as k approaches infinity. If a metric space (X, d) supports an algebra with a suitable notion of 'addition', this idea can be generalised to a more abstract setting.

Definition 9.1 Pointwise Convergence

Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions $f : X \to Y$. The function f is the *pointwise limit* of the function sequence if and only if

$$\lim_{n\to\infty} f_n(x_0) = f(x_0) \text{ for any } x_0 \in X. \tag{9.7}$$

This equivalence allows us to study convergence in an arbitrary metric space by examining convergence in the codomain Y.

Definition 9.2 The Sequential Characterisation of Continuity

Let $f : X \to Y$ be a function from (X, d) to (Y, d^*). Then, f is *continuous at* $x_0 \in X$ if and only if $f(x_n)$ converges to $f(x_0)$ whenever the sequence generated by x_n converges to x_0.

The function f is *continuous* if and only if f is continuous at every point $x_0 \in X$.

Definition 9.3 Cauchy Sequence

Let $(x_n)_{n=1}^{\infty}$ be a sequence. Then, $(x_n)_{n=1}^{\infty}$ is a *Cauchy sequence* if and only if, for any $\epsilon > 0$, there exists an $N > 0$ such that

$$d(x_n, x_m) < \epsilon \text{ for all } n, m > N \tag{9.8}$$

The terms of Cauchy sequences grow arbitrarily close, and Cauchy sequences need not be convergent.

Example 9.2 The Cauchy condition does not imply convergence

Take any sequence of rationals p_n/q_n for which $(p_n/q_n)^2 \to 2$ as $n \to \infty$. Note the following:

- $(p_n/q_n)_{n=1}^{\infty}$ is Cauchy;
- There is no $x \in \mathbb{Q}$ such that $\lim_{n\to\infty} (p_n/q_n) = x$, since $\sqrt{2} \notin \mathbb{Q}$.

These facts lead to the well-known result regarding the incompleteness of the rationals.

Theorem 9.2 Convergence implies Cauchy

Let (X, d) be a metric space and let $(x_n)_{n=1}^{\infty}$ be a convergent sequence such that $\lim_{n\to\infty} x_n = x$. Then, $(x_n)_{x=1}^{\infty}$ is Cauchy.

Proof. Let $\epsilon > 0$ be given, and set $\delta := \epsilon/2$. Given that $(x_n)_{n=1}^{\infty}$ is convergent, there exists an $N(\delta) > 0$ such that $d(x_n, x) < \delta$ for all $n > N$, according to definition 8.2. By the triangle inequality, for all $n, m > N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) \tag{9.9}$$

$$< \delta + \delta = \epsilon \tag{9.10}$$

This is precisely the criterion for the Cauchy property on $(x_n)_{n=1}^{\infty}$ by definition 9.3. □

This result, combined with the counterexample shown in example 9.2, demonstrates that while convergence implies Cauchy, Cauchy does not necessarily imply convergence.

Lecture 10: Results on Cauchy Sequences, Subsequences, and Completeness

Lecture Ten establishes more useful relationships between the properties of a Cauchy sequence and a convergent sequences, and introduces the concept of subsequences. The property of completeness is finally defined, along with a statement of the Completion Theorem.

26th October 2023

TODO S'cd

TODO S&V

Example 10.1 The Harmonic Series Sequence is not Convergent

Consider $(\mathbb{R}, d)$ and the Harmonic series $\sum_{n=1}^{\infty}(1/n)$. Define a sequence $(S_N)_{N=1}^{\infty}$, such that

$$S_N := \sum_{n=1}^{N} \frac{1}{n}. \tag{10.1}$$

We claim that $(S_N)_{N=1}^{\infty}$ is not convergent; this can be shown by considering the contrapositive of the statement given by theorem 9.2: "not Cauchy" implies "not convergent". Consider the difference between $S_{(2N)}$ and S_N:

$$S_{(2N)} - S_N = \left(\frac{1}{2N} + \frac{1}{2N-1} + \ldots + \frac{1}{N+1} + \frac{1}{N} + \frac{1}{N-1} + \ldots + 1\right) - \left(\frac{1}{N} + \frac{1}{N-1} + \ldots + 1\right) \tag{10.2}$$

$$\geq \frac{N}{2N} = \frac{1}{2} \tag{10.3}$$

By definition 9.3, given an $\epsilon > 0$, we need to find $K > 0$ such that

$$|S_M - S_N| < \epsilon \text{ for all } M, N > K. \tag{10.4}$$

Given equations 10.2 to 10.3, if $M := 2N$, then $|S_M - S_N| > \epsilon$, and the Cauchy condition does not hold. By the initial statement, the sequence is therefore not convergent.

Definition 10.1 Subsequence

Given a metric space (X, d) containing a sequence $(x_n)_{n=1}^{\infty}$, a *subsequence* of $(x_n)_{n=1}^{\infty}$ is a sequence of elements $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$ such that $n_1 < n_2 < \ldots < n_k < \ldots$. Intuitively, we are generating a subsequence by selecting elements from an 'ambient sequence' while preserving the order of indices.

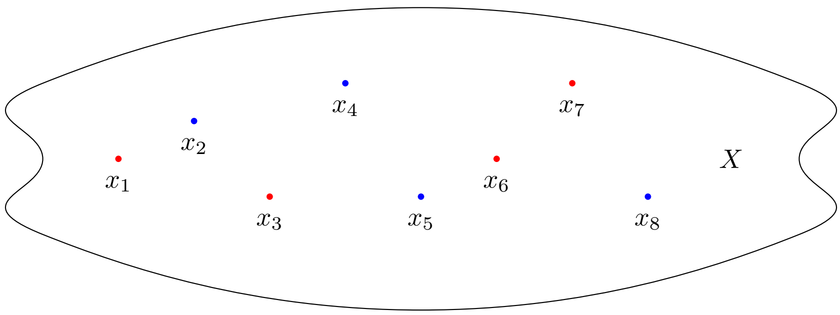

Figure 10.1: An ambient sequence (red and blue points) generated by x_n over $n = 1, \ldots, 8$, and a subsequence (blue points only) x_{n_k} with $k = 2, 4, 5, 8$.

Theorem 10.1 Convergence in Subsequences

If $(x_n)_{n=1}^{\infty}$ is a sequence in (X, d) such that $\lim_{n\to\infty} x_n = x$, then any subsequence of $(x_n)_{n=1}^{\infty}$ shares the limit. The converse also holds; this induces a useful contraposition test for non-convergence: if any subsequence is not convergent, then the ambient sequence is neither convergent.

Example 10.2 Determining Divergence by Testing Subsequences

Consider he sequence generated by $x_n := (\pm x)^n$ in $\mathbb{R}$. Then consider the subsequences generated by extracting terms at even and odd indices and examine their respective limits:

$$n_1 = 1,\ n_2 = 3,\ n_3 = 5,\ \ldots \quad \mapsto \quad (-1, -1, -1, \ldots) \quad \to \quad -1 \quad (\text{as } n \to \infty) \tag{10.5}$$

$$n_1 = 2,\ n_2 = 4,\ n_3 = 6,\ \ldots \quad \mapsto \quad (1, 1, 1, \ldots) \quad \to \quad 1 \quad (\text{as } n \to \infty) \tag{10.6}$$

By the contraposition test described by theorem 10.1, we can see immediately that $(x_n)_{n=1}^{\infty}$ diverges, given that its subsequences do not converge to a consistent limit.

Definition 10.2 Completeness

Let (X, d) be a metric space. Then, X is *complete* if and only if any Cauchy sequence converges to a point in X. We may assume that $(\mathbb{R}, d)$ is complete.

If a sequence is Cauchy, and its resident metric space is complete, then the sequence is convergent. To show that a space is *not complete*, we must find a Cauchy sequence in the space which does not converge to another point in the space.

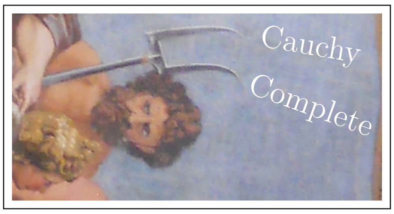

Figure 10.2: Pluto, the Greek God of Convergence, brandishing his topological bident

Theorem 10.2 The Completion Theorem

Let (X, d) be a metric space. There is a metric space (X^*, d^*) and an isometry $\psi\colon X \to X^*$ such that the following properties hold:

- X^* is complete (c.f. definition 10.2); and

- $\psi(X)$ is dense in X^* (c.f. definition 7.3).

X^* is called a *completion* of X; all completions of X are isometric to X^*.

Proof. Proof sketch:

1. If X is already complete, we can set $X^* := X$ and $\psi(x) := x$ and finish. Therefore, we assume that X is not complete.
2. Collate the set of all Cauchy sequences in X, and denote the set as $\mathcal{G}(X)$. This set is trivially nonempty: for the degenerate case of a singleton $X = \{x\}$, we can extract the Cauchy sequence $(x, x, \ldots)$.
3. Define an equivalence relation $\sim$ on $\mathcal{G}(X)$ such that
$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} d(x_n, y_n) = 0 \text{ for all sequences in } \mathcal{G}(X). \tag{10.7}$$
4. Construct the set of all equivalence classes under the $\sim$ relation: the quotient set $\mathcal{G}(X)/\sim$.
5. Extend the metric d on X to the metric d^* on X^* by defining a map $\widetilde{d}\colon \mathcal{G}(X)/\sim \to \mathbb{R}$ such that
$$([(x_n)], [(y_n)]) \mapsto \lim_{n\to\infty} d(x_n, y_n) \in \mathbb{R}. \tag{10.8}$$
6. Define the isometry map $\psi\colon X \to X^*$ such that
$$x \mapsto [(x)_{n=1}^{\infty}] \in \mathcal{G}(X)/\sim \text{ where } [(x)_{n=1}^{\infty}] = (x, x, \ldots), \tag{10.9}$$
and prove that ψ is an isometry.
7. Show that $\overline{\psi(X)} = \mathcal{G}(X)/\sim = X^*$, thereby confirming density.
8. Show that any Cauchy sequence in $\mathcal{G}(X)/\sim$ converges to a common limit in the same quotient set, and prove uniqueness.

The detailed proof is completed in Section 1.5 of [SV06]. □

Theorem 10.3 A real-valued vector space is complete under the sup-metric

The space $(\mathbb{R}^k, d_\infty)$ where $k \geq 2$ is complete.

Proof. By definition 10.2, and in the same vein as theorem 10.2, we take an arbitrary Cauchy sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ from $(\mathbb{R}^k, d_\infty)$, and let $\epsilon > 0$ be given. Since $(\mathbf{x}_n)_{n=1}^{\infty}$ is Cauchy, there exists an $N(\epsilon) > 0$ such that, for all $n, m > N$,

$$d_\infty(\mathbf{x}_n, \mathbf{x}_m) = \max_{1 \le i \le k} \left\{ \left| x_i^{(n)} - x_i^{(m)} \right| \right\} \quad \text{[Definition 1.2]} \tag{10.10}$$

$$< \epsilon \quad \text{[Definition 9.3]} \tag{10.11}$$

Given that $\mathbb{R}$ is axiomatised to be complete, each real subsequence converges to a real limit. Since we know the limit to exist, we can construct a candidate limit $\mathbf{x}$ to be the component-wise limits, as pictured below.

$$\begin{array}{c|c}
\mathbf{x}_1 & \left(x_1^{(1)}, \; x_1^{(2)}, \; \ldots, \; x_1^{(k)} \right) \\
\mathbf{x}_2 & \left(x_2^{(1)}, \; x_2^{(2)}, \; \ldots, \; x_2^{(k)} \right) \\
\vdots & \downarrow \quad \downarrow \quad \ddots \quad \downarrow \\
\mathbf{x} & \left(x_1, \; x_2, \; \ldots, \; x_k \right) \; \Big\} \; N(\epsilon)
\end{array}$$

After the index N, $\mathbf{x}_n$ and $\mathbf{x}_m$ are sufficiently close by the d_∞ metric; that is, $\mathbf{x} \in \mathbb{R}^k$ is composed of the component-wise limits of $\mathbf{x}_i$ for $i \in \mathbb{N}$. We can use theorem 8.1 to show that the candidate limit $\mathbf{x}$ is valid, given that the coordinate sequences converge component-wise (by construction of $\mathbf{x}$).

Thus, any arbitrary Cauchy sequence $(\mathbf{x}_n)_{n=1}^\infty \in \left(\mathbb{R}^k\right)^{\mathbb{N}}$ converges to $\mathbf{x} \in \mathbb{R}^k$, so $\left(\mathbb{R}^k, d_\infty\right)$ is complete by definition 10.2. □

Lecture 11: Generalising Continuity

Lecture Eleven introduces a general form of continuity according to the ϵ–δ definition and hence motivates five superficially different, albeit equivalent, notions of a continuous map in terms of highly abstract open and closed sets. The sets of bounded and continuous functions are revisited with greater care, and an example helps to illustrate $B(X, Y)$ with the d_∞ metric.

7th November 2023
TODO S'cd
TODO S&V

Definition 11.1 Inverse Image (Traditional Interpretation)

Consider $f\colon X \to Y$, and let $A \subseteq Y$. Then, the *inverse image* of A is defined as

$$f^{-1}(A) \coloneqq \{x \in X : f(x) \in A\}. \tag{11.1}$$

Definition 11.2 Inverse Image (Fibreoptic Interpretation)

Following from definition 11.1, we can define a *fibre* of y to be the inverse image of the singleton set $\{y\} \subseteq Y$, denoted F_y:

$$F_y := \{x \in X : f(x) = y\}. \tag{11.2}$$

Then, the *inverse image* of A is the union over all fibres of the singletons in A:

$$f^{-1}(A) = \bigcup_{y \in A} F_y. \tag{11.3}$$

Definition 11.3 Continuity on the Real Line

A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* if and only if, for all $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon. \tag{11.4}$$

This is the least abstract, yet formally correct, definition of continuity, as it applies only to the set of real numbers. Over the next set of definitions, we define increasingly abstract—albeit equivalent—notions of *continuity in a metric space*.

Definition 11.4 Continuity in an Abstract Metric Space

Let (X, d) and $(Y, \widetilde{d})$ be metric spaces. The function $f : X \to Y$ is continuous at $x_0 \in X$ if any of the following equivalent definitions holds:

1. **(Analytic Definition)** For any $\epsilon > 0$, there exists a $\delta > 0$ such that $\widetilde{d}(f(x), f(x_0)) < \epsilon$ whenever $d(x, x_0) < \delta$.
2. **(Open Ball Definition)** For any $\epsilon > 0$, there exists a $\delta > 0$ such that $f(B(x_0, \delta)) \subseteq B(f(x_0, \epsilon))$. See figure 11.1.
3. **(Open Set with Ball Definition)** For any open set $V \subseteq Y$ with $f(x_0) \in V$, there exists a *ball* $B \subseteq X$ such that $x_0 \in B$ and $f(B) \subseteq V$.
4. **(Purely Topological Definition)** For any open set $V \subseteq Y$ with $f(x_0) \in V$, there exists an *open set* $U \subseteq X$ with $x_0 \in U$ and $f(U) \subseteq V$.
5. **(Sequence Definition)** For any sequence $(x_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = x_0$, the sequence generated by $f(x_n)$ is such that $\lim_{n\to\infty} f(x_n) = f(x_0)$.

To prove the equivalence between these definitions, we need to demonstrate the following cycle of implications:

$$\#1 \implies \#2 \implies \#3 \implies \#4 \implies \#5 \implies \#1. \tag{11.5}$$

In particular, we will show that #4 $\implies$ #5 (theorem 11.1), followed by #5 $\implies$ #1 (theorem 11.2). The equivalences of statements #1 through #4 can be trivially established through previously established results on the connexions between open balls and open sets, such as those enumerated throughout lecture 5.

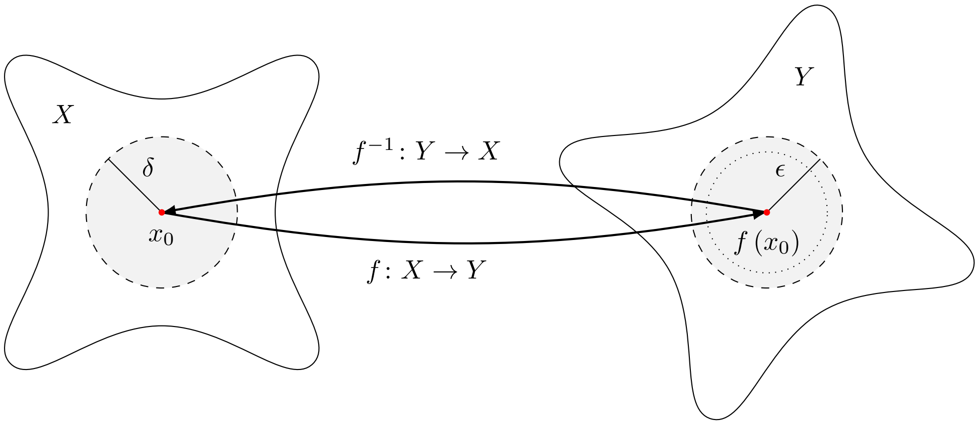

Figure 11.1: An open ball $B(x_0, \delta) \subseteq X$ with two open balls in Y such that $f(B(x_0, \delta)) \subseteq B(f(x_0), \epsilon) \subseteq Y$, marked by dotted, dashed, and solid paths respectively.

Theorem 11.1 The sequential statement of continuity follows from the topological definition

Let $x_0 \in X$ and let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x_0$. If, for any open set $V \subseteq Y$ with $f(x_0) \in V$, there exists an open set $U \subseteq X$ with $x_0 \in U$ and $f(U) \subseteq V$, then the sequence generated by $f(x_n)$ is such that $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Proof. Take an arbitrary open set $V \subseteq Y$ with $f(x_0) \in V$; thus, there exists an open set $U \subseteq X$ such that $x_0 \in U$ and $f(U) \subseteq V$. Given that U is open, there exists an open ball by theorem 5.2: $\exists \epsilon > 0$ such that $B(x_0, \epsilon) \subseteq U$. As $(x_n)_{n=1}^{\infty}$ tends to x_0, we can find an index point $N > 0$ such that $d(x_n, x_0) < \epsilon$ for all $n > N$ (by definition 8.2).

Thus, $f(x_n) \in f(U) \subseteq V$ for all $n > N$. Given that V was chosen arbitrarily, any sequence generated by $f(x_n)$ must converge to $f(x_0)$. □

Theorem 11.2 The analytical statement of continuity follows from the sequential definition

For any sequence $(x_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = x_0$, if the sequence $(f(x_n))_{n=1}^{\infty}$ is such that $\lim_{n\to\infty} f(x_n) = f(x_0)$, then given an $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f(x), f(x_0)) < \epsilon$ whenever $\widetilde{d}(x, x_0)$.

Proof. We will prove this using contraposition (modus tollens); the argument is visualised below with the premise P and conclusion Q, to argue that $\neg Q \implies \neg P$.

Premise P		Conclusion Q
For an arbitrary sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \to x_0$ as $n \to \infty$, the sequence $f(x_n) \to f(x_0)$ as $n \to \infty$.	$\implies$	For any $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f(x), f(x_0)) < \epsilon$ whenever $\widetilde{d}(x, x_0)$.

$$\Updownarrow$$

Negated Conclusion $\neg Q$		Negated Premise $\neg P$
There exists some $\epsilon > 0$ such that, for all $\delta > 0$, $\widetilde{d}(f(x), f(x_0)) \geq \epsilon$ for some $x \in X$ with $d(x, x_0) < \delta$.	$\implies$	There exists a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \to x_0$ as $n \to \infty$, but $f(x_n) \not\to f(x_0)$.

We will assume $\neg Q$ and derive $\neg P$. For each $n \in \mathbb{N}$, define the set

$$A_n := \left\{ x' \in X : d(x', x_0) < \frac{1}{n} \wedge \widetilde{d}(f(x'), f(x_0)) \geq \epsilon \right\}. \tag{11.6}$$

By the assumption of $\neg Q$, A_n is non-empty for all $n \in \mathbb{N}$. Thus, for each n, pick an element $x_n \in A_n$ and note that $\lim_{n\to\infty} x_n = x_0$, but $\lim_{n\to\infty} \widetilde{d}(f(x_n), f(x_0)) \geq \epsilon$, so the sequence generated by $f(x_n)$ does not converge to $f(x_0)$. Hence, P does not hold, as required. □

Lecture 12: Applications of Continuity

Lecture Twelve reframes much of the theoretical discussion around continuity into a more applied context, in which the equivalent definitions of a continuous map are considered in bounded and continuous function spaces. Some remarks are made in the realm of closed sets, particularly with regard to the concept of *global continuity*, where we have previously focussed on manipulations with open sets to show *local* continuity.

9th November 2023
TODO S'cd
TODO S&V

Definition 12.1 Global continuity (by way of local continuity)

Let $f : X \to Y$. Then, f is *globally continuous* if and only if it is (locally) continuous, by definition 11.4, at every point $x_0 \in X$.

Definition 12.2 Global continuity (topological perspective)

If $V \subseteq Y$ is open, then f is *globally continuous* if and only if $f^{-1}(V) \subseteq X$ is also open; this is justified formally by theorem 12.3. We can eventually restate this definition in terms of a closed set $F \subseteq Y$, justified by justified by theorem 12.4:

$$f \text{ is globally continuous } \iff V \in T_e \implies f^{-1}(V) \in T_d \tag{12.1}$$

$$\iff f^{-1}(F) \text{ is closed.} \tag{12.2}$$

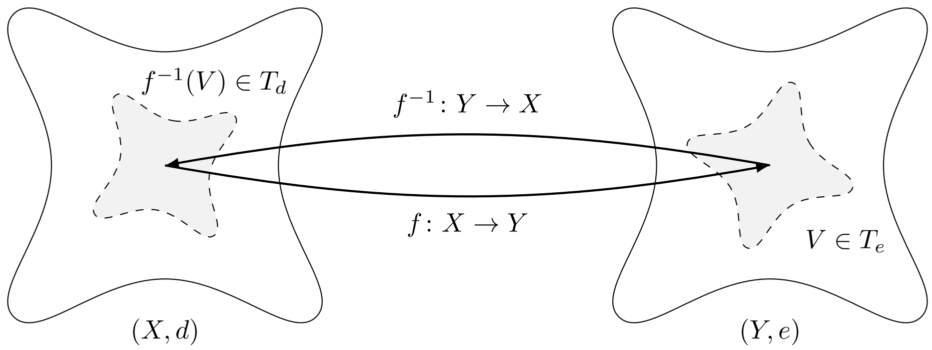

Figure 12.1: Considering the membership of the pre-image of V under f in the topology of the domain space, versus membership in the topology of the codomain space, can reveal f to be a globally continuous map.

Example 12.1 Global continuity in the plane

Consider $f \colon \mathbb{R}^2 \to \mathbb{R}$ such that $(x, y) \mapsto x^2 + e^y$ and a curve Γ such that

$$\Gamma := \left\{(x, y) \in \mathbb{R}^2 \colon f(x, y) = 1\right\}. \tag{12.3}$$

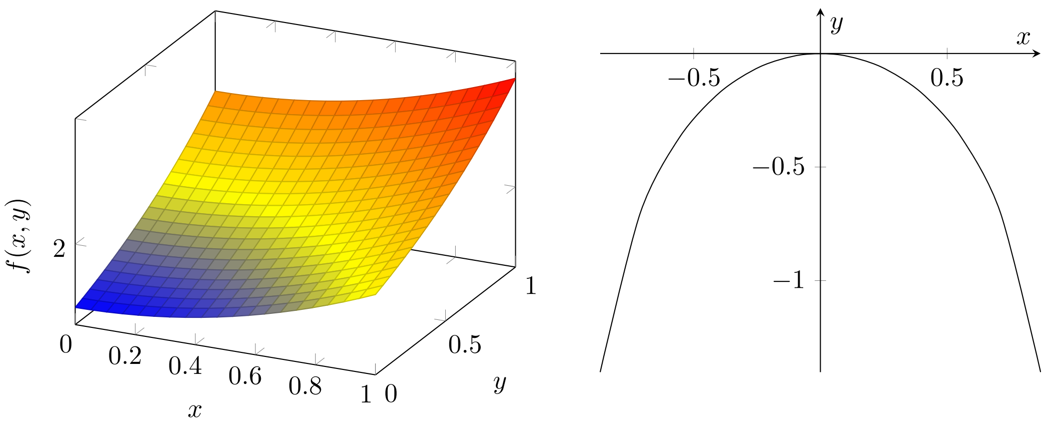

Figure 12.2: The mesh $z = f(x, y)$ with the level set Γ.

By definition 12.2 it is easy to verify that Γ is closed in $\mathbb{R}^2$: given that f is globally continuous, and given that $\{1\}$ is closed—as are all singletons—its inverse image is also closed. Since $\Gamma = f^{-1}(\{1\})$, Γ is closed.

Theorem 12.1 Constant functions are globally continuous

Let $f\colon X \to Y$ be such that $x \mapsto k$, where k is fixed. Then, f is continuous.

Proof. The singleton $\{k\}$ is known to be closed, and the entire domain space $f^{-1}(\{k\}) = X$ is known to be clopen, and thus closed. Hence, by definition 12.2, f is continuous. □

Theorem 12.2 Continuity is preserved under function composition

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be a pair of continuous functions. Then, $g \circ f\colon X \to Z$ is also continuous.

Proof. Let $V \subseteq Z$ be open. Given that f and g are continuous, $g^{-1}(V) \subseteq Y$ and $f^{-1}\left(g^{-1}(V)\right) \subseteq X$ are open in their respective spaces, by definition 12.2. It is known from set theory that

$$(g \circ f)^{-1}(V) = f^{-1}\left(g^{-1}(V)\right), \tag{12.4}$$

and hence the composition is continuous. It follows immediately that $g \circ f$ is continuous. □

Example 12.2 Continuous Composition of Non-Continuous Constituents

Consider the pair of functions f and g from $\mathbb{R}$ to $[-1, 1]$ such that

$$f(t) := \begin{cases} 1 \text{ if } t \geq 0; \\ -1 \text{ otherwise.} \end{cases} \tag{12.5}$$

$$g(t) := 0 \text{ for all } t \in \mathbb{R}. \tag{12.6}$$

f is clearly non-continuous. Despite that, $(g \circ f)(t) = 0$ for all $t \in \mathbb{R}$, which is continuous. Therefore, continuity on a composition does not generally imply continuity on the individual components.

Theorem 12.3 Global continuity is equivalent to open pre-images

Let $f\colon X \to Y$. Then, f is globally continuous if and only if, for any open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.

Proof. $\boxed{\Longrightarrow}$ **(Using fibres.)** Suppose that $f\colon X \to Y$ is globally continuous, and consider an open set $V \subseteq Y$. If $V = \emptyset$, then $f^{-1}(V) = \emptyset$, and we are done, since the empty set is open. Assuming that $V \neq \emptyset$, take $y \in V$, and note the multiple possibilities for the nature of y:

- If $y \notin f(X)$, then $F_y = \emptyset$, and we are done.
- If $y \in f(X)$, then $F_y \neq \emptyset$; that is, there exists an $x \in X$ such that $f(x) = y$. Given that V is open and f is continuous, by theorem 5.2, there exists an open ball $B \subseteq X$ such that $x \in B$ and $f(B) \subset V$. Take the union over all such balls to acquire the entire inverse image of V in X:

$$f^{-1}(V) = \bigcup_{\substack{y \in V \\ F_y \neq \emptyset}} F_y = \bigcup_{x \in f^{-1}(V)} B_x \tag{12.7}$$

By theorem 5.5 with theorem 5.3, $f^{-1}(V)$ is a union of open balls, and is therefore open, as required. (This is an example of an *open cover*.)

$\boxed{\Longleftarrow}$ **(Using epsilon–deltas.)** Conversely, suppose that $f^{-1}(V) \subseteq X$ is open for all open sets $V \subseteq V$. Take an $x \in X$, and note that $B\left(f(x), \epsilon\right)$ is open in Y, and $f^{-1}\left(B\left(f(x), \epsilon\right)\right)$ is open in X. Since $x \in f^{-1}\left(B\left(f(x), \epsilon\right)\right)$, there exists a $\delta(\epsilon) > 0$ such that $B(x, \delta) \subseteq f^{-1}\left(B\left(f(x), \epsilon\right)\right)$. Hence, f is continuous at x. Given that $x \in X$ was chosen arbitrarily, f is globally continuous. □

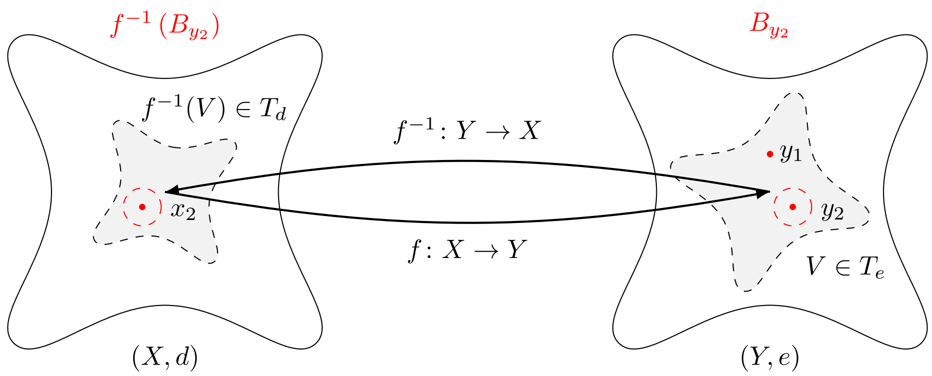

Figure 12.3: Mapping V with open balls in Y, and their respective pre-images in X.

Theorem 12.4 Global continuity is equivalent to closed pre-images

Let $f\colon X \to Y$. Then, f is globally continuous if and only if, for any closed set $F \subseteq Y$, $f^{-1}(F)$ is closed in X.

Proof. Recall from theorem 12.3 that the property of global continuity on f is equivalent to the openness of $V \subseteq Y$ implying the openness of $f^{-1}(V) \subseteq X$.

$\boxed{\Longrightarrow}$ Suppose that $F \subseteq Y$ is closed. By theorem 5.1, F^c is closed, and by standard results from set theory, for a globally continuous f,

$$F^c \text{ is closed } \implies f^{-1}(F^c) \text{ is open} \tag{12.8}$$
$$\implies f^{-1}(F) \text{ is closed.} \tag{12.9}$$

$\boxed{\Longleftarrow}$ Suppose that V is open, so V^c is closed. As above, $f^{-1}(V^c)$ is closed, and $f^{-1}(V)$ is open. ☐

Lecture 13: The Contraction Mapping Theorem

Lecture Thirteen introduces the concepts of Lipschitz constants, Lipschitz functions, (strict) contractions, and the Contraction Mapping Theorem. It also restates the existence of the spaces of bounded and continuous function, and considers a stronger notion of continuity induced by the d_∞ metric. *Note:* this lecture was unusually short.

14th November 2023
TODO
TODO

Definition 13.1 The space of continuous bounded functions

The entire space of all continuous functions from (X, d) to (Y, e) can be difficult to handle. From definition 2.3, we can consider $B(X, Y)$ to be set of all bounded functions $f\colon X \to Y$: there exists an open ball $B \subseteq Y$ such that $f(X) \subseteq B$. Now consider the *set of continuous bounded functions* from X to Y, denoted $\mathcal{C}(X, Y)$, such that

$$\mathcal{C}(X, Y) = B(X, Y) \cap C(X, Y). \tag{13.1}$$

We will normally impose the d_∞ metric on this space, such that

$$d_\infty(f, g) = \sup\{e(f(x), g(x)) : x \in X\} \tag{13.2}$$

for all $f, g \in \mathcal{C}(X, Y)$.

Definition 13.2 Uniform Convergence

Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from $(\mathcal{C}(X, \mathbb{K}), d_\infty)$, where $\mathbb{K}$ is a field ($\mathbb{R}$ or $\mathbb{C}$). The sequence $(f_n)_{n=1}^{\infty}$ is *uniformly convergent* on X with a limit $f\colon X \to \mathbb{K}$ if and only if, for every $\epsilon > 0$, there exists an $N > 0$ such that

$$d_\infty(f_n, f) \text{ for all } n > N. \tag{13.3}$$

The central point of this definition is that we are using the d_∞ metric, sometimes called the *uniform metric*, to measure distances between continuous functions from X to $\mathbb{K}$ and hence classify convergence properties.

Definition 13.3 Fixed Points

A point $x \in X$ is called a *fixed point* of the mapping $T: X \to X$ if $T(x) = x$.

Definition 13.4 Lipschitz Function and Lipschitz Constants

Suppose that (X, d) and (Y, e) are metric spaces, and that $f: X \to Y$. If there exists a $k > 0$ such that

$$e(f(a), f(b)) \leq kd(a, b) \text{ for all } a, b \in X, \tag{13.4}$$

then f is called a *Lipschitz function* on X with a *Lipschitz constant* of k.

Definition 13.5 Contractions

A *strict contraction*, henceforth called a *contraction*, is a Lipschitz function for which the Lipschitz constant k is such that $0 < k < 1$. In such cases, the Lipschitz constant of the contraction map is sometimes called the *contraction factor*.

Theorem 13.1 The Contraction Mapping Theorem

Let (X, d) be a complete metric space and consider a contraction $f: X \to X$. Then,

- f has a unique fixed point $y \in X$; and
- for any $x_0 \in X$, the sequence $(x_n)_{n=1}^{\infty}$ converges to y, where $x_n = f(x_{n-1})$ for $n \geq 1$.

This result is due to Stefan Banach[a], and is often called the Banach Fixed Point Theorem.

Proof. TODO: see lecture XIV. □

[a] 1892–1945; responsible for the discovery of Functional Analysis

Recommended Texts & Further Reading

[OSe07] Mícheál O'Searcoid. *Metric Spaces*. Springer Undergraduate Mathematics Series. **Core module text**. London, United Kingdom: Springer, 2007. ISBN: 978-1-846283697.

[SV06] Satish Shirali and Harkrishan Lal Vasudeva. *Metric Spaces*. **Core module text**. London, United Kingdom: Springer, 2006. ISBN: 978-1-852339227.

[Hut81] John Hutchinson. "Fractals and Self Similarity". In: *Indiana University Mathematics Journal* 30.5 (1981). **For interest only**, pp. 713–747. ISSN: 0022-2518, 1943-5258. URL: https://www.jstor.org/stable/24893080.

[Bee13] Gerald Beer. "The Structure of Extended Real-Valued Metric Spaces". In: *Set-Valued and Variational Analysis* 21 (2013). **For interest only**, pp. 591–602. ISSN: 1877-0541. DOI: 10.1007/s11228-013-0255-2.

[Kop88] Ralph Kopperman. "All Topologies Come From Generalized Metrics". In: *The American Mathematical Monthly* 95.2 (1988). **For interest only**, pp. 89–97. ISSN: 0002-9890, 1930-0972. URL: https://www.jstor.org/stable/2323060.