An Investigation of Elementary Category Theory

. . . with Selected Applications in PM and TCS.

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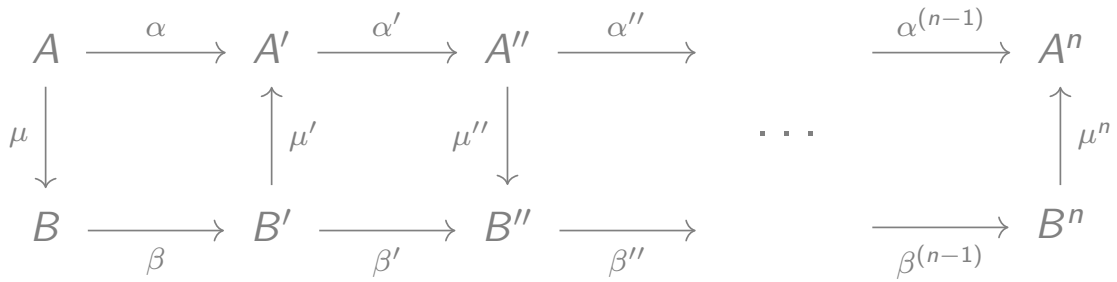

Presentation Overview

The Intuition: What are Categories? (1)

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- The power of categories, and the associated theories, lie in their generality: objects can be any mathematically definable structure, and morphisms may be any process or algorithm by which you move from one object to another.
- At first glance, morphisms appear indistinguishable from functions. Whilst traditional sets and their functions certainly do form a category, as we will see later, the expanse of Category Theory extends far beyond naive Set Theory!

The Intuition: What are Categories? (2)

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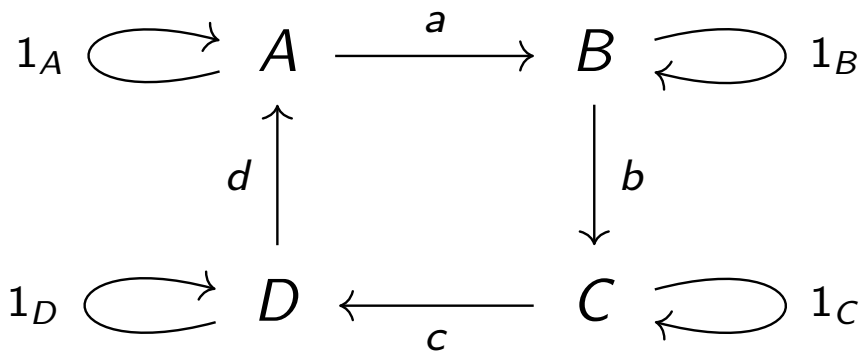

Figure 1.1: A commutative diagram for a simple, four-object category.

Identity morphisms are shown explicitly here, however they are usually omitted to maintain brevity.

The Definition: The Axioms of a Category

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- The composition of morphisms, where applicable, must be an associative operation.
- The usual unit law must hold: $f \circ 1_A = f = 1_B \circ f$ for all $f : A \to B$.

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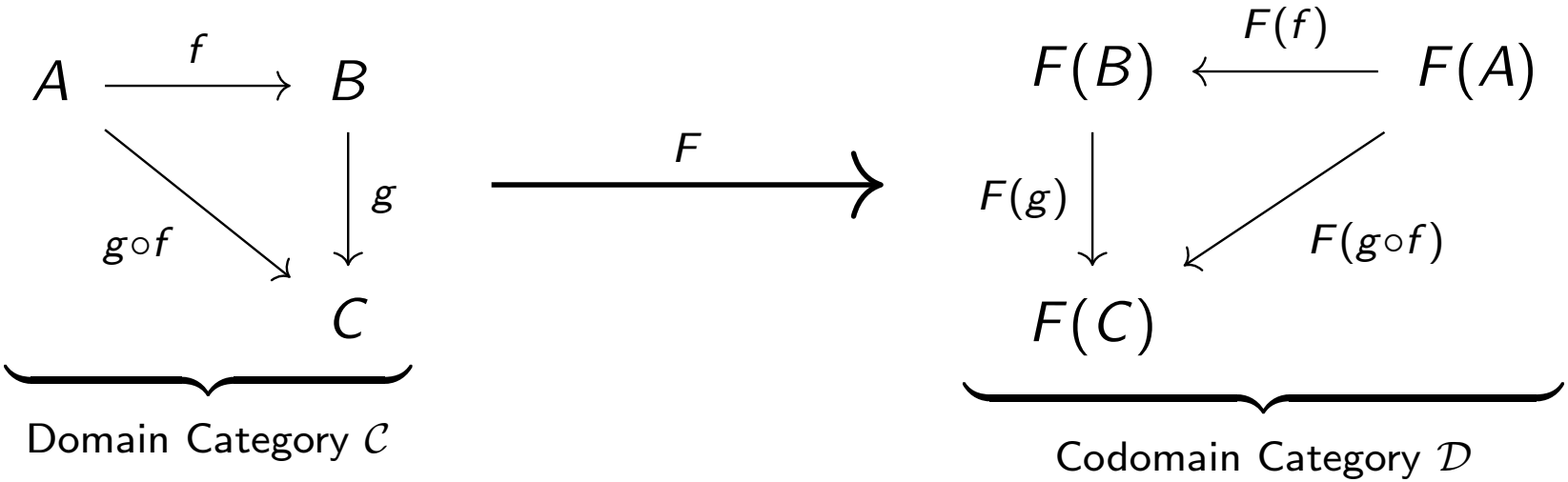
A pair of connected commutative diagrams can pictorially demonstrate the role of a functor $F : \mathcal{C} \to \mathcal{D}$:



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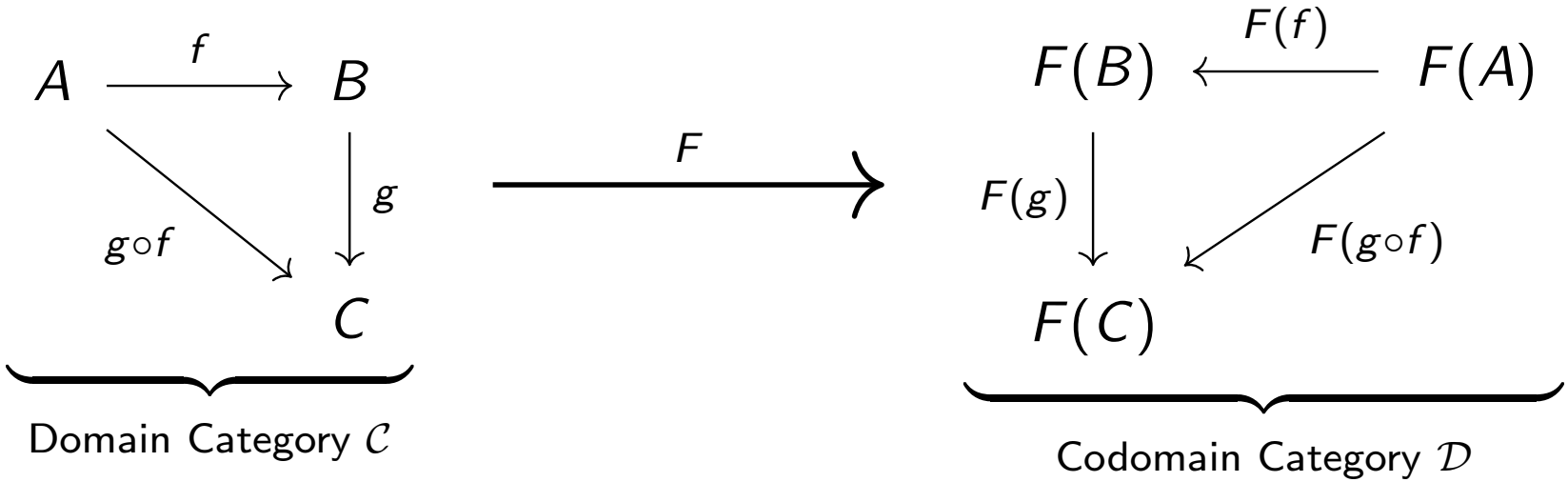

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- An *endofunctor* is a functor whose domain and codomain is the same. This concept is simple, but it will become very important shortly!

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- The objects of $\mathcal{C} \times \mathcal{D}$ have the form (c, d) for $c \in \operatorname{ob}\mathcal{C}$ and $d \in \operatorname{ob}\mathcal{D}$. Morphisms have the form $(f, g) \colon (c, d) \to (c', d')$, where $f : c \to c' \in \operatorname{ob}\mathcal{C}$ and $g : d \to d' \in \operatorname{ob}\mathcal{D}$.

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- $\mathcal{C} \times \mathcal{D}$ can be easily shown to satisfy the category axioms; hence, $\times$ is capable of producing a new category.
- We also have two obvious *projection functors* $\pi_{\mathcal{C}}$ and $\pi_{\mathcal{D}}$ from the product to the original constituent categories:

$$\mathcal{C} \xleftarrow[\pi_{\mathcal{C}}]{} \mathcal{C} \times \mathcal{D} \xrightarrow[\pi_{\mathcal{D}}]{} \mathcal{D}$$

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- The formal definition of natural transformations is slightly technical, however, for intuition, it suffices to think of natural transformations as morphisms between functors in $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$.
- As usual, these morphisms meet certain coherence conditions; most of the time these are trivially satisfiable.

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- Let A, B, C, D be sets, and let f, g, h be (set-theoretic) functions such that $f\colon A \to B$, $g\colon B \to C$, and $h\colon C \to D$.

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- Hence, the category axioms are satisfied, and there is a category whose objects are sets and morphisms are the functions between those sets. We'll denote this category **Set** moving forwards.

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- Analogous connexions can be drawn for sets equipped with preording relations (**Ord**), pointed sets (**Based**), etc.

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$$f(\vec{u}+\vec{v}) = f(\vec{u}) + f(\vec{v}); \qquad \textbf{[Additivity]}$$
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  where $\vec{u}, \vec{v} \in K^n$ and $\lambda \in K$.
- This only applies to *finite-dimensional* vector spaces. Banach spaces and Functional Analysis are out of the scope of this talk! ($n \in \mathbb{N}$)

The Category of Deduction and Deducability

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- A richer category is the Category of Deduction, **Deduce**. For every pair of objects, a corresponding morphism in the homset represents a *specific manner of deduction* between its domain and codomain. Deductive cancelling is equivalent to morphism composition!

The True Isomorphism! (1)

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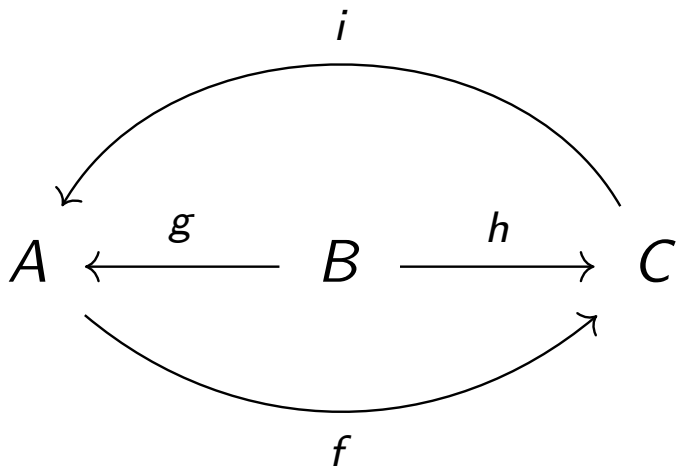

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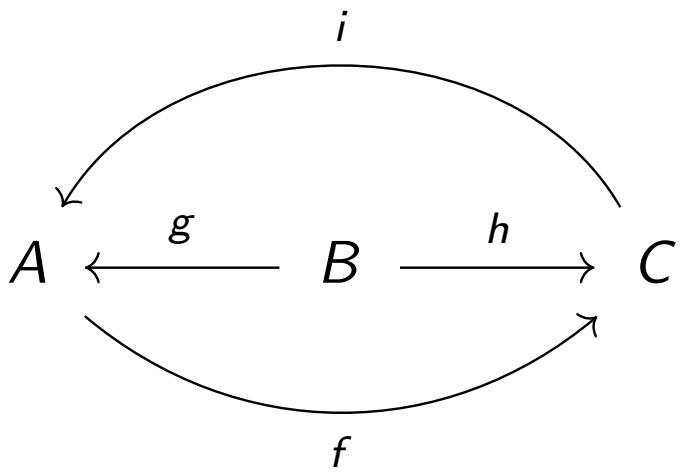

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Isomorphisms are easy to spot with well-drawn commutative diagrams!

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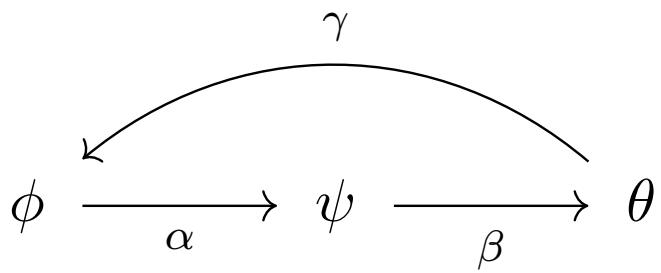

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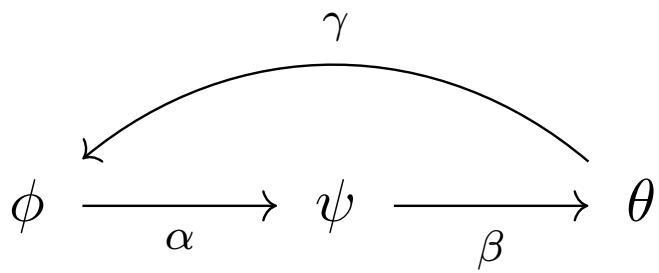

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Using deductive cancelling, α and β can be forged into a single morphism, with ψ being embedded into the environmental axioms of ϕ. Given our knowledge of the nature of ob (**Deduce**), we know that

$$(\phi \vdash \theta) \wedge (\theta \vdash \phi) \iff \phi \equiv \theta \iff \phi \cong \theta.$$

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Then, $\mathcal{C} = (\mathcal{C}_0, \otimes, I, \alpha, \lambda, \rho)$, or just $\mathcal{C} = (\mathcal{C}_0, \otimes, I)$.

Monoidal Categories (2)

$$\begin{array}{ccccc}
[(A \otimes B) \otimes C] \otimes D & \xrightarrow{\alpha} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha} & A \otimes [B \otimes (C \otimes D)] \\
{\scriptstyle \alpha \otimes \mathrm{id}\, D}\downarrow & & & & \uparrow{\scriptstyle \mathrm{id}\, A \otimes \alpha} \\
[A \otimes (B \otimes C)] \otimes D & & \xrightarrow[\alpha]{} & & A \otimes [(B \otimes C) \otimes D]
\end{array}$$

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$$\begin{array}{ccc}
(B \otimes I) \otimes C & \xrightarrow{\alpha} & B \otimes (I \otimes C) \\
& {\scriptstyle \rho \otimes \mathrm{id}\, C} \searrow \quad \swarrow {\scriptstyle \mathrm{id}\, B \otimes \lambda} & \\
& B \otimes C &
\end{array}$$

Figure 3.2: Left- and right-identities induced by λ and ρ

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A monoid (M, μ, η) is composed of:

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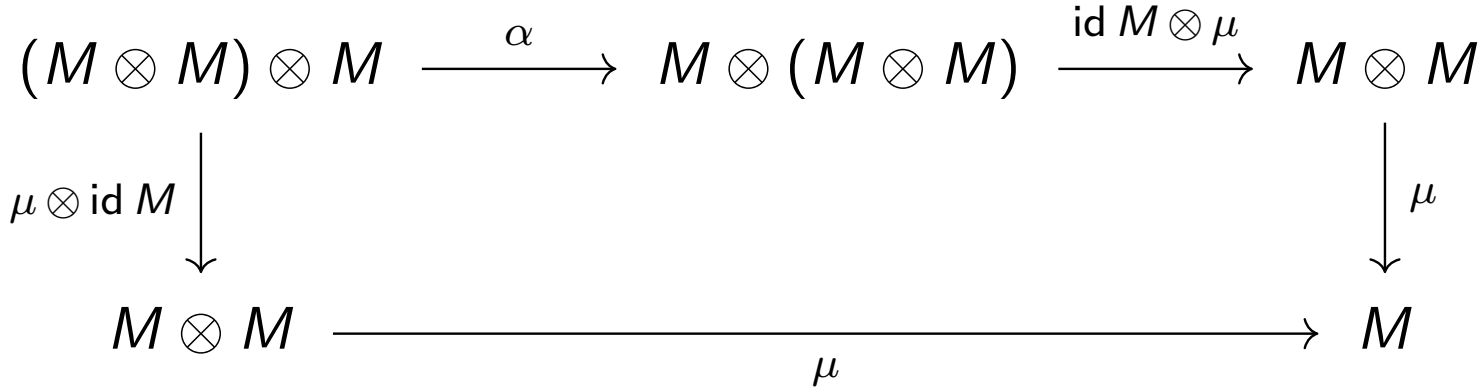

Figure 3.3: Associativity of the monoid

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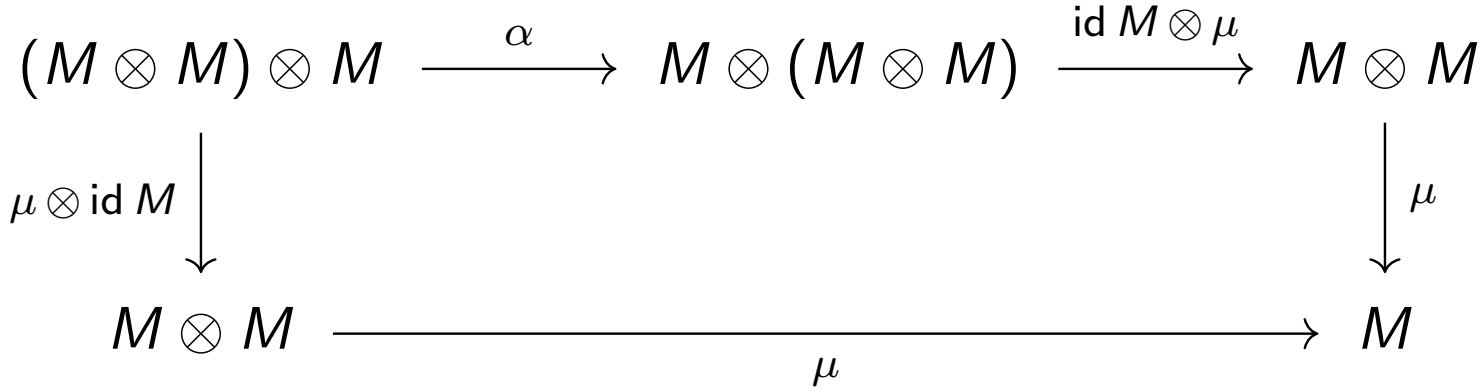

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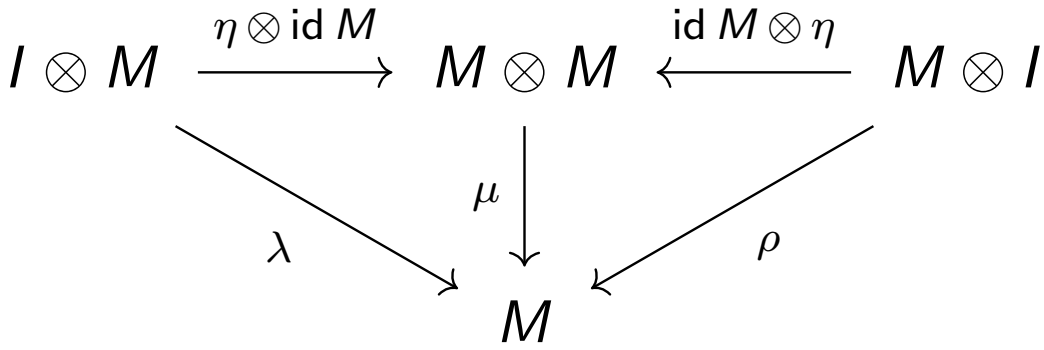

Figure 3.4: Left- and right-identities of the monoid

Functional Programming Languages

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- How can monoids and monodial categories help us with printing "Hello World"?

An Especially Useful Monoidal Category

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- A *monad* is a monoid in this category.
- We can specify the base category $\mathcal{C}_0$ with $\mathbf{Endo}\,(\mathcal{C}_0)$.

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- Thus, endofunctors on **Hask** with natural transformations μ and η form monads.

In Closing: Practical Notes

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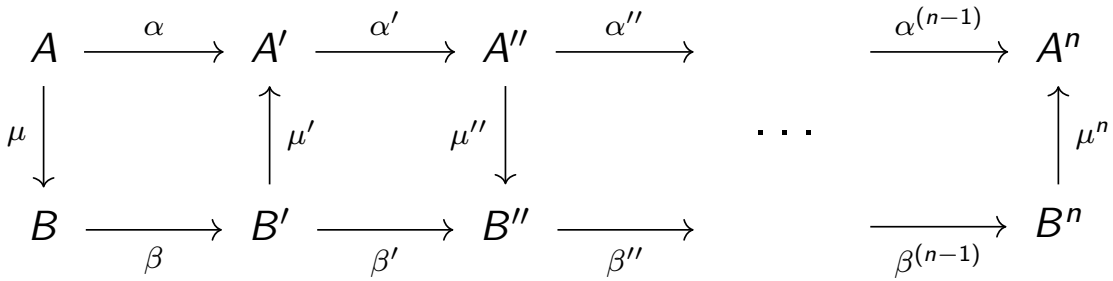
- In practice, monads allow stateful calculation. Unlike the mindless emulation of imperative languages, this is achieved without breaking function purity. This can be seen intuitively as *threading state though a chain of functions*.

Any Questions?



"*The Categorical Snake of Morphistic Infinitude*"