Theory 3: Practical Week 8

Introduction

In the practical, we will be looking at natural deduction proofs in the propositional calculus. First, we remind you of the symbols, and various deduction laws that are available. Remember that the numeric subscripted versions are combined into a single rule in Carnap.

Connective	Keyboard
→	`->`, `=>`, `>`
∧	`/\`, `&`, `and`
∨	`\/`, `|`, `or`
↔	`<->`, `<=>`
¬	`-`, `~`, `not`
⊥	`!?`, `_|_`

Rule	Abbreviation
Conditional-Intro.	`→I`
Conditional-Elim. (Modus Ponens)	`→E`
Not-Intro (By Contradiction)	`¬I`
Not-Elim (Indirect Proof)	`¬E`
And-Intro.	`∧I`
And-Elim.	`∧E`
Or-Intro	`∨I`
Or-Elim (Case Analysis)	`∨E`

Rule	Abbreviation
Biconditional-Intro	`↔I`
Biconditional-Elim	`↔E`
Reiteration	`R`
Modus Tollens	`MT`
Double Negation Elim.	`¬¬E`
DeMorgan's Laws	`DeM`
Law of Excluded Middle	`LEM`

Be aware that the bottom elimination law `⊥E` is called `X` in Carnap. It does the same thing: starting from `⊥` we can prove any proposition. Moreover, sometimes in proofs we need to repeat a previous statement, which is called a *reiteration*. This can be done using the rule `R` and citing the previous line. This is needed when we have a subproof in Carnap whose conclusion is the same as the assumption.

Basic Proofs

Proofs in Carnap are nearly identical to those considered in the lectures, but there are a couple of differences. Each proof line has a colon followed by the name of the proof rule that is used. The rule `PR` is used to introduce premises. All the other rules have the same names as those shown in the lectures, though there are no subscripted versions. For example, Carnap does not distinguish `∧E1` and `∧E2`, which are both contained in the same law `∧E`; similarly for `∨I` and `DeM`. We can use the same ASCII-art like notation in rule names.

Try and prove the properties below, some of which were considered already in the problem classes. Each Carnap proof needs to have the same premises and conclusion as the specified argument. If you get parse errors, try adding brackets according to the precedences of the propositional calculus. The Carnap parser appears to be quite conservative in handling precedences.

1.1

$((P \land Q) \to R), P, Q \vdash R$ ✓

```
1. (P & Q) → R :PR          +
2. P :PR                    +
3. Q :PR                    +
4. P & Q :&I 2, 3           +
5. R :→E 4, 1               +
```

1.	$((P \land Q) \to R)$	PR
2.	P	PR
3.	Q	PR
4.	$(P \land Q)$	∧I 2, 3
5.	R	→E 4, 1

1.2

$((P \land Q) \to R), \neg R \vdash (\neg P \lor \neg Q)$ ✓

```
1. (P & Q) → R :PR          +
2. ~R :PR                   +
3. ~(P & Q) :MT 1, 2        +
4. ~P | ~Q :DeM 3           +
```

1.	$((P \land Q) \to R)$	PR
2.	$\neg R$	PR
3.	$\neg(P \land Q)$	MT 1, 2
4.	$(\neg P \lor \neg Q)$	DeM 3

1.3

$((P \wedge Q) \wedge R), (S \wedge T) \vdash (Q \wedge S)$ ✓

```
1. (P & Q) & R :PR      +
2. S & T :PR            +
3. P & Q :&E 1          +
4. Q :&E 3              +
5. S :&E 2              +
6. Q & S :&I 4, 5       +
```

1.	$((P \wedge Q) \wedge R)$	PR
2.	$(S \wedge T)$	PR
3.	$(P \wedge Q)$	$\wedge$E 1
4.	Q	$\wedge$E 3
5.	S	$\wedge$E 2
6.	$(Q \wedge S)$	$\wedge$I 4, 5

Subproofs

Subproofs in Carnap are given by indenting the proof lines where the subproof occurs. Assumptions are introduced with the rule `AS`, which functions in a similar way to `PR`. If multiple subproofs are required, for example in a case analysis proof with `∨E`, the subproofs should be separated with two dashes "--", on a reduced indent. Subproofs can also be nested by increasing the indent.

Below is an example proof from the previous problem class:

2.1

(P → Q) ⊢ (¬Q → ¬P) ✓

```
1. P → Q :PR
2.   ~Q :AS
3.   ~P :MT 1,2
4. ~Q → ~P :→I 2-3
```

1.	(P → Q)	PR
2.	¬Q	AS
3.	¬P	MT 1, 2
4.	(¬Q → ¬P)	→I 2-3

The subproof is on lines 2-3, and so each of the lines is indented by 2 spaces. Carnap preserves the indentation when you hit enter. Another example is below, this time using a separator between two subproofs. This one effectively has no premises, which is indicated by the `⊤` formula.

2.2

⊤ ⊢ ((A ∨ B) → (B ∨ A)) ✓

```
1.   A ∨ B:AS
2.     A:AS
3.     B ∨ A:∨I 2
4.   --
5.     B:AS
6.     B ∨ A:∨I 5
7.   B ∨ A:∨E 1,2-3,5-6
8. (A ∨ B) → (B ∨ A):→I 1-7
```

1.	(A ∨ B)	AS
2.	A	AS
3.	(B ∨ A)	∨I 2
5.	B	AS
6.	(B ∨ A)	∨I 5
7.	(B ∨ A)	∨E 1, 2-3, 5-6
8.	((A ∨ B) → (B ∨ A))	→I 1-7

Now complete the proofs below.

2.3

(P → (Q → R)) ⊢ ((P ∧ Q) → R) ✓

```
1. P → (Q → R) :PR                +
2.     P & Q :AS                  +
3.     P :&E 2                    +
4.     Q :&E 2                    +
5.     Q → R :→E 1, 3             +
6.     R :→E 5, 4                 +
7. (P & Q) → R :→I 2-6            +
```

1.	(P → (Q → R))	PR
2.	│ (P ∧ Q)	AS
3.	│ P	∧E 2
4.	│ Q	∧E 2
5.	│ (Q → R)	→E 1, 3
6.	│ R	→E 5, 4
7.	((P ∧ Q) → R)	→I 2-6

2.4

(P → Q), (P → ¬Q) ⊢ ¬P ✓

```
1. P → Q :PR                      +
2. P → ~Q :PR                     +
3.     P :AS                      +
4.     Q :→E 1, 3                 +
5.     ~Q :→E 2, 3                +
6.     !? :~E 4, 5                +
7. ~P :~I 3-6                     +
```

1.	(P → Q)	PR
2.	(P → ¬Q)	PR
3.	│ P	AS
4.	│ Q	→E 1, 3
5.	│ ¬Q	→E 2, 3
6.	│ ⊥	¬E 4, 5
7.	¬P	¬I 3-6

2.5

$((P \wedge Q) \rightarrow R) \vdash (P \rightarrow (Q \rightarrow R))$ ✓

```
1. (P & Q) → R :PR              +
2.     P :AS                    +
3.         Q :AS                +
4.         P & Q :&I 2, 3       +
5.         R :→E 4, 1           +
6.     Q → R :→I 3-5            +
7. P → (Q → R) :→I 2-6          +
```

1.	$((P \wedge Q) \rightarrow R)$	PR
2.	│ P	AS
3.	│ │ Q	AS
4.	│ │ $(P \wedge Q)$	∧I 2, 3
5.	│ │ R	→E 4, 1
6.	│ $(Q \rightarrow R)$	→I 3-5
7.	$(P \rightarrow (Q \rightarrow R))$	→I 2-6

Knights and Knaves

Provide proofs to support validity of the following arguments, each of which is a knights and knaves problem, that we've considered already.

A: "We are both knaves."

3.1

$(A \leftrightarrow (\neg A \land \neg B)) \vdash (\neg A \land B)$ ✓

```
 1. A ⟷ (~A & ~B) :PR                 +
 2.     A :AS                         +
 3.     ~A & ~B :⟷E 1, 2              +
 4.     ~A :&E 3                      +
 5.     !? :~E 2, 4                   +
 6. ~A :~I 2-5                        +
 7.     ~B :AS                        +
 8.     ~A & ~B :&I 6, 7              +
 9.     A :⟷E 1, 8                    +
10.     !? :~E 6, 9                   +
11. ~~B :~I 7-10                      +
12. B :~~E 11                         +
13. ~A & B :&I 6, 12                  +
```

1.	$(A \leftrightarrow (\neg A \land \neg B))$	PR
2.	$\quad A$	AS
3.	$\quad (\neg A \land \neg B)$	$\leftrightarrow$E 1, 2
4.	$\quad \neg A$	$\land$E 3
5.	$\quad \bot$	$\neg$E 2, 4
6.	$\neg A$	$\neg$I 2-5
7.	$\quad \neg B$	AS
8.	$\quad (\neg A \land \neg B)$	$\land$I 6, 7
9.	$\quad A$	$\leftrightarrow$E 1, 8
10.	$\quad \bot$	$\neg$E 6, 9
11.	$\neg\neg B$	$\neg$I 7-10
12.	B	DNE 11
13.	$(\neg A \land B)$	$\land$I 6, 12

C: "There are no knaves here.", D: "Don't believe C, he's lying!"

3.2

```
(C ↔ (C ∧ D)), (C ↔ ¬D) ⊢ (¬C ∧ D)   ✓

 1. C ↔ (C & D) :PR           +
 2. C ↔ ~D :PR                +
 3.     C :AS                 +
 4.     C & D :↔E 1, 3        +
 5.     D :&E 4               +
 6.     ~D :↔E 2, 3           +
 7.     !? :~E 5, 6           +
 8. ~C :~I 3-7                +
 9.     ~D :AS                +
10.     C :↔E 2, 9            +
11.     !? :~E 8, 10          +
12. ~~D :~I 9-11              +
13. D :~~E 12                 +
14. ~C & D :&I 8, 13          +
```

1.	(C ↔ (C ∧ D))	PR
2.	(C ↔ ¬D)	PR
3.	C	AS
4.	(C ∧ D)	↔E 1, 3
5.	D	∧E 4
6.	¬D	↔E 2, 3
7.	⊥	¬E 5, 6
8.	¬C	¬I 3-7
9.	¬D	AS
10.	C	↔E 2, 9
11.	⊥	¬E 8, 10
12.	¬¬D	¬I 9-11
13.	D	DNE 12
14.	(¬C ∧ D)	∧I 8, 13

Case Analysis Proofs

The next proofs use case analysis, which requires us to delineate the two subproofs with "--" on a new line with a dropped indentation. Case analysis proofs are performed using the rules `∨E` and `LEM` (Law of Excluded Middle). `LEM` in Carnap is different to the one in LiCS, as it requires two subproofs of the form `ϕ ... χ` and `¬ϕ ... χ`, i.e. a case analysis on `ϕ` to prove a conclusion `χ`. This is equivalent to the `LEM` in LiCS of the form `ϕ ∨ ¬ϕ`.

The "Labyrinth" door puzzle. L: "R would say that D leads to the castle".

4.1

```
(L ⇔ ¬R), (L ⇔ (R ⇔ D)) ⊢ ¬D                                  ✓

 1. L  ⟷  ~R :PR                                               +
 2. L  ⟷  (R  ⟷  D) :PR                                        +
 3.     L :AS                                                  +
 4.     ~R :⟷E 1, 3                                            +
 5.     R  ⟷  D :⟷E 2, 3                                       +
 6.     ~D :BMT 5, 4                                           +
 7. --
 8.     ~L :AS                                                 +
 9.     ~~R :BMT 1,8                                           +
10.     R :~~E 9                                               +
11.     ~(R  ⟷  D) :BMT 2, 8                                   +
12.     R  ⟷  ~D :NBD 11                                       +
13.     ~D :⟷E 12, 10                                          +
14. ~D :LEM 3-6, 8-13                                          +
```

1.	$(L \leftrightarrow \neg R)$	PR
2.	$(L \leftrightarrow (R \leftrightarrow D))$	PR
3.	L	AS
4.	$\neg R$	$\leftrightarrow$E 1, 3
5.	$(R \leftrightarrow D)$	$\leftrightarrow$E 2, 3
6.	$\neg D$	BMT 5, 4
8.	$\neg L$	AS
9.	$\neg\neg R$	BMT 1, 8
10.	R	DNE 9
11.	$\neg(R \leftrightarrow D)$	BMT 2, 8
12.	$(R \leftrightarrow \neg D)$	NBD 11
13.	$\neg D$	$\leftrightarrow$E 12, 10
14.	$\neg D$	LEM 3-6, 8-13

"If Smith has installed central heating, then he has sold his car or he has not paid his mortgage. Smith has installed central heating. He has not sold his car. Therefore, he must not have paid his mortgage."

4.2

(P → (Q ∨ ¬R)), P, ¬Q ⊢ ¬R ✓

```
 1. P → (Q | ~R) :PR                         +
 2. P :PR                                    +
 3. ~Q :PR                                   +
 4. Q | ~R :→E 1, 2                          +
 5.     Q :AS                                +
 6.     !? :~E 3, 5                          +
 7.     ~R :X 6                              +
 8. --
 9.     ~R :AS                               +
10.     ~R :R 9                              +
11. ~R :|E 4, 5-7, 9-10                      +
```

1.	(P → (Q ∨ ¬R))	PR
2.	P	PR
3.	¬Q	PR
4.	(Q ∨ ¬R)	→E 1, 2
5.	Q	AS
6.	⊥	¬E 3, 5
7.	¬R	⊥E 6
9.	¬R	AS
10.	¬R	R 9
11.	¬R	∨E 4, 5-7, 9-10

"If Dick met Jane yesterday, they had a cup of coffee together, or they took a walk in the park. Dick neither had a cup of coffee with Jane, nor took a walk in the park. Therefore Dick did not meet Jane yesterday."

4.3

$(P \to (Q \lor R)), \neg Q, \neg R \vdash \neg P$ ✓

```
1. P → (Q | R) :PR            +
2. ~Q :PR                      +
3. ~R :PR                      +
4. ~Q & ~R :&I 2, 3            +
5. ~(Q | R) :DeM 4             +
6. ~P :MT 1, 5                 +
```

1.	$(P \to (Q \lor R))$	PR
2.	$\neg Q$	PR
3.	$\neg R$	PR
4.	$(\neg Q \land \neg R)$	∧I 2, 3
5.	$\neg(Q \lor R)$	DeM 4
6.	$\neg P$	MT 1, 5

F: "At least one us is a knave."

4.4

(F ↔ (¬E ∨ ¬F)) ⊢ (¬E ∧ F) ✓

```
 1. F ⟷ (~E | ~F) :PR                     +
 2.     ~F :AS                             +
 3.     ~(~E | ~F) :BMT 1, 2               +
 4.     ~~E & ~~F :DeM 3                   +
 5.     ~~F :&E 4                          +
 6.     F :~~E 5                           +
 7.     !? :~E 2, 6                        +
 8. ~~F :~I 2-7                            +
 9. F :~~E 8                               +
10. ~E | ~F :⟷E 1, 9                       +
11.     ~E :AS                             +
12.     ~E & F :&I 11, 9                   +
13. --
14.     ~F :AS                             +
15.     !? :~E 9, 14                       +
16.     ~E & F :X 15                       +
17. ~E & F :|E 10, 11-12, 14-16            +
```

1.	$(F \leftrightarrow (\neg E \lor \neg F))$	PR
2.	│ $\neg F$	AS
3.	│ $\neg(\neg E \lor \neg F)$	BMT 1, 2
4.	│ $(\neg\neg E \land \neg\neg F)$	DeM 3
5.	│ $\neg\neg F$	∧E 4
6.	│ F	DNE 5
7.	│ $\bot$	¬E 2, 6
8.	$\neg\neg F$	¬I 2-7
9.	F	DNE 8
10.	$(\neg E \lor \neg F)$	↔E 1, 9
11.	│ $\neg E$	AS
12.	│ $(\neg E \land F)$	∧I 11, 9
14.	│ $\neg F$	AS
15.	│ $\bot$	¬E 9, 14
16.	│ $(\neg E \land F)$	⊥E 15
17.	$(\neg E \land F)$	∨E 10, 11-12, 14-16

Derived Laws

The following properties correspond to derived laws of the propositional calculus. Prove each of them using **only** the core laws, namely: `∧I`, `∧E`, `∨I`, `∨E`, `→I`, `→E`, `¬I`, `¬E`, `¬¬E`, `⊥E`.

5.1

(P → Q), ¬Q ⊢ ¬P ✓

```
1. P → Q :PR          +
2. ~Q :PR             +
3.     P :AS          +
4.     Q :→E 1, 3     +
5.     !? :~E 2, 4    +
6. ~P :~I 3-5         +
```

1.	(P → Q)	PR
2.	¬Q	PR
3.	P	AS
4.	Q	→E 1, 3
5.	⊥	¬E 2, 4
6.	¬P	¬I 3-5

5.2

¬(P ∨ Q) ⊢ (¬P ∧ ¬Q) ✓

```
 1. ~(P | Q) :PR          +
 2.     P :AS             +
 3.     P | Q :|I 2       +
 4.     !? :~E 1, 3       +
 5. --
 6.     Q :AS             +
 7.     P | Q :|I 6       +
 8.     !? :~E 1, 7       +
 9. ~P :~I 2-4            +
10. ~Q :~I 6-8            +
11. ~P & ~Q :&I 9, 10     +
```

1.	$\neg(P \lor Q)$	PR
2.	| P	AS
3.	| $(P \lor Q)$	∨I 2
4.	| $\bot$	¬E 1, 3
6.	| Q	AS
7.	| $(P \lor Q)$	∨I 6
8.	| $\bot$	¬E 1, 7
9.	$\neg P$	¬I 2-4
10.	$\neg Q$	¬I 6-8
11.	$(\neg P \land \neg Q)$	∧I 9, 10

5.3

¬(P ∧ Q) ⊢ (¬P ∨ ¬Q) ✓

```
 1. ~(P & Q) :PR                    +
 2.     P :AS                       +
 3.         Q :AS                   +
 4.         P & Q :&I 2, 3          +
 5.         !? :~E 1, 4             +
 6.     ~Q :~I 3-5                  +
 7. --
 8.     Q :AS                       +
 9.         P :AS                   +
10.         P & Q :&I 8, 9          +
11.         !? :~E 1, 10            +
12.     ~P :~I 9-11                 +
13. P → ~Q :→I 2-6                  +
14.     P :AS                       +
15.     ~Q :→E 13, 14               +
16.     ~P | ~Q :|I 15              +
17. --
18.     ~P :AS                      +
19.     ~P | ~Q :|I 18              +
20. ~P | ~Q :LEM 14-16 18-19        +
```

1.	¬(P ∧ Q)	PR
2.	P	AS
3.	Q	AS
4.	(P ∧ Q)	∧I 2, 3
5.	⊥	¬E 1, 4
6.	¬Q	¬I 3-5
8.	Q	AS
9.	P	AS
10.	(P ∧ Q)	∧I 8, 9
11.	⊥	¬E 1, 10
12.	¬P	¬I 9-11
13.	(P → ¬Q)	→I 2-6
14.	P	AS
15.	¬Q	→E 13, 14
16.	(¬P ∨ ¬Q)	∨I 15
18.	¬P	AS
19.	(¬P ∨ ¬Q)	∨I 18
20.	(¬P ∨ ¬Q)	LEM 14-16, 18-19

5.4

(¬P → ⊥) ⊢ P ✓

```
1. ~P → !? :PR              +
2.     ~P :AS               +
3.     !? :→E 1, 2          +
4. ~~P :~I 2-3              +
5. P :~~E 4                 +
```

1.	$(\neg P \rightarrow \bot)$	PR
2.	$\quad \neg P$	AS
3.	$\quad \bot$	→E 1, 2
4.	$\neg\neg P$	¬I 2-3
5.	P	DNE 4

5.5

```
(P → Q), (¬P → Q) ⊢ Q                                        ✓

  1. P → Q :PR                                              +
  2. ~P → Q :PR                                             +
  3.     ~Q :AS                                             +
  4.         P :AS                                          +
  5.         Q :→E 1, 4                                     +
  6.         !? :~E 3, 5                                    +
  7.     ~P :~I 4-6                                         +
  8.     Q :→E 2, 7                                         +
  9.     !? :~E 3, 8                                        +
 10. ~~Q :~I 3-9                                            +
 11. Q :~~E 10                                              +
```

1.	$(P \rightarrow Q)$	PR
2.	$(\neg P \rightarrow Q)$	PR
3.	$\quad \neg Q$	AS
4.	$\quad\quad P$	AS
5.	$\quad\quad Q$	$\rightarrow$E 1, 4
6.	$\quad\quad \bot$	$\neg$E 3, 5
7.	$\quad \neg P$	$\neg$I 4-6
8.	$\quad Q$	$\rightarrow$E 2, 7
9.	$\quad \bot$	$\neg$E 3, 8
10.	$\neg\neg Q$	$\neg$I 3-9
11.	Q	DNE 10

5.6

(P → Q) ⊢ (¬P ∨ Q) ✓

```
1. P → Q :PR                    +
2.     ~P :AS                   +
3.     ~P | Q :|I 2             +
4. --
5.     P :AS                    +
6.     Q :→E 1 5                +
7.     ~P | Q :|I 6             +
8. ~P | Q :LEM 2-3 5-7          +
```

1.	(P → Q)	PR
2.	¬P	AS
3.	(¬P ∨ Q)	∨I 2
5.	P	AS
6.	Q	→E 1, 5
7.	(¬P ∨ Q)	∨I 6
8.	(¬P ∨ Q)	LEM 2-3, 5-7

5.7

(¬P ∨ Q) ⊢ (P → Q) ✓

```
1. ~P | Q :PR                  +
2.     P :AS                   +
3.         ~P :AS              +
4.         !? :~E 2 3          +
5.         Q :X 4              +
6.     --
7.         Q :AS               +
8.         Q :R 7              +
9.     Q :|E 1 3-5 7-8         +
10. P → Q :→I 2-9              +
```

1.	(¬P ∨ Q)	PR
2.	P	AS
3.	¬P	AS
4.	⊥	¬E 2, 3
5.	Q	⊥E 4
7.	Q	AS
8.	Q	R 7
9.	Q	∨E 1, 3-5, 7-8
10.	(P → Q)	→I 2-9

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