Logic: Deduction Rules

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Unless stated otherwise, sentences in the propositional calculus are denoted by ϕ, ψ, and χ. The logical bottom is denoted by $\bot$, and T indicates an 'empty set' of premises that universally hold true. Logical name labels are set in a sans serif typeface (e.g. MT for modus tollens). Boxed premises represents sub-proofs.

In the predicate calculus, predicates are represented by capital Latin or lowercase Greek letters, using parentheses for constant application and brackets for substitution of a hypothetical individual; e.g. $P(a)$, or $\phi[x \mapsto x_0]$. Hoare triples are composed of the precondition, followed by the (partial) program/program name, followed by the postcondition; e.g. $\{x > 0\}\, x \coloneqq x + 1\, \{x \geq 1\}$. A fixed invariant is notated as I.

1 Propositional Calculus

1.1 Conjunction

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathsf{I} \qquad \frac{\phi \wedge \psi}{\phi}\ \wedge\mathsf{E}_1 \qquad \frac{\phi \wedge \psi}{\psi}\ \wedge\mathsf{E}_2$$

1.2 Disjunction

$$\frac{\phi}{\phi \vee \psi}\ \vee\mathsf{I}_1 \qquad \frac{\psi}{\phi \vee \psi}\ \vee\mathsf{I}_2 \qquad \frac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathsf{E}$$

1.3 Negation and Contradiction

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg\mathsf{I} \qquad \frac{\neg\neg\phi}{\phi}\ \neg\neg\mathsf{E} \qquad \frac{\phi \quad \neg\phi}{\bot}\ \neg\mathsf{E} \qquad \frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathsf{I} \qquad \frac{\bot}{\psi}\ \bot\mathsf{E}$$

1.4 Conditional

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \to \psi}\ {\to}\mathsf{I} \qquad \frac{\phi \quad \phi \to \psi}{\psi}\ {\to}\mathsf{E} \qquad \frac{\phi \to \psi \quad \neg\psi}{\neg\phi}\ \mathsf{MT}$$

1.5 De Morgan's Laws

$$\frac{\neg(\phi \vee \psi)}{\neg\phi \wedge \neg\psi}\ \mathsf{DeM}_1 \qquad \frac{\neg(\phi \wedge \psi)}{\neg\phi \vee \neg\psi}\ \mathsf{DeM}_2$$

1.6 Proof-by-Contradiction and the Law of the Excluded Middle

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathsf{PBC} \qquad \frac{\mathsf{T}}{\phi \vee \neg\phi}\ \mathsf{LEM} \qquad \frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \chi\end{array}}}{\chi}\ \mathsf{LEM\text{-}Carnap}$$

1.7 Biconditional

$$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \phi\end{array}}}{\phi \leftrightarrow \psi}\ \leftrightarrow\mathsf{I} \qquad \dfrac{\phi \leftrightarrow \psi \quad \phi}{\psi}\ \leftrightarrow\mathsf{E}_1 \qquad \dfrac{\phi \leftrightarrow \psi \quad \psi}{\phi}\ \leftrightarrow\mathsf{E}_2 \qquad \dfrac{\phi \leftrightarrow \psi \quad \neg\phi}{\neg\psi}\ \mathsf{BMT}$$

$$\dfrac{\neg(\phi \leftrightarrow \psi)}{\neg\phi \leftrightarrow \psi}\ \mathsf{NBD}_1 \qquad \dfrac{\neg(\phi \leftrightarrow \psi)}{\phi \leftrightarrow \neg\psi}\ \mathsf{NBD}_2$$

2 Predicate Calculus

2.1 Equality

$$\dfrac{\mathsf{T}}{t = t}\ {=}\mathsf{I} \qquad \dfrac{t_1 = t_2 \quad \phi[x \mapsto t_1]}{\phi[x \mapsto t_2]}\ {=}\mathsf{E}$$

2.2 Universal Introduction and Elimination

$$\dfrac{\boxed{\begin{array}{c}x_0\\ \vdots\\ \phi[x \mapsto x_0]\end{array}}}{\forall x\, \phi}\ \forall\mathsf{I} \qquad \dfrac{\forall x\, \phi}{\phi[x \mapsto t]}\ \forall\mathsf{E}$$

2.3 Existential Introduction and Elimination

$$\dfrac{\phi[x \mapsto t]}{\exists x\, \phi}\ \exists\mathsf{I} \qquad \dfrac{\exists x\, \phi \quad \boxed{\begin{array}{c}x_0 \quad \phi[x \mapsto x_0]\\ \vdots\\ \chi\end{array}}}{\chi}\ \exists\mathsf{E}$$

2.4 Induction and Strong Induction

$$\dfrac{P[n \mapsto 0] \quad \boxed{\begin{array}{c}k \quad P[n \mapsto k]\\ \vdots\\ P[n \mapsto k+1]\end{array}}}{\forall n \in \mathbb{N}\, P}\ \mathbb{N}\mathsf{I} \qquad \dfrac{\boxed{\begin{array}{c}\forall k < m \quad P[n \mapsto k]\\ \vdots\\ P[n \mapsto m]\end{array}}}{\forall n \in \mathbb{N}\, P}\ \mathbb{N}\mathsf{SI}$$

3 Hoare Logic

3.1 Skip

$$\frac{P \to Q}{\{P\}\,\text{skip}\,\{Q\}}\ \textsf{hskip}$$

3.2 Assignment

$$\frac{P \to Q\,[x \mapsto e]}{\{P\}\,x \coloneqq e\,\{Q\}}\ \textsf{hassign}$$

3.3 Conditional

$$\frac{\{P \land B\}\,C_1\,\{Q\} \quad \{P \land \neg B\}\,C_2\,\{Q\}}{\{P\}\,\text{if } B \text{ then } C_1 \text{ else } C_2\,\{Q\}}\ \textsf{hcond}$$

3.4 Sequential Composition

$$\frac{\{P\}\,C_1\,\{Q\} \quad \{Q\}\,C_2\,\{R\}}{\{P\}\,C_1;\,C_2\,\{R\}}\ \textsf{hseq}$$

3.5 Iteration

$$\frac{P \to \mathsf{I} \quad \{I \land B\}\,C\,\{\mathsf{I}\} \quad \mathsf{I} \land \neg B \to Q}{\{P\}\,\text{while } B \text{ do } C\,\{Q\}}\ \textsf{hwhile}$$

END.