Mathematical Modelling
Lecture 10 – Difference Equations

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Overview of Course

- Model construction $\longrightarrow$ dimensional analysis
- Experimental input $\longrightarrow$ fitting
- Finding a 'best' answer $\longrightarrow$ optimisation
- Tools for constructing and manipulating models $\longrightarrow$ networks, differential equations, integration
- Tools for constructing and simulating models $\longrightarrow$ randomness
- Real world difficulties $\longrightarrow$ chaos and fractals

A First Course in Mathematical Modeling by Giordano, Weir & Fox, pub. Brooks/Cole. Today we're in chapter 1.

Change?!

We often have to model dynamic systems.

- Discrete $\longrightarrow$ difference equations
- Continuous $\longrightarrow$ differential equations

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Discrete systems

Some systems are genuinely discrete.

- E.g. savings account, 1% interest/month
- Invest £1000 initially
- What is balance after a year?

Savings account

Define initial amount $a_0 = 1000$. Then the next value in the sequence is $a_1 = a_0 + \Delta a_0$, where Δa_0 is the amount due to the monthly interest.

i.e. in this case we have $\Delta a_0 = 0.01 a_0$, and:

$$\begin{aligned} a_1 &= a_0 + 0.01 a_0 \\ a_2 &= a_1 + 0.01 a_1 \\ &\vdots \\ a_{12} &= a_{11} + 0.01 a_{11} \end{aligned}$$

Discrete systems

- The savings account example led to a simple series
- We may have other actions – e.g. regular withdrawals
- In general we don't have a precise formula
  $\longrightarrow$ have to fit change to data

Approximating change

- In practice continuous systems are often modelled as discrete processes
- Experimental data is usually discrete
- Often need to guess an approximate form for model and fit to data

Population analysis

Remember this from the first lecture? We had:

- Data on population every 10 years $\longrightarrow$ discrete changes
- Had to deduce functional form
  - Simplest was Malthus

$$a_{n+1} = a_n + ka_n$$

  - Verhulst model saturates – finite carrying capacity

$$a_{n+1} = a_n + k\left(1 - \frac{a_n}{a_\infty}\right)a_n$$

Extensions to model

- Competition – species a and b compete for resources

$$a_{n+1} = a_n + k_1 a_n - k_3 a_n b_n$$
$$b_{n+1} = b_n + k_2 b_n - k_4 a_n b_n$$

- Predator-prey – species b eats species a

$$a_{n+1} = a_n + k_1 a_n - k_3 a_n b_n$$
$$b_{n+1} = b_n - k_2 b_n + k_4 a_n b_n$$

- War!

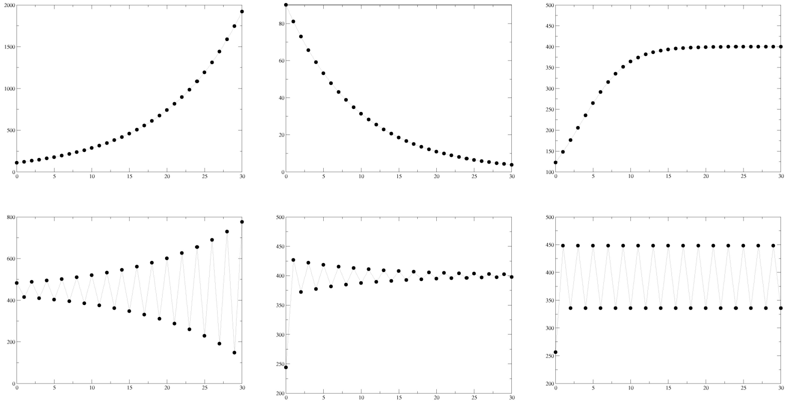
$$a_{n+1} = a_n - k_3 b_n$$
$$b_{n+1} = b_n - k_4 a_n$$

Population analysis

There are lots of questions we might want to ask about how these models behave, e.g.:

- What is the long-time behaviour?
- How sensitive are the solutions to the initial conditions?
- Can we have sustainable hunting/farming?

Long-time behaviour



Malthus

Recall the simplest model we looked at, the Malthus model:

- $a_{n+1} = a_n + ka_n = (1+k)a_n = ra_n$
- How does its behaviour depend on r?
  - $r = 0 \rightarrow a_{n+1} = 0$
  - $r = 1 \rightarrow a_{n+1} = a_n$
  - $r < 0 \rightarrow$ oscillatory
  - $|r| < 1 \rightarrow$ decay
  - $|r| > 1 \rightarrow$ growth

Malthus

What is the equilibrium value? At equilibrium:

$$\begin{aligned} a_{n+1} &= a_n \\ \Rightarrow r = 1 \quad &\text{or} \quad a_n = 0 \end{aligned}$$

Savings account

Back to our savings account.

- Same as Malthus!
- Include regular withdrawls: $a_{n+1} = ra_n + b$
- Equilibrium:

$$\begin{aligned} a_{n+1} &= a_n \\ \Rightarrow a_n &= \frac{b}{1-r} \end{aligned}$$

Savings account

- Equilibrium:

$$a_n = \frac{b}{1-r}$$

- $r = 1 \Rightarrow$ no equilibrium
- Otherwise an equilibrium a_∞ exists
- Are the equilibria all the same?

Savings account

- $|r| < 1$
  - stable equilibrium
  - different a_0 converge to a_∞
- $|r| > 1$
  - unstable equilibrium
  - different a_0 diverge
  - only get equilibrium if $a_0 = a_\infty$

Non-linear case

The logistic equation and Verhulst equations are non-linear, e.g.:

$$a_{n+1} = r(1 - a_n)a_n$$

Their behaviour is interesting:

- $0 < r < 3$ stable equilibrium
- $r = 3$ oscillation between 2 different values
- $r = 3.6$ oscillation between 4 different values – period doubling
- $r = 3.7$ chaos! No pattern or long-term prediction possible

Summary

- We can use difference equations to
  - model discrete processes
  - approximate continuous processes
- Long-time behaviour is often of interest
  - Does the model decay or grow?
  - Does the model tend to a limit?
  - Does the model oscillate?
- Non-linearity $\longrightarrow$ chaos!