Mathematical Modelling
Lecture 2 – Dimensional Analysis

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Overview of Course

- Model construction ⟶ dimensional analysis
- Experimental input ⟶ fitting
- Finding a 'best' answer ⟶ optimisation
- Tools for constructing and manipulating models ⟶ networks, differential equations, integration
- Tools for constructing and simulating models ⟶ randomness
- Real world difficulties ⟶ chaos and fractals

A First Course in Mathematical Modeling by Giordano, Weir & Fox, pub. Brooks/Cole. Today we're in chapter 8.

Aim

- To identify the relevant parameters and relationships for real-world problems and hence guide experimental design

Functions of several variables

Suppose we have a function of x, y and z and we know that it is linear in x, y and z – i.e. if we fix y and z, and plot f against x we get a straight line; and the same if we fix x and z and vary y etc. How many different terms are there?

$$f(x) = mx + c$$

A straight line has 2 parameters (slope and intercept), and we have 3 variables (x,y,z). Does this means we have $2 \times 3 = 6$ parameters in total?

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Functions of several variables

$$f(x, y, z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 xz + a_7 yz + a_8 xyz$$

We have $2^3 = 8$.

This gets worse very quickly if it isn't linear, but quadratic ($3^3 = 27$), cubic ($4^3 = 64$) or even higher order.

To get these parameters from experiment, we need at least one experimental measurement per parameter.

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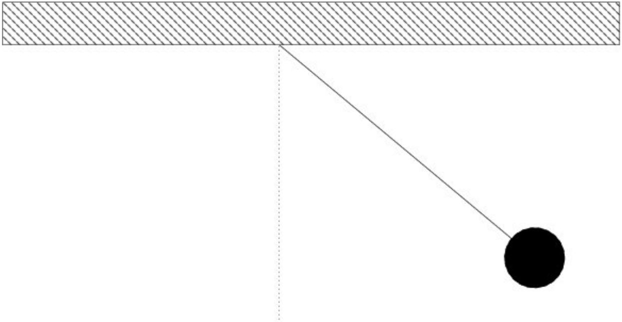
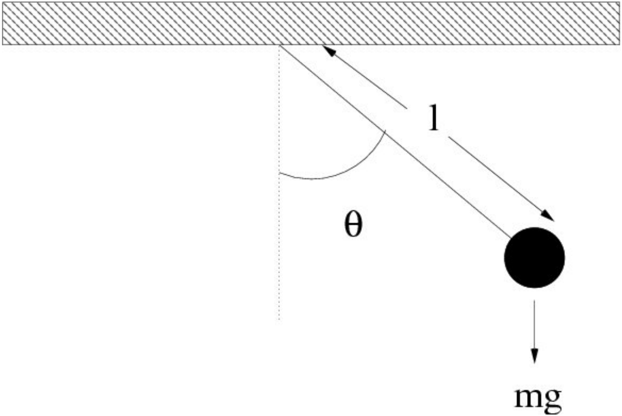
You are feeling very sleepy...



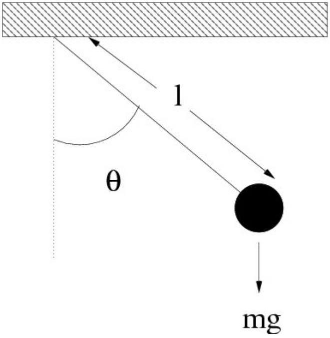
What affects the period τ of the pendulum?

You are feeling very sleepy...



Perhaps $\tau = f\,(l, m, g, \theta)$. What is f?

Pendula



$\tau = f(l, m, g, \theta)$

Could use experiments to determine f. How many measurements do we need?

- Quadratic $\Rightarrow$ 3 parameters
- 4 independent variables
- $\Rightarrow 3^4 = 81$ total parameters
- $\Rightarrow$ 81 expt measurements!

Dimensional analysis

P parameters per variable, V variables $\Rightarrow P^V$ total parameters.

But what if we can reduce the number of variables that need to be studied? Then we have a big saving!

Dimensional analysis does this by considering *dimensionless products*.

NB Dimensions are not the same as units!

Dimensional analysis

Dimensions M, L, T (and K, C, etc)

Product (includes quotients) e.g. [area]=L^2, [energy]=ML^2T^{-2}, etc.

Dimensionless product = combination s.t. dimensions are $M^0L^0T^0$

Dimensional compatibility

When adding terms in an equation they must all have the same dimension.

$$s = ut + \frac{1}{2}at^2$$

- u is a velocity, LT^{-1}, t is a time T
  $\Rightarrow$ ut is $LT^{-1}.T = L$, i.e. a length
- a is acceleration, LT^{-2}
  $\Rightarrow$ at^2 is also a length

You cannot add apples and oranges! Terms must be *dimensionally compatible*.

Dimensional homogeneity

The equation should be true regardless of units. This is achieved if the left- and right-hand sides have the same dimensions.

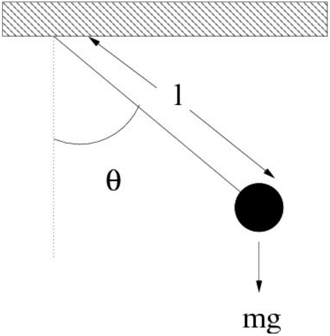
- $s = \frac{1}{2}gt^2$
  g is an acceleration LT^{-2}, so $\frac{1}{2}gt^2$ is L
  $\rightarrow$ *dimensionally homogeneous*
- $s = 4.6t^2$
  Dimensionally inhomogeneous $\rightarrow$ different answer if measure time in seconds, minutes, hours...

Dimensionless products

Products of variables which are dimensionless are always dimensionally homogeneous.

If a real world problem can be modelled by a dimensionally homogeneous equation (and no logarithms) then we can find the form of that equation using dimensional analysis.

Pendulum Analysis



Variable	Dimension
m	M
g	LT^{-2}
τ	T
l	L
θ	$M^0L^0T^0$

What are the dimensions of a general product $m^\alpha g^\beta \tau^\gamma l^\delta \theta^\epsilon$?

$M^\alpha L^{\beta+\delta} T^{\gamma-2\beta}$

Pendulum analysis

$M^{\alpha} L^{\beta+\delta} T^{\gamma-2\beta}$ is dimensionless iff

$$\begin{aligned} M: &\quad \alpha &= 0 \\ L: &\quad \beta+\delta &= 0 \\ T: &\quad \gamma-2\beta &= 0 \end{aligned}$$

which gives an infinite set of solutions – not enough equations!

- $\alpha = 0 \implies m$ cannot appear in the model
- θ has no units $\implies$ value is arbitrary

Dimensionless products

There are three basic rules when forming dimensionless products:

1. Choose the dependent variable to appear once
2. Choose any variable that always appears in each dimensional equation
3. Choose any variable that always has zero exponent (e.g. θ)

Pendulum – dimensionless product 1

- Our dependent variable is τ, so choose $\gamma = 1$
- But $\gamma - 2\beta \implies \beta = \frac{1}{2}$
- $\delta = -\beta = -\frac{1}{2}$
- ϵ is arbitrary, so choose $\epsilon = 0$

Thus our first dimensionless product is

$$\Pi_1 = m^0 g^{\frac{1}{2}} \tau l^{-\frac{1}{2}} \theta^0 = \tau \sqrt{\frac{g}{l}}$$

Pendulum – dimensionless product 2

- Already have τ in first product
- For second product choose $\gamma = 0$
- $\implies \beta = 0$
- $\implies \delta = 0$
- ϵ arbitrary – choose $\epsilon = 1$ (already used $\epsilon = 0$)

Thus our second dimensionless product is

$$\Pi_2 = m^0 g^0 \tau^0 l^0 \theta^1 = \theta$$

Pendulum analysis

Our dimensionless pendulum equation will relate the dps in some way

$$
\begin{aligned}
\Pi_1 &= f(\Pi_2) \\
\Rightarrow \tau\sqrt{\frac{g}{l}} &= f(\theta) \\
\tau &= \sqrt{\frac{l}{g}} f(\theta)
\end{aligned}
$$

Quick check:

- LHS has dimension T
- RHS has dimension $\sqrt{\frac{L}{LT^{-2}}} = T$
- Equation is *dimensionally homogeneous*

Pendulum analysis

$$\tau = \sqrt{\frac{l}{g}} f(\theta)$$

What have we learned?

- τ does not depend on $m \Rightarrow$ 'only' $5^3 = 125$ experiments
- changing units of time cannot change the actual period τ – there's a corresponding change in 'g' $\Rightarrow$ equation is dimensionally homogeneous.

We shall make this a bit more rigorous in a moment.

But for now...

Pendulum analysis

$$\tau = \sqrt{\frac{l}{g}}.h(\theta)$$

If we keep $\theta = \theta_0 =$ *constant* and vary l then

$$\frac{\tau_1}{\tau_2} = \sqrt{\frac{l_1}{l_2}}$$

i.e.

- $\tau \propto \sqrt{l}$ *regardless of h*.
- $\Rightarrow$ graph of τ against $\sqrt{l}$ should be linear
- $\Rightarrow$ simple test requiring only 5 points!

If test fails, we go back and check our assumptions...

Pendulum analysis

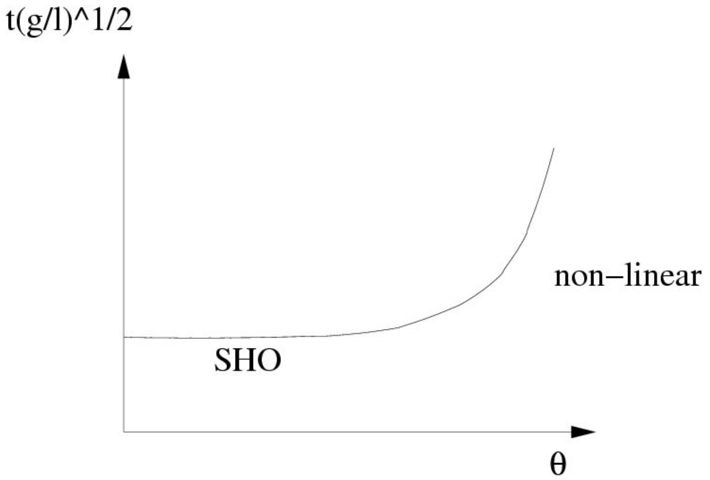
What about $h(\theta)$? Fix $l = l_0$ and vary θ:

$$\frac{\tau_1}{\tau_2} = \frac{h(\theta_1)}{h(\theta_2)}$$

- $\Rightarrow$ plot a graph of τ vs. θ (or better, $\tau\sqrt{\frac{g}{l}}$ vs. θ)
- Can get $h(\theta)$directly! Another insight gained ...

NB Whilst can do SHO analytically, cannot do the general case for arbitrary θ as it is non-linear ...

A pendulum



Dimensional analysis

- Original problem had 1 dependent and 4 independent variables
- We had 3 dimensional constraints
- Hence need $5 - 3 = 2$ dimensionless products
- Result was an equation determined up to an unknown function of 1 dimensionless product

Buckingham's Π-Theorem

- A problem with n variables and m independent dimensional constraints can be written in dimensionally homogeneous form using $(n - m)$ dimensionless products (dps) as

$$f\left(\Pi_1, \Pi_2, \ldots \Pi_{n-m}\right) = 0$$

Example 1

$n = 4, m = 3$

- $\Rightarrow$ need 1 product including the dependent variable
- $\Rightarrow f(\Pi_1) = 0$ so can solve to get $\Pi_1 = \textit{constant}$
- $\Rightarrow$ can get dependent variable = (unknown constant) $\times$ (other variables)

Example 2

$n = 5, m = 3$

- $\Rightarrow$ need 2 products
- $\Rightarrow f(\Pi_1, \Pi_2) = 0$
- can solve to get $\Pi_1 = h(\Pi_2)$
- $\Rightarrow$ can get equation up to an unknown function of a single dimensionless product, as with the pendulum

Example 3

$n = 6, m = 3$

- $\Rightarrow$ need 3 products
- $\Rightarrow f(\Pi_1, \Pi_2, \Pi_3) = 0$
- so can solve to get $\Pi_1 = h(\Pi_2, \Pi_3)$
- $\Rightarrow$ can get equation up to an unknown function of two dimensionless products, etc.

We always need to construct (n-m) independent dps, with the dependent variable only appearing once, e.g. in Π_1.

NB Good to put the most sensitive variables into the dp with the independent variable (e.g. Π_1) to minimise the amount of unknown behaviour and simplify experiments.

Buckingham's Π-Theorem Example

- Predict the period of 2 masses (m_1 & m_2) orbiting each other at a distance R apart, in vacuum

Identify variables:

Variable	Dimensions
τ	T
m_1	M
m_2	M
R	L
G	$L^3 M^{-1} T^{-2}$

So we have 5 variables and 3 dimensions $\Rightarrow$ need 2 dps

Buckingham's Π-Theorem Example

General form of dp:

$$\begin{aligned} \Pi &= \tau^{\alpha} m_1^{\beta} m_2^{\gamma} R^{\delta} G^{\epsilon} \\ &= T^{\alpha} M^{\beta} M^{\gamma} L^{\delta} L^{3\epsilon} M^{-\epsilon} T^{-2\epsilon} \\ &= M^{\beta+\gamma-\epsilon} L^{3\epsilon+\delta} T^{\alpha-2\epsilon} \end{aligned}$$

i.e. coefficients:

$$\begin{aligned} T: &\quad \alpha - 2\epsilon &= 0 \\ M: &\quad \beta + \gamma - \epsilon &= 0 \\ L: &\quad \delta + 3\epsilon &= 0 \end{aligned}$$

Buckingham's Π-Theorem Example

$$
\begin{aligned}
T: &\quad \alpha - 2\epsilon &= 0 \\
M: &\quad \beta + \gamma - \epsilon &= 0 \\
L: &\quad \delta + 3\epsilon &= 0
\end{aligned}
$$

Π_1: include τ once $\Rightarrow \alpha = 1 \Rightarrow \epsilon = \frac{1}{2} \Rightarrow \delta = -\frac{3}{2} \Rightarrow \beta + \gamma = \frac{1}{2}$ so free choice, e.g. $\beta = 1/2$ and $\gamma = 0$

$$\Rightarrow \Pi_1 = \tau m_1^{1/2} R^{-3/2} G^{1/2} = \tau \sqrt{\frac{m_1 G}{R^3}}$$

Buckingham's Π-Theorem Example

$$\begin{aligned} T: &\quad \alpha - 2\epsilon &= 0 \\ M: &\quad \beta + \gamma - \epsilon &= 0 \\ L: &\quad \delta + 3\epsilon &= 0 \end{aligned}$$

Π_2 : set $\alpha = 0 \Rightarrow \epsilon = 0 \Rightarrow \delta = 0 \Rightarrow \beta + \gamma = 0$ so free choice except must not choose same as before, e.g. $\beta = 1$ and $\gamma = -1$

$$\Rightarrow \Pi_2 = \frac{m_1}{m_2}$$

Buckingham's Π-Theorem Example

Hence $f\left(\Pi_1, \Pi_2\right) = 0$

$$\Rightarrow \tau = \sqrt{\frac{R^3}{m_1 G}}.h\left(\frac{m_1}{m_2}\right)$$

Exact analytic answer: $\tau = 2\pi\sqrt{\frac{R^3}{G(m_1+m_2)}}$

$$\Rightarrow h\left(\frac{m_1}{m_2}\right) = \frac{1}{\sqrt{1 + m_1/m_2}}$$

Summary of Methodology

1. Decide your n variables, hence m dimensional constraints
2. Form complete set of $(n - m)$ dimensionless products (dps):
    - dependent variable only appears once (e.g. in Π_1)
    - put most sensitive variables into same dp
    - check each dp found has no dimensions!
3. Apply Buckingham's Π-Theorem and hence solve for dependent variable
4. Test assumptions made (e.g. $\tau \propto \sqrt{l}$ for pendulum)
5. Conduct further experiments necessary to find any unknown functions, or further computations based upon the dps found.