Mathematical Modelling
Lecture 7 – Linear Programming

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Overview of Course

- Model construction $\longrightarrow$ dimensional analysis
- Experimental input $\longrightarrow$ fitting
- Finding a ‘best’ answer $\longrightarrow$ optimisation
- Tools for constructing and manipulating models $\longrightarrow$ networks, differential equations, integration
- Tools for constructing and simulating models $\longrightarrow$ randomness
- Real world difficulties $\longrightarrow$ chaos and fractals

A First Course in Mathematical Modeling by Giordano, Weir & Fox, pub. Brooks/Cole. Today we’re in chapter 7.

Aim

- Last lecture we looked at optimising general functions subject to general constraints
- Today we're concentrating on the special case of linear functions

What is a linear function?

A linear function is any function that depends linearly on its inputs.

E.g.

$$\begin{aligned} f(x) &= mx + c \\ f(x_0, x_1) &= a_0 x_0 + a_1 x_1 + c \end{aligned}$$

Such functions often appear in relation to economic and industrial applications.

Linear programming

Today we'll be optimising linear functions with linear constraints using a technique called linear programming.

NB this is not computer programming!

Example – Painting the town red

Suppose a decorating shop has:

- 20,000 litres of red paint, sells for £2.45 per litre
- 10,000 litres of green paint, sells for £2.00 per litre

and it also sells

- brown paint (half red, half green) for £2.40 a litre

How can the shop maximise its income?

Example – Painting the town red

We now have three constraints:

- Amount of red paint $\Rightarrow 20000 - x \geq 0$
- Amount of green paint $\Rightarrow 10000 - x \geq 0$
- Amount of brown paint $\Rightarrow 2x \geq 0$

How can the shop maximise its income?

We can see this on a graph...

Example – Painting the town red

Painting the town red

With this example we only had one variable, x, so could plot income on the graph easily.

With more variables plotting becomes trickier. Let's look at a slightly more complicated example...

Example – Painting the town navy and brown

Suppose a smaller decorating shop has:

- 300 litres of red paint, sells for £2.00 per litre
- 200 litres of blue paint, sells for £2.00 per litre
- 200 litres of green paint, sells for £2.00 per litre

and it also sells:

- Brown paint (half red, quarter blue, quarter green) for £4.00 a litre.
- Navy blue paint (half blue, quarter red, quarter green) for £5.00 a litre.

How can the shop maximise its income?

Example – Painting the town navy and brown

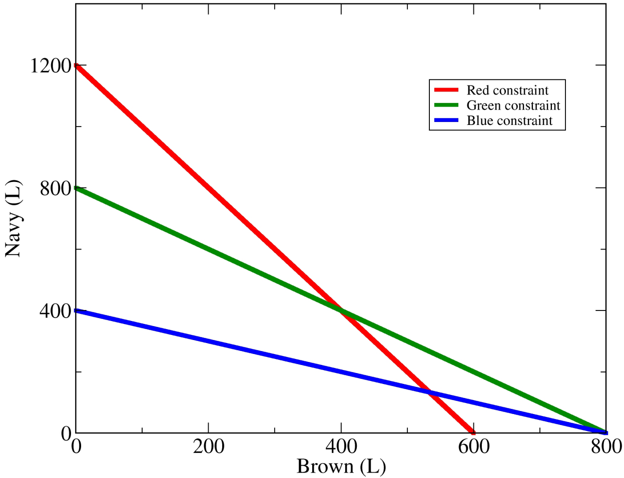
Now have 5 constraints

- Amount of red $\Rightarrow 0.5x_1 + 0.25x_2 \leq 300$
- Amount of blue $\Rightarrow 0.25x_1 + 0.5x_2 \leq 200$
- Amount of green $\Rightarrow 0.25x_1 + 0.25x_2 \leq 200$
- Amount of brown $x_1 \geq 0$
- Amount of navy blue $x_2 \geq 0$

Income is $1400 + 2x_1 + 3x_2$

How can the shop maximise its income?

Example – Painting the town navy and brown



Example – Painting the town navy and brown

Income is $1400 + 2x_1 + 3x_2$. Optimum is at point B, where red and blue constraints meet.

$$\begin{aligned} 2x_1 + x_2 &= 1200 \\ x_1 + 2x_2 &= 800 \end{aligned}$$

Maximum income I is:

$$I = 1400 + 2\left(\frac{1600}{3}\right) + 3\left(\frac{400}{3}\right)$$

i.e. $I =$£2866.66.

Corners

Because the function is linear, the function at any point in the allowed region can be expressed in terms of the function value at the corners.

$\longrightarrow$ we only need to look at the corners!

Slicing and dicing

- We had 2 variables
- $\Rightarrow$ 2-D parameter space
- Each constraint sliced space up $\longrightarrow$ allowed and disallowed regions
- Final allowed region was a 2-D shape (a polygon)
- Only the corners matter

Slicing and dicing

In general:

- N variables
- $\Rightarrow$ N-D parameter space
- m constraints $\longrightarrow$ allowed and disallowed regions
- Final allowed region is an N-D shape called a *simplex*
- Only the vertices matter

Example - Chebyshev fitting

Recall problem of fitting a model to some data. Rather than minimising the total error (S^2 or χ^2) we could minimise the worst error - this is the Chebyshev fitting criterion.

- Residual at each point $R_i = (y_i - f(x_i))$
- Biggest residual is $max(R_i)$
- $\Rightarrow$ want to minimise $max(R_i)$
- $\Rightarrow max(R_i) - R_i \geq 0$ and $max(R_i) + R_i \geq 0$
- Can write as a linear program!
- NB N data points gives 2N constraints

Simplex method

1. Start at a point in allowed region
   (i.e. satisfies all constraints)
   Often this will be the origin
2. Move along an edge of the simplex to find a vertex

   Good idea to try all the edges and find the one giving biggest objective function
3. Evaluate objective function at vertex
4. Repeat steps 2 and 3 until all vertices have been evaluated

Summary

- Can use linear programming whenever the objective function and constraints are linear in the N variables
- Constraints create an N-D allowed region called a simplex
- Maxima and minima must be at the vertices of the simplex