Rotational motion and moments of inertia

Introduction and definition

Moment of inertia is to rotational motion as mass is to linear motion:

For a point on a rotating body, the linear speed is:

$v = r\,\omega$ where r is distance from rotation axis (m) and ω is angular speed (radians/s)

The tangential acceleration is:

$$a = \frac{dv}{dt} = r\,\frac{d\omega}{dt} = r\,\alpha \qquad \text{where } \alpha = \text{angular acceleration}$$

The equivalent of a force in rotational motion is a torque, i.e. a turning force: $\tau = r\,F$

Use Newton's 2nd law of motion for constant mass: $\boxed{F = m\,a}$

so that: $\tau = r\,m\,a = m\,r^2\,\alpha$

Now define moment of inertia of a point mass as $\boxed{I = m\,r^2}$ so that $\boxed{\tau = I\,\alpha}$

This is the rotational equivalent of Newton's 2nd law. It shows that a torque alters angular acceleration just as a force alters linear acceleration and that moment of inertia corresponds to mass.

Moment of inertia of a thin rod about its centre

We have a formula for the moment of inertia of a point mass, m, a distance r from the rotation axis: $I = m\,r^2$

To calculate the moment of inertia of an extended body, split it into an infinite number of point masses and add their moments of inertia together. This is done using an integral.

Length of rod = ℓ (m)

Density of rod = ρ (kg/m)

Mass of rod = $M = \rho\,\ell$ (kg)

Now take an element of the rod of length dx a distance x from the centre.

Mass of element = $\rho\,dx$ Moment of inertia of element = $\rho\,dx\,x^2$

So total moment of inertia of rod = $I = \int_{-l/2}^{l/2} \rho\,x^2\,dx = \frac{1}{3}\rho\left[x^3\right]_{-l/2}^{l/2} = \frac{1}{3}\rho\,\frac{2\,l^3}{8} = \frac{\rho\,l^3}{12}$

But $M = \rho\,\ell$, therefore $I = \frac{1}{12}\,M\,l^2$ (kg m^2)

Moment of inertia of a disc

The moment of inertia always depends upon the position and direction of the rotation axis. For the disc, we will calculate the moment of inertia for the axis normal to the disc and going through its centre.

Using the same strategy as before, the disc is split into an infinite number of simple parts whose moments of inertia are known and all the moments of inertia are added together using an integral.

Density of disc = ρ (kg/m^2)
Radius of disc = R (m)
Therefore, mass of disc = M = area × density = $\pi R^2 \rho$ (kg)

Split the disc into elementary rings of radius r and width dr:

Mass of ring = area × density = $2 \pi r\, dr\, \rho$

As all the mass is at the same radius, moment of inertia of ring = $2 \pi r\, dr\, \rho\, r^2 = 2 \pi \rho\, r^3\, dr$

Therefore, moment of inertia of disc = $\int_0^R 2 \pi \rho\, r^3\, dr = \frac{\pi}{2} \rho\, [r^4]_0^R = \frac{\pi}{2} \rho\, R^4 = \frac{1}{2} M R^2$

Moment of inertia of a sphere about a diameter: first method

We already have a formula for the moment of inertia of a disc, so we can regard the sphere as being composed of an infinite number of infinitesimally thin discs parallel to the xz-plane.

Density of sphere = ρ (kg/m^3)
Radius of sphere = R (m)
So, mass of sphere = $M = \frac{4}{3} \pi R^3 \rho$ (kg)

Radius of elementary disc = $(R^2 - y^2)^{1/2}$

Thickness of elementary disc = dy

Mass of elementary disc = $\pi\, (R^2 - y^2)\, \rho\, dy$

Moment of inertia of elementary disc = $\frac{1}{2} \pi\, (R^2 - y^2)\, \rho\, dy\, (R^2 - y^2)$

Moment of inertia of sphere = $\frac{\pi}{2} \rho \int_{-R}^{R} (R^2 - y^2)^2\, dy = \frac{\pi}{2} \rho \int_{-R}^{R} (R^4 - 2R^2 y^2 + y^4)\, dy$

$$= \frac{\pi}{2} \rho \left[R^4 y - \frac{2}{3} R^2 y^3 + \frac{1}{5} y^5\right]_{-R}^{R} = \frac{\pi}{2} \rho \left(2R^5 - \frac{4}{3} R^5 + \frac{2}{5} R^5\right) = \frac{8}{15} \pi R^5 \rho$$

i.e. moment of inertia of sphere $= \frac{2}{5} \left(\frac{4}{3} \pi R^3 \rho\right) R^2 = \frac{2}{5} M R^2$

Moment of inertia of a sphere about a diameter: second method

There is nearly always more than one way of doing a calculation. This time we split the sphere into an infinite number of point masses. The moment of inertia of each point mass is easily calculated and they are all added together using an integral.

Use same sphere, so that density = ρ (kg/m^3); radius = R (m); mass = $M = \frac{4}{3} \pi R^3 \rho$ (kg)

Using spherical polar coordinates, consider a point mass at (r, θ, ϕ) with $\theta = 0$, i.e. the z-axis, as the rotation axis.

Element of volume = $r^2 \sin\theta\, dr\, d\theta\, d\phi$ Mass of element = $r^2 \sin\theta\, dr\, d\theta\, d\phi\, \rho$

Distance of (r, θ, ϕ) from the rotation axis = $r \sin\theta$

Therefore, moment of inertia of point mass = $r^2 \sin\theta\, dr\, d\theta\, d\phi\, \rho\, r^2 \sin^2\theta = \rho\, r^4 \sin^3\theta\, dr\, d\theta\, d\phi$

So, moment of inertia of sphere = $I = \rho \int_0^R r^4\, dr \int_0^{\pi} \sin^3\theta\, d\theta \int_0^{2\pi} d\phi$

Now use the trig identity: $\sin 3\theta = 3 \sin\theta - 4 \sin^3\theta$

$$\therefore I \;=\; \rho\,\tfrac{1}{5}\,[r^5]_0^R\,\tfrac{1}{4}\int_0^{\pi}(3\,sin\theta - sin3\theta)\,d\theta\,[\phi]_0^{2\pi} \;=\; \rho\,\tfrac{1}{20}\,R^5\left[-3\,cos\theta + \tfrac{1}{3}cos3\theta\right]_0^{\pi}\,2\pi$$

$$= \;\rho\,\tfrac{\pi}{10}\,R^5\,2\left(3 - \tfrac{1}{3}\right) \;=\; \tfrac{8}{15}\,\pi\,\rho\,R^5 \;=\; \tfrac{2}{5}\left(\tfrac{4}{3}\pi R^3\rho\right)R^2 \;=\; \tfrac{2}{5}\,M\,R^2$$

Moment of inertia of a hollow cylinder

The moment of inertia will be calculated about the cylinder axis.

Density of cylinder = ρ (kg/m^3)
Height of cylinder = *h* (m)
External and internal radii are R_1 and R_2 respectively
Therefore, mass of cylinder = M = volume × density = $\pi\,h\,(R_1^2 - R_2^2)\,\rho$ (kg)

Split the cylinder into cylindrical shells of radius *r* and thickness *dr*

Mass of shell = $2\,\pi\,r\,dr\,h\,\rho$

As all the mass is at the same radius, mmt of inertia of shell = $2\,\pi\,r\,dr\,h\,\rho\,r^2$ = $2\,\pi\,\rho\,h\,r^3\,dr$

So, moment of inertia of hollow cylinder = $2\,\pi\,\rho\,h\int_{R_2}^{R_1} r^3\,dr \;=\; \frac{\pi}{2}\,\rho\,h\,[R^4]_{R_2}^{R_1}$

$$= \;\tfrac{\pi}{2}\,\rho\,h\,(R_1^4 - R_2^4) \;=\; \tfrac{1}{2}\,\pi\,\rho\,h\,(R_1^2 - R_2^2)\,(R_1^2 + R_2^2) \;=\; \tfrac{1}{2}\,M\,(R_1^2 + R_2^2)$$

For a solid cylinder, $R_2 = 0$, so that $I \;=\; \frac{1}{2}\,M\,R_1^2$

Comment It can be seen from these formulae that the moment of inertia of a hollow cylinder is greater than that of a solid cylinder of the same external radius and mass. This is because, in a hollow cylinder, the mass is distributed further away from the rotation axis.

In order to illustrate the use of moments of inertia, it is necessary to calculate the kinetic energy of a rotating body.

Kinetic energy

In linear motion, work is given by *force × distance moved*. When the body is moving, this corresponds to translational kinetic energy:

$$W \;=\; \int F\,dx \;=\; \int m\,a\,dx \;=\; m\int \frac{dv}{dt}\,dx \;=\; m\int \frac{dx}{dt}\,dv \;=\; m\int v\,dv \;=\; \boxed{\tfrac{1}{2}\,m\,v^2}$$

In rotational motion, work is given by *torque × angle turned through*. When the body is rotating, this corresponds to rotational kinetic energy:

$$W \;=\; \int \tau\,d\theta \;=\; \int I\,\alpha\,d\theta \;=\; I\int \frac{d\omega}{dt}\,d\theta \;=\; I\int \frac{d\theta}{dt}\,d\omega \;=\; I\int \omega\,d\omega \;=\; \boxed{\tfrac{1}{2}\,I\,\omega^2}$$

Cylinder rolling down an incline

A solid cylinder of mass *M* and radius *R* rolls down an incline from a height *h*. What is its linear speed when it reaches the bottom?

Potential energy at the top of the incline = kinetic energy at the bottom

$$Mgh \;=\; \tfrac{1}{2}\,M\,v^2 + \tfrac{1}{2}\,I\,\omega^2$$

For a solid cylinder, $I \;=\; \frac{1}{2}MR^2$. Also, $v = \omega R$ so that $\omega = v / R$.

$$\therefore Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}\frac{1}{2}MR^2\frac{v^2}{R^2} = \frac{1}{2}Mv^2 + \frac{1}{4}Mv^2 = \frac{3}{4}Mv^2$$

$$\therefore \quad v^2 = \frac{4}{3}gh$$

An object that slides down a frictionless plane without rolling will have a speed given by:

$$mgh = ½mv^2 \quad \text{so that} \quad v^2 = 2gh$$

This is greater than the speed of a rolling cylinder because no energy is used in rotating the sliding object. All the initial potential energy is therefore converted into translational kinetic energy.

Question If a solid cylinder and a sphere of the same mass and radius roll down an incline, which will reach the bottom first if they are released simultaneously from the same height?

Answer The sphere will reach the bottom first because it has a lower moment of inertia than the cylinder. This means less rotational kinetic energy is needed for the sphere, allowing for more translational kinetic energy, so it goes faster.

First order differential equations

Differential equations are used to describe the behaviour of physical systems. They are therefore used when things change – a voltage or current, a temperature, a position (i.e. something is moving) or some quantity changes value with time or distance. Here are some examples of things that change and the mathematical equations that describe the change.

Radioactive decay

The radioactivity of a substance (number of decays per second) depends upon the number of atoms present. If the number of atoms doubles, then so does the count rate.

Let N be the number of atoms, then the rate of change of N, dN/dt, is a measure of the radioactivity. This is proportional to N, so we can write

$$\frac{dN}{dt} = -k\,N$$

where k is the constant of proportionality. dN/dt gives the rate of increase of the number of atoms, whereas we know the number is decreasing. Hence the minus sign.

This is a differential equation because it contains a derivative and it can be solved by separation of the variables:

$$\frac{dN}{N} = -k\,dt$$

Notice that all the Ns are on one side of the equation and all the ts are on the other. Integrating both sides gives

$$\ln(N) = -k\,t + const$$

which includes a constant of integration. Evaluation of the constant requires some extra information. For example, we can assume that at the very beginning, i.e. at t = 0, the number of atoms is N_0. Putting this into the equation gives the value of the constant

$$\ln(N_0) = const$$

so we now have $\ln(N) = -k\,t + \ln(N_0) \quad \Rightarrow \quad \ln(N) - \ln(N_0) = \ln\left(\frac{N}{N_0}\right) = -k\,t$

Take the exponential of both sides so that $\frac{N}{N_0} = \exp(-k\,t) \quad \Rightarrow \quad N = N_0 \exp(-k\,t)$

A graph of N against t shows an exponential decay. The constant k is often expressed in terms of the half-life of the substance, $t_{½}$, i.e. the time taken for half of the atoms to decay:

$$\frac{N}{N_0} = \frac{1}{2} = \exp(-k\,t_{½}) \quad \Rightarrow \quad k = \frac{\ln(2)}{t_{½}}$$

Discharging a capacitor

A capacitor, C, is discharged through a resistor, R. How does the charge on the capacitor vary with time?

During the discharge, a current must flow round the circuit. The current, I, is the rate of flow of the charge, so we have $I = \frac{dQ}{dt}$ where Q is the charge.

Now sum the electric potential around the circuit:

Potential drop across R $= \; I\,R \; = \; \frac{dQ}{dt}\,R$

Potential drop across C $= \; \frac{Q}{C}$

There is no applied potential, so we must have $\quad R\,\frac{dQ}{dt} + \frac{Q}{C} = 0 \quad \Rightarrow \quad \frac{dQ}{dt} + \frac{Q}{RC} \; = \; 0$

This is a differential equation whose solution gives Q as a function of time.

It can be solved by separation of variables: $\qquad \frac{dQ}{Q} \; = \; -\frac{dt}{R\,C}$

and integrating both sides gives: $\qquad \ln(Q) \, = \, -\frac{t}{R\,C} + \, const$

When $t = 0$, $Q = Q_0$. Putting this into the equation gives: $\qquad \ln(Q_0) = const$

$$\therefore \; \ln(Q) = \, -\frac{t}{R\,C} + \ln(Q_0) \quad \Rightarrow \quad \ln(Q) - \ln(Q_0) = \, \ln\left(\frac{Q}{Q_0}\right) \; = \, -\frac{t}{R\,C}$$

Take the exponential of both sides: $\quad \frac{Q}{Q_0} \; = \; \exp\left(-\frac{t}{R\,C}\right) \quad \Rightarrow \quad Q \; = \; Q_0 \exp\left(-\frac{t}{R\,C}\right)$

This shows there is an exponential decay of charge on the capacitor as it discharges. The quantity RC governs the rate of discharge and it is known as the time constant of the circuit.

Charging a capacitor

Charging a capacitor is similar to discharging, except there is an applied potential from a battery or power supply. The differential equation is therefore similar to the one above with the battery potential, V_0, on the right-hand-side:

$$R\,\frac{dQ}{dt} + \frac{Q}{C} = V_0 \quad \Rightarrow \quad \frac{dQ}{dt} + \frac{Q}{RC} \; = \; \frac{V_0}{R}$$

The variables are not separable this time, but the equation can be solved using an integrating factor. To solve $\frac{dy}{dx} + P(x)y \; = \; Q(x)$ the integrating factor is $\exp(\int P(x)dx)$. Multiply throughout by the integrating factor and integrate to express y as a function of x.

$$y\,\exp(\int P(x)dx) \;\; = \;\; \int Q(x)\exp(\int P(x)dx)$$

For the capacitor equation above, the integrating factor is $\exp\left(\int \frac{dt}{R\,C}\right) \; = \; \exp\left(\frac{t}{R\,C}\right)$

Apply this to the equation and integrate:

$$Q\,\exp\left(\frac{t}{R\,C}\right) \; = \; \frac{V_0}{R} \int \exp\left(\frac{t}{R\,C}\right)\,dt \; = \; \frac{V_0}{R}\,\frac{\exp(t/RC)}{1/RC} + const \; = \; V_0 C\,\exp\left(\frac{t}{R\,C}\right) + const$$

The constant of integration can be evaluated with the information that at $t = 0$, the capacitor is uncharged so $Q = 0$.

Substituting these values into the equation gives: $\quad 0 = V_0\,C + const$

so that $\quad Q\,\exp\left(\frac{t}{R\,C}\right) \; = \; V_0\,C\,\exp\left(\frac{t}{R\,C}\right) - V_0\,C \quad \Rightarrow \quad \frac{Q}{C} \; = \; V \; = \; V_0\,\left(1 - \exp\left(-\frac{t}{R\,C}\right)\right)$

where $Q/C = V$, the potential difference across the capacitor.

The potential across the capacitor starts from zero and rises asymptotically towards V_0.

Newton's law of cooling

Newton's law of cooling states that the rate of cooling of a body is proportional to its difference in temperature from its surroundings.

Let us start with a hot cup of tea at 90°C in a room at 20°C. Given that after ¼ hour, it has cooled to 40°C, Newton's law of cooling will let us answer the following questions. What will be its temperature after ½ hour? How long will it take to reach 60°C?

The rate of cooling is given by $-d\theta/dt$ where θ is temperature and *t* is time – note the minus sign.

Temperature of tea above its surroundings is $(\theta - 20)$°C.

Newton's law of cooling gives $\frac{d\theta}{dt} = -k\,(\theta - 20)$ where *k* is the constant of proportionality.

The variables are separable so that $\frac{d\theta}{\theta - 20} = -k\,dt \quad \Rightarrow \quad \ln(\theta - 20) = -k\,t + const$

At $t = 0$, $\theta = 90$°C, so the above equation gives $\ln(70) = const$

$$\therefore\ \ln(\theta - 20) = -k\,t + \ln(70) \quad \Rightarrow \quad \ln\left(\frac{\theta - 20}{70}\right) = -k\,t \quad \Rightarrow \quad \theta = 70\exp(-k\,t) + 20$$

The unknown constant *k* still has to be evaluated and this is done with the additional information that when $t = ¼$ hrs, $\theta = 40$°C.

$$\therefore\ 40 = 70\exp\left(-\frac{k}{4}\right) + 20 \quad \Rightarrow \quad \frac{20}{70} = \exp\left(-\frac{k}{4}\right) \quad \Rightarrow \quad \frac{k}{4} = \ln\left(\frac{7}{2}\right) \quad \Rightarrow \quad k = 4\ln\left(\frac{7}{2}\right)$$

Now we can answer the question, "What is θ when $t = ½$ hour?"

$$\theta = 70\exp\left(-4\ln\left(\frac{7}{2}\right)\frac{1}{2}\right) + 20 = 70\exp\left(-\ln\left(\frac{7^2}{2^2}\right)\right) + 20 = 70\,\frac{2^2}{7^2} + 20 = 25.7^o\mathrm{C}$$

Time to reach 60°C is given by:

$$60 = 70\exp\left(-4\ln\left(\frac{7}{2}\right)t\right) + 20 \quad \Rightarrow \quad \frac{40}{70} = \exp\left(-\ln\left(\frac{7^4}{2^4}\right)t\right) \quad \Rightarrow \quad \ln\left(\frac{7}{4}\right) = \ln\left(\frac{7^4}{2^4}\right)t$$

$$t = \frac{\ln(7/4)}{4\ln(7/2)} = 0.1117\ \mathrm{hrs} = 6\mathrm{mins}\ 42\mathrm{secs}$$

<u>Heat conduction</u>

The rate of heat flow, *Q* (measured in watts), through a slab of material is proportional to its area, *A*, and inversely proportional to its thickness, Δx. It is also proportional to the temperature difference, $\Delta\theta$, and the heat must travel from the hotter side to the cooler. All of this is built into the formula $Q = -k\,A\,\frac{\Delta\theta}{\Delta \mathrm{x}}$ where the constant of proportionality, *k*, is known as the thermal conductivity. This formula applies when both the thermal gradient, $\frac{\Delta\theta}{\Delta \mathrm{x}}$, and the area, *A*, are constant throughout the slab of material. If either varies, then the heat flow must be described locally at a particular value of *x*. That is, the heat flows through a slab of infinitesimal thickness, *dx*, with a thermal gradient of $\frac{d\theta}{dx}$ and an area, *A*, appropriate to that value of *x*.

This is illustrated by the calculation of heat loss from a lagged pipe. A cross section of the pipe is shown in the diagram, where it is clear that the area of pipe is different from the outside area of lagging, making the above formula unsuitable. The internal and external radii, R_1 and R_2, and the internal and external temperatures, θ_1 and θ_2, are all known.

Consider a cylindrical shell of lagging of radius r, thickness dr and length ℓ. Its area is $2\pi r\ell$, so the heat conduction formula is written as:

$$Q \;=\; -k\;2\pi r l\;\frac{d\theta}{dr}$$

which is now a differential equation whose variables (θ and r) are separable. The equation is therefore written as:

$$Q\;\frac{dr}{r} \;=\; -2\pi k l\;d\theta$$

To obtain the heat flow through the lagging, this must now be integrated over all values of r from $r = R_1$ to $r = R_2$. These values correspond to $\theta = \theta_1$ and $\theta = \theta_2$ respectively:

$$Q\int_{R_1}^{R_2}\frac{dr}{r} \;=\; -2\pi k l\int_{\theta_1}^{\theta_2} d\theta \quad\Rightarrow\quad Q\ln\left(\frac{R_2}{R_1}\right) \;=\; -2\pi k l\;(\theta_2-\theta_1)$$

So the rate of heat flow is:

$$Q \;=\; \frac{2\pi k l\;(\theta_1-\theta_2)}{\ln(R_2/R_1)}$$

The harmonic oscillator and resonance

Resonance in an oscillator occurs when one of its parameters, for example the amplitude or average power, is maximised by adjusting the frequency of oscillation. To study resonance, therefore, we need to examine the behaviour of an oscillating system.

Equation of motion of a damped harmonic oscillator

A good example of an oscillator is the vertical mass-spring system shown in the diagram.

For a displacement *x* from the equilibrium position, the spring exerts a restoring force of $-kx$ where *k* is the spring constant. The restoring force is always in the opposite direction from the displacement, hence the minus sign.

Forces such as friction or air resistance dissipate the oscillation energy, causing a reduction of the amplitude, and the motion is said to be damped. In many systems, the damping force is proportional to the speed, so the force is $-\alpha \frac{dx}{dt}$ where α is the damping constant. The damping force always opposes the motion, hence the minus sign.

Newton's 2nd law of motion for constant mass is $F = m\,a$ where *a* is acceleration. The forces on the oscillator are the restoring force and the damping force, so Newton's law becomes

$$-k\,x - \alpha\,\frac{dx}{dt} = m\,a = m\,\frac{d^2x}{dt^2}$$

which is more conveniently written as $$\frac{d^2x}{dt^2} + b\,\frac{dx}{dt} + \frac{k}{m}\,x = 0$$

where $b = \alpha/m$ = damping factor per unit mass. This is the equation of motion of a damped harmonic oscillator. It is a second order differential equation whose solution gives the displacement, *x*, as a function of time, *t*.

Solving the equation

The trial solution is $x = A\,e^{at}$ in which *A* and *a* have to be determined. Substitute this into the equation of motion to obtain the auxiliary equation: $a^2 + b\,a + \frac{k}{m} = 0$

whose solution is: $$a = \frac{-b \pm \sqrt{b^2 - 4\,k/m}}{2} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \frac{k}{m}}$$

This gives two solutions to the differential equation and the general solution is the sum of the two, giving:

$$\begin{aligned} x &= A\exp\left\{\left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{k}{m}}\right)t\right\} + B\exp\left\{\left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{k}{m}}\right)t\right\} \\ &= \exp\left(-\frac{b}{2}\,t\right)\left\{A\exp\left(\sqrt{\frac{b^2}{4} - \frac{k}{m}}\,t\right) + B\exp\left(-\sqrt{\frac{b^2}{4} - \frac{k}{m}}\,t\right)\right\} \end{aligned}$$

There are two constants of integration, *A* and *B*, because we are dealing with a second order differential equation. They are normally evaluated from initial conditions.

If the system is heavily damped so that $\frac{b^2}{4} > \frac{k}{m}$, the square roots give real results so the expression describes exponential decay. The condition $\frac{b^2}{4} = \frac{k}{m}$ corresponds to critical damping whereby the mass returns to the equilibrium position in the shortest possible time without overshooting.

With a lightly damped system for which $\frac{b^2}{4} < \frac{k}{m}$, the square roots become imaginary, resulting in complex exponentials that describe oscillations:

$$x = \exp\left(-\frac{b}{2}\,t\right)\left\{A\exp\left(i\sqrt{\frac{k}{m}-\frac{b^2}{4}}\;t\right) + B\exp\left(-i\sqrt{\frac{k}{m}-\frac{b^2}{4}}\;t\right)\right\}$$

$$= \exp\left(-\frac{b}{2}\,t\right)\{(A+B)\cos(\nu\,t) + i\,(A-B)\sin(\nu\,t)\} \quad \text{where } \nu = \sqrt{\frac{k}{m}-\frac{b^2}{4}}$$

The constants *A* and *B* can be replaced by new constants *C* and *D* such that

$$x = \exp\left(-\frac{b}{2}\,t\right)\{\mathrm{C}\cos(\nu\,t) + D\sin(\nu\,t)\}$$

which describes damped simple harmonic motion.

When there is no damping (*b* = 0), $\nu = \sqrt{k/m}$ which is the angular frequency of undamped natural oscillations.

Forced oscillations

To counteract the damping and maintain the oscillations, an oscillating external force may be applied. This is added to the equation of motion which becomes:

$$\frac{d^2x}{dt^2} + b\,\frac{dx}{dt} + \omega_0^2\,x = f\cos(\omega t)$$

where *f* is the applied force per unit mass, ω is its angular frequency (= $2\pi \times$ *frequency*) and $\sqrt{k/m}$ is written as ω_0 – the frequency of natural oscillations.

This is now the equation of motion of a damped, driven harmonic oscillator.

Mathematically, the solution consists of two parts – the complementary function and the particular integral. The complementary function is the general solution of the equation with *f* = 0 and this has already been done. It is the expression for damped simple harmonic motion obtained above.

The particular integral must look like the right-hand-side of the equation, so we can try the expression $x = p\cos(\omega t) + q\sin(\omega t)$ where *p* and *q* are chosen to satisfy the equation. To substitute this into the equation, we need:

$$\frac{dx}{dt} = -p\,\omega\sin(\omega t) + q\,\omega\cos(\omega t) \quad \text{and} \quad \frac{d^2x}{dt^2} = -p\,\omega^2\cos(\omega t) - q\,\omega^2\sin(\omega t)$$

so the equation becomes:

$$-p\,\omega^2\cos(\omega t) - q\,\omega^2\sin(\omega t) - p\,b\,\omega\sin(\omega t) + q\,b\,\omega\cos(\omega t) + p\,\omega_0^2\cos(\omega t) + q\,\omega_0^2\sin(\omega t) = f\cos(\omega t)$$

Now equate the coefficients of cos(ωt) and sin(ωt) separately:

Coefficients of cos: $\quad -p\,\omega^2 + q\,b\,\omega + p\,\omega_0^2 = f \quad \Rightarrow \quad (\omega_0^2 - \omega^2)\,p + b\,\omega\,q = f$

Coefficients of sin: $\quad -q\,\omega^2 - p\,b\,\omega + q\,\omega_0^2 \;=\; 0 \quad\Rightarrow\quad -b\,\omega\,p + (\omega_0^2 - \omega^2)\,q \;=\; 0$

This pair of equations can be solved to give: $\quad p \;=\; \frac{f\,(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+b^2\,\omega^2} \quad q \;=\; \frac{f\,b\,\omega}{(\omega_0^2-\omega^2)^2+b^2\,\omega^2}$

The particular integral is therefore: $\quad x \;=\; \frac{f\,(\omega_0^2-\omega^2)\cos(\omega t)+f\,b\,\omega\,\sin(\omega t)}{(\omega_0^2-\omega^2)^2+b^2\,\omega^2}$

This can be simplified by putting the numerator into the form:

$A\sin(\omega t+\delta) = A\sin(\omega t)\cos(\delta) + A\cos(\omega t)\sin(\delta) = (\omega_0^2-\omega^2)\cos(\omega t) + b\omega\sin(\omega t)$

Comparing coefficients of cos and sin: $\quad A\sin(\delta) = \omega_0^2 - \omega^2 \quad$ and $\quad A\cos(\delta) = b\omega$

Squaring and adding gives: $\;A^2 = (\omega_0^2-\omega^2)^2 + b^2\,\omega^2 \quad$ Dividing gives: $\;\tan(\delta) = \frac{\omega_0^2-\omega^2}{b\,\omega}$

The particular integral is therefore: $\quad x \;=\; \frac{f\sin(\omega t+\delta)}{\left[(\omega_0^2-\omega^2)^2+b^2\,\omega^2\right]^{1/2}}$

and the complete solution of the equation of motion is:

$$x \;=\; \exp\left(-\frac{b}{2}\,t\right)\{\mathrm{C}\cos(\nu t) + D\sin(\nu t)\} + \frac{f\sin(\omega t+\delta)}{[(\omega_0^2-\omega^2)^2+b^2\,\omega^2]^{1/2}}$$

with $\quad \omega_0^2 \;=\; \frac{k}{m} \qquad \nu \;=\; \sqrt{\frac{k}{m}-\frac{b^2}{4}} \qquad \tan(\delta) = \frac{\omega_0^2-\omega^2}{b\,\omega}$

The first term (complementary function) represents damped simple harmonic motion and so becomes insignificant over time. The second term (particular integral) is not damped and it represents oscillations of constant amplitude at an angular frequency of ω. These are known as forced oscillations and they are the steady state of the oscillator. Because the two terms represent oscillations of different frequencies (ν and ω) the initial oscillations will be irregular until the steady state is established.

Resonance

The amplitude of the steady state term, i.e. $\frac{f}{\left[(\omega_0^2-\omega^2)^2+b^2\,\omega^2\right]^{1/2}}$, is a function of the forcing frequency, ω, which is plotted in the diagram. The maximum is located by differentiating with respect to ω, equating the derivative to zero and solving for ω. It is found to be at $\sqrt{\omega_0^2 - b^2/2}$. Since this maximises the amplitude, it is the resonance frequency and the curve in the diagram is known as the resonance curve of the oscillator.

The amplitude at the resonance frequency is $\frac{f}{b\sqrt{\omega_0^2-b^2/4}}$, which can be very large indeed for a lightly damped system (small *b*).

The width of the resonance curve is measured at the half power points, i.e. where the (amplitude)^2 is half the maximum. The calculation produces a complicated expression with an approximate value of *b*. A lightly damped system, therefore, resonates with a very large amplitude, but the frequency that drives it must be adjusted within narrow limits. On the other hand, a heavily damped system resonates with a much smaller amplitude and it produces this response over a much wider frequency range.

Electric and magnetic fields

Coulomb's law

In the late 18th century, the French scientist Charles Coulomb investigated the force between static electric charges and his results are summarised in Coulomb's law:

$$\mathbf{F} = \frac{Q\,q}{4\pi\varepsilon_0\, r^2}\,\hat{\mathbf{r}}$$

where **F** is the force experienced by the charge q in the electric field of the charge Q. The force is written as a vector parallel to the unit vector $\hat{\mathbf{r}}$, which points from Q towards q as shown in the diagram. In addition, the force is inversely proportional to the square of the distance, r, between the point charges. The constant $4\pi\varepsilon_0$ is required by the SI system of units to match the magnitudes and physical dimensions on both sides of the equation. ε_0 is the electric constant with a defined value of 8.854×10^{-12} $C^2/N.m^2$

When the two charges are of the same sign, the force on q is in the same direction as $\hat{\mathbf{r}}$ so it pushes the charges apart. When the charges are of opposite sign, the numerical value of $\frac{Q\,q}{4\pi\varepsilon_0\, r^2}$ is negative, indicating that the force is in the opposite direction to $\hat{\mathbf{r}}$ and the charges attract each other.

Electric field of a point charge

The force on a point charge q in an electric field **E** is

$$\mathbf{F} = q\,\mathbf{E}$$

where **F** and **E** are parallel vectors. They point in the same direction when q is a positive charge and in opposite directions when q is negative. The equation can be rearranged to give an expression for the field **E**, i.e. **E** = **F** / q. **F** is given by Coulomb's law, so the expression for the electric field a distance r away from a point charge is

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0\, r^2}\,\hat{\mathbf{r}}$$

The unit vector $\hat{\mathbf{r}}$ always points away from the charge. Note that the field has a direction as well as a magnitude and it behaves as a vector quantity.

The field of several point charges

When there are several point charges, each has its own associated field and the fields superimpose to produce the total electric field. Since each field has a direction as well as a magnitude, vector addition must be used to combine the fields:

$$\mathbf{E} = \sum_i \mathbf{E}_i$$

Example The diagram shows the positions of two point charges – q_1 of 68 μC at (0, 1) and q_2 of -34 μC at (2, 0) where the scales on the x and y axes are in metres. Calculate the electric field at the point (2, 2).

For q_1: (distance)^2 from (0, 1) to (2, 2) = $2^2 + 1^2$ = 5 metres^2

Unit vector in the direction (0, 1) to (2, 2) = $\frac{1}{\sqrt{5}}\,(2\mathbf{i} + \mathbf{j})$

Field at (2, 2) due to q_1 = $\frac{68\times10^{-6}}{4\pi\varepsilon_0\, 5}\,\frac{2\mathbf{i}+\mathbf{j}}{\sqrt{5}}$ N.C^{-1}

For q_2: (distance)^2 from (2, 0) to (2, 2) = 4 metres^2

Unit vector in the direction (2, 0) to (2, 2) = **j**

Field at (2, 2) due to q_2 = $\frac{-34\times10^{-6}}{4\pi\varepsilon_0\, 4}\,\mathbf{j}$ N.C^{-1}

Total electric field $= \frac{10^{-6}}{4\pi\varepsilon_0}\left(\frac{68}{5\sqrt{5}}\,(2\mathbf{i}+\mathbf{j}) - \frac{34}{4}\,\mathbf{j}\right) = \frac{10^{-6}}{4\pi\varepsilon_0}\left(\frac{136}{5\sqrt{5}}\,\mathbf{i} + \left(\frac{68}{5\sqrt{5}} - \frac{34}{4}\right)\mathbf{j}\right)$

$= 10^5\,(1.09\,\mathbf{i} - 0.22\,\mathbf{j})$ N.C^{-1}

Calculate the force on a point charge of 15 μC at (2, 2):

$$\mathbf{F} = q\,\mathbf{E} = 15\times10^{-6} \times 10^5\,(1.09\,\mathbf{i} - 0.22\,\mathbf{j}) = 1.64\,\mathbf{i} - 0.33\,\mathbf{j}\ \text{N}$$

This result could also be obtained using Coulomb's law with a vector addition of forces.

Field of a distributed charge

When the electric charge is distributed over a region of space, Coulomb's law is no longer suitable, neither is the formula to calculate the field. One way of dealing with this is to divide the charge into an infinite number of point charges. The field for each point charge can be calculated and all the fields added together using integration. As with the previous example, the addition has to be done vectorially.

For an infinitesimal point charge *dq*, the electric field is: $\mathbf{dE} = \frac{dq}{4\pi\varepsilon_0\, r^2}\,\hat{\mathbf{r}}$

The total field is given by: $\mathbf{E} = \int \mathbf{dE} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\,\hat{\mathbf{r}}$

where the integration is carried out over the whole of the distributed charge.

Example A thin rod of length ℓ metres contains a total charge of *Q* coulombs distributed uniformly along its length. Determine the electric field at the origin in the diagram, a distance *a* from the end of the rod.

For a rod of length ℓ containing a charge *Q*, the charge density is Q / ℓ C.m^{-1}

Consider a small length of rod *dx* a distance *x* from the origin.

The charge associated with this point $= dq = \frac{Q}{l}\,dx$ C.

The field at $x = 0$ due to this point charge $= \mathbf{dE} = \frac{{}^{Q}/_{l}\,dx}{4\pi\varepsilon_0\, x^2}\,(-\mathbf{i})$

where $-\mathbf{i}$ is the unit vector directed from *dq* towards the origin.

The total field at $x = 0$ is the sum of the fields due to all such point charges along the rod:

$$\mathbf{E} = \int \mathbf{dE} = \frac{Q}{4\pi\varepsilon_0\, l}\int_a^{a+l}\frac{dx}{x^2}\,(-\mathbf{i})$$

where the limits on the integral cover the entire rod and it is recognised that the unit vector $-\mathbf{i}$ is the same for all points along the rod.

$$\therefore\ \mathbf{E} = \frac{-\mathbf{i}\,Q}{4\pi\varepsilon_0\, l}\left[-\frac{1}{x}\right]_a^{a+l} = \frac{-\mathbf{i}\,Q}{4\pi\varepsilon_0\, l}\left(\frac{1}{a} - \frac{1}{a+l}\right) = \frac{-\mathbf{i}\,Q}{4\pi\varepsilon_0\, l}\,\frac{l}{a\,(a+l)} = \frac{-\mathbf{i}\,Q}{4\pi\varepsilon_0\, a(a+l)}\ \text{N.C}^{-1}$$

The Biot-Savart law

Shortly after the magnetic effects of electric currents were discovered, the French scientists Jean Baptiste Biot and Félix Savart experimentally determined the form of the magnetic field arising from a steady current. Their results are embodied in the Biot-Savart law:

$$\mathbf{dB} \;=\; \frac{\mu_0}{4\pi}\, I\, \frac{\mathbf{dl}\times\hat{\mathbf{r}}}{r^2}$$

which gives the contribution **dB** to the magnetic field at the point P due to the current element *I***dl** in the wire carrying a steady current *I* in the direction of **dl**. P is a distance *r* from the wire in the direction of the unit vector $\hat{\mathbf{r}}$ and **dB** is inversely proportional to r^2. The constant $\mu_0/4\pi$ is required by the SI system of units to match the magnitudes and physical dimensions on both sides of the equation. μ_0 is the magnetic constant with a defined value of exactly $4\pi\times10^{-7}\ \mathrm{N.A^{-2}}$.

The current element *I***dl** has the dimensions of *charge* × *velocity* (current is measured in amps which is coulombs/second, then multiplying by metres gives coulomb metres/second). This shows that a magnetic field is produced by a moving charge as opposed to an electric field which is produced by a static charge. The velocity of the charge introduces an extra vector and the combination of vectors requires a cross product. This results in the direction of **dB** being perpendicular to both **dl** and $\hat{\mathbf{r}}$, in accordance with the findings of Biot and Savart, so it is into the page as seen in the diagram.

Since a current element cannot exist on its own, the Biot-Savart law gives only a small contribution to the magnetic field and the complete field is the sum of the contributions from all current elements:

$$\mathbf{B} \;=\; \int \mathbf{dB} \;=\; \frac{\mu_0\, I}{4\pi}\int \frac{\mathbf{dl}\times\hat{\mathbf{r}}}{r^2}$$

where the integration is taken over the entire electrical circuit.

Example – the magnetic field of an infinitely long straight wire carrying a current *I*.

Let the wire coincide with the *x*-axis with the current travelling in the positive *x* direction.

It is required to calculate the magnetic field at point P on the *y*-axis a distance *a* from the wire.

The current element at the point *x* is *I***dx** and the unit vector $\hat{\mathbf{r}}$ points towards P.

Since **dB** is perpendicular to both **dx** and $\hat{\mathbf{r}}$, it is directed out of the page and this is the same for all current elements. As all the vectors **dB** are parallel, it is necessary to sum only their magnitudes.

If θ is the angle between **dx** and $\hat{\mathbf{r}}$, then

$$|\mathbf{dx}\times\hat{\mathbf{r}}| \;=\; dx\times1\times\sin(\theta) = \; dx\,\frac{a}{\sqrt{a^2+x^2}} \qquad \text{also} \qquad r^2 \;=\; a^2+x^2$$

$$\therefore\; dB \;=\; \frac{\mu_0\, I}{4\pi}\,\frac{|\mathbf{dx}\times\hat{\mathbf{r}}|}{r^2} \;=\; \frac{\mu_0\, I}{4\pi}\,\frac{a\,dx}{(a^2+x^2)^{3/2}}$$

$$\therefore\; B \;=\; \int dB \;=\; \frac{\mu_0\, I}{4\pi}\int_{-\infty}^{\infty}\frac{a\,dx}{(a^2+x^2)^{3/2}}$$

Let $x = a\tan(\phi)$ then $a^2 + x^2 = a^2(1+\tan^2(\phi)) = a^2\sec^2(\phi)$ also $dx = a\sec^2(\phi)\, d\phi$

In addition, when $x = -\infty,\ \phi = -\pi/2$ and when $x = \infty,\ \phi = \pi/2$

$$\therefore\ B = \frac{\mu_0\, I}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{a^2\sec^2(\phi)\, d\phi}{a^3\sec^3(\phi)} = \frac{\mu_0\, I}{4\pi a}\int_{-\pi/2}^{\pi/2}\cos(\phi)\, d\phi = \frac{\mu_0\, I}{4\pi a}[\sin(\phi)]_{-\pi/2}^{\pi/2} = \frac{\mu_0\, I}{2\pi a}$$

AC circuits

The most convenient mathematical description of AC circuit elements (capacitor and inductor) makes use of complex numbers. Normally, imaginary numbers use the symbol *i* $\left(= \sqrt{-1}\right)$, but in electricity *i* is often used to represent a current. It is standard practice, therefore, to use *j* instead and that will be followed here.

Capacitor

Consider the circuit shown in the diagram in which an AC voltage, $V_0\sin(\omega t)$, is applied across a capacitor. Capacitance, by definition, is $C = \frac{q}{V}$ where q is the charge on the capacitor and V is the potential difference across it. This can be rearranged to give

$$q = C\,V$$

Differentiate with respect to time: $\frac{dq}{dt} = I = C\,\frac{dV}{dt}$

where the current, I, is given by the rate of change of charge.

But V is given by the power supply so that $V = V_0 \sin(\omega t)$

$$\therefore\ I = C\,\frac{d}{dt}(V_0 \sin(\omega t)) = \omega C V_0 \cos(\omega t) = \omega C V_0 \sin\left(\omega t + \frac{\pi}{2}\right)$$

where the current is expressed using a sine for direct comparison with the voltage. It can be seen that the phase of the current is $\pi/2$ in advance of the voltage phase. An excellent animation of this can be seen at http://www.animations.physics.unsw.edu.au/jw/AC.html

From the above equation, the current amplitude $= I_0 = \omega\,C\,V_0$

Comparison with Ohm's law, $I = \frac{V}{R}$, shows that the 'resistance' of the capacitor is $1/\omega C$. As resistance applies to resistors, this requires a different name and it is referred to as 'impedance'.

Mathematically, the voltage can be represented in amplitude and phase by a complex number as shown in the diagram. The relative phase of the current is $\pi/2$ in advance of this, which is achieved by multiplying the voltage by j $\left(= \sqrt{-1}\right)$. This is also shown in the diagram. Note that multiplying any complex number by j rotates it by $\pi/2$ in the complex plane. We can express this as $I \propto j\,V$ or $V \propto -j\,I$. Rewriting Ohm's law ($V = IR$) with this additional behaviour gives $V = \frac{-j\,I}{\omega\,C}$. Therefore, the impedance of the capacitor is more completely represented by $\frac{-j}{\omega\,C}$ or $\frac{1}{j\,\omega\,C}$.

Inductor

Faraday's law of electromagnetic induction relates the emf in an inductor to the rate of change of current: $\mathcal{E} = -L\,\frac{dI}{dt}$ where L is the inductance. The direction of the emf opposes changes in the inductor current, hence the minus sign.

Consider the circuit shown in the diagram in which an AC voltage, $V_0\sin(\omega t)$, is applied across an inductor. Adding potential differences around the circuit gives

$$V_0 \sin(\omega t) - L\,\frac{dI}{dt} = 0$$

Integrate with respect to time: $\int V_0 \sin(\omega t)\, dt \;=\; \int L \frac{dI}{dt}\, dt \;=\; \int L\, dI$

$$\therefore\; -\frac{V_0 \cos(\omega t)}{\omega} \;=\; L\, I$$

where the constants of integration are all zero as there is no DC emf.

$$\therefore\; I \;=\; -\frac{V_0 \cos(\omega t)}{\omega\, L} \;=\; \frac{V_0}{\omega\, L} \sin\left(\omega t - \frac{\pi}{2}\right)$$

where the current is expressed using a sine function for direct comparison with the voltage. It can be seen that the phase of the current lags behind the voltage phase by $\pi/2$.

From the above equation, the current amplitude $= I_0 = \frac{V_0}{\omega\, L}$ and comparison with Ohm's law, $I = \frac{V}{R}$, shows that the impedance of the inductor is ωL. Using the same mathematical logic as with the capacitor, the inductor impedance is more completely represented by $j\omega L$.

LRC in series

Determine the complex impedance of the circuit elements L, R and C in series:

using a phasor diagram: For the given current, I, include in the phasor diagram the voltages across the circuit elements L, R and C. V_R is in phase with I, V_C lags I by 90^o and V_L leads I by 90^o. Now add the voltages vectorially:

Pythagoras' theorem gives $V^2 = (V_L - V_C)^2 + V_R^2 = (I\omega L - I/\omega C)^2 + (IR)^2$

$$\therefore\; (impedance)^2 \;=\; \frac{V^2}{I^2} \;=\; Z^2 \;=\; \left(\omega L - \frac{1}{\omega C}\right)^2 + R^2$$

The voltage leads the current by ϕ where $\tan\phi \;=\; \frac{V_L - V_C}{V_R} \;=\; \frac{\omega L - {}^1\!/_{\omega C}}{R}$

using complex numbers: The total impedance is the sum of the impedances of the individual circuit elements: $Z \;=\; j\omega L + R + \frac{1}{j\omega C} \;=\; R + j\left(\omega L - \frac{1}{\omega C}\right)$

The voltage leads the current by ϕ where $\tan\phi \;=\; \frac{imaginary\ part\ of\ Z}{real\ part\ of\ Z} \;=\; \frac{\omega L - {}^1\!/_{\omega C}}{R}$

Note that the magnitude of the complex impedance Z is the same as the magnitude of Z obtained from the phasor diagram.

Example

Calculate the complex impedance at 50Hz with L = 650mH, R = 250Ω and C = 1.5μF.

$$Z \;=\; 250 + j\left(2\pi 50 \times 0.650 - {}^1\!/_{2\pi 50 \times 1.5 \times 10^{-6}}\right) \;=\; 250 - j1918\Omega$$

The voltage leads the current by $\arctan\left(-\frac{1918}{250}\right) \;=\; -82.6^o$, i.e. the voltage lags the current by 82.6^o.

LRC in parallel

Determine the complex impedance of the circuit elements L, R and C in parallel:

Recall that the resistance of resistors, R_1 and R_2, in parallel is given by $\frac{1}{R} \;=\; \frac{1}{R_1} + \frac{1}{R_2}$

The same formula is used for complex impedances, so the impedance, Z, of L, R and C in parallel is given by: $\frac{1}{Z} = \frac{1}{j\omega L} + \frac{1}{R} + j\omega C = \frac{R + j\omega L - \omega^2 LRC}{j\omega LR}$

$$\therefore \; Z = \frac{j\omega LR}{R - \omega^2 LRC + j\omega L} = \frac{j\omega LR(R - \omega^2 LRC - j\omega L)}{(R - \omega^2 LRC)^2 + \omega^2 L^2} = \frac{\omega^2 L^2 R + j\omega LR^2(1 - \omega^2 LC)}{R^2(1 - \omega^2 LC)^2 + \omega^2 L^2}$$

Example

Calculate the complex impedance at 50Hz with L = 650mH, R = 250Ω and C = 1.5μF.

Putting these values into the above formula and using a calculator gives Z = 112.4 + j124.4Ω

The magnitude of Z is: $|Z| = \sqrt{112.4^2 + 124.4^2} = 167.6\Omega$

The voltage leads the current by $\phi = \arctan\left(\frac{124.4}{112.4}\right) = 48^o$

Oscillations in an LRC circuit

The circuit in the diagram has an inductor, a resistor and a capacitor in series. The capacitor is charged and the switch is closed. What is the subsequent behaviour of the circuit?

Add the potentials across each element: $L\frac{dI}{dt} + IR + \frac{q}{C} = 0$

But current is rate of flow of charge: $\therefore \; I = \frac{dq}{dt}$ and $\frac{dI}{dt} = \frac{d^2q}{dt^2}$

So the sum of potentials becomes: $\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$

Compare this with the equation of motion of the damped harmonic oscillator:

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega^2 x = 0$$

These equations are of the same mathematical form and so both describe oscillations. It can be seen that the LRC circuit supports electrical oscillations with a frequency of

$$\omega = \frac{1}{\sqrt{LC}}$$

The damping is caused by resistance and inductance corresponds to mass. The frequency of oscillation coincides with the imaginary components of impedance cancelling, i.e.

$$j\omega L + \frac{1}{j\omega C} = 0 \quad \Longrightarrow \quad \omega L = \frac{1}{\omega C} \quad \Longrightarrow \quad \omega^2 = \frac{1}{LC}$$

The impedance of the circuit becomes a pure resistance at this frequency so the voltage and current are exactly in phase.