Calculus of functions of more than one variable

Nine lectures for the Maths II course given to
first year physics students in the Spring Term.

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About the course

This course shows you how to differentiate and integrate functions of several variables. It is presented as an extension of the calculus you already know which deals with a single variable. It is vital mathematics for physicists since we live in a 3-dimensional world, giving us at least three variables – x, y and z – to describe it. You will make good use of this maths in the mathematical physics courses in the second year.

> The great book of nature is written in the language of mathematics.
> **Galileo Galilei**

http://www-users.york.ac.uk/~pm1/PMweb/teaching.htm

Partial differentiation

So far, you can differentiate functions of one variable. For functions of several variables, we may proceed as follows:

For a cylinder of radius r and height h, the volume is $V = \pi r^2 h$

If h is constant, the rate of change of V wrt r is $\dfrac{dV}{dr} = 2\pi r h$

If r is constant, the rate of change of V wrt h is $\dfrac{dV}{dh} = \pi r^2$

If both h and r are variables, we can still use these results and designate the derivatives as partial derivatives.

These are written as:

If $V = \pi r^2 h$ where r and h are variables, then $\left(\dfrac{\partial V}{\partial r}\right)_h = 2\pi r h$ and $\left(\dfrac{\partial V}{\partial h}\right)_r = \pi r^2$

The operation we have carried out is that of partial differentiation. The recipe is easy – to differentiate partially wrt one variable, regard all other variables as constants.

Notation

Functions: $z = f(x, y)$ $z(x, y)$

Partial derivatives: $\left(\dfrac{\partial f}{\partial x}\right)_y$ $\left(\dfrac{\partial f}{\partial y}\right)_x$ $\left(\dfrac{\partial z}{\partial x}\right)_y$ $\left(\dfrac{\partial z}{\partial y}\right)_x$

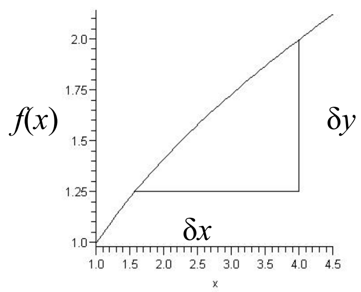
A more shorthand notation is often adopted:

$$\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \qquad f_x \quad f_y \qquad \frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y} \qquad z_x \quad z_y$$

Definition

For a function of one variable:

if $y = f(x)$ then $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x}$



Similarly, we can write for $f(x, y)$:

$$\left(\frac{\partial f}{\partial x}\right)_y = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

$$\left(\frac{\partial f}{\partial y}\right)_x = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

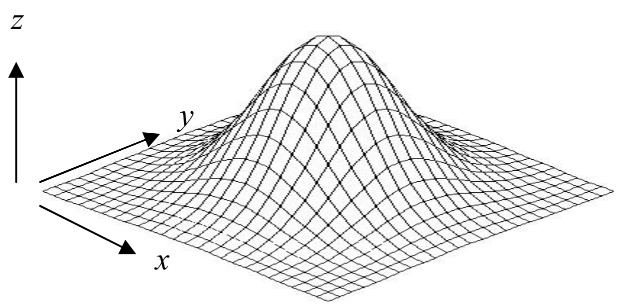
i.e. partial differentiation is achieved by differentiating wrt one variable while keeping the others constant. The generalisation to functions of any number of variables should be obvious.

Geometrical interpretation

$z = f(x, y)$ gives a surface, i.e. the value of the function is plotted in the z-direction.

$\dfrac{\partial f}{\partial x}$ gives the slope of the lines at constant y

$\dfrac{\partial f}{\partial y}$ gives the slope of the lines at constant x



Example If $f(x,y) = x^2 e^{a y} \cos(b y)$, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$

Solution:

$$\frac{\partial f}{\partial x} = 2 x e^{a y} \cos(b y)$$

$$\frac{\partial f}{\partial y} = x^2 \left(a e^{a y} \cos(b y) - b e^{a y} \sin(b y)\right) = x^2 e^{a y} \left(a \cos(b y) - b \sin(b y)\right)$$

Caution!

If something can go wrong it will. Here's an example of where you can easily make a big mistake in partial differentiation.

The relationships between Cartesian (x, y) and polar coordinates (r, θ) will give us some simple expressions to differentiate:

$$x = r\cos(\theta) \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin(\theta) \qquad \theta = \arctan\left(y/x\right)$$

$$\frac{\partial x}{\partial r} = \cos(\theta) \qquad \frac{\partial r}{\partial x} = \frac{1}{2} 2x(x^2+y^2)^{-1/2} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{r\cos(\theta)}{r} = \cos(\theta)$$

It appears, therefore, that $\dfrac{\partial x}{\partial r} = \dfrac{\partial r}{\partial x} = \cos(\theta)$ **(!!)**

The paradox occurs because of <u>slack notation</u>.

Let us use the complete symbol for the partial derivatives:

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \cos(\theta) \qquad \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{\sqrt{x^2+y^2}} = \cos(\theta)$$

and there is no reason to have any particular relationship between these two derivatives since different variables are held constant and different functions have been differentiated.

<u>More correctly:</u> from $x = r\cos(\theta)$ we obtain $r = x\sec(\theta)$

$$\therefore \left(\frac{\partial r}{\partial x}\right)_\theta = \sec(\theta) \quad \text{and we find} \quad \left(\frac{\partial x}{\partial r}\right)_\theta = 1\Big/\left(\frac{\partial r}{\partial x}\right)_\theta \quad \text{as expected.}$$

$\dfrac{\partial x}{\partial r} = 1\Big/\dfrac{\partial r}{\partial x}$ **only if the same variables are held constant during the differentiation**.

Higher order partial derivatives

Just as for functions of one variable, we may define higher order derivatives:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$$

The first order derivatives of $f(x, y)$ are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, so that

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}$$ which can also be written as f_{xx}

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}$$ which can also be written as f_{yy}

Mixed derivatives are also possible because f_x, f_y are both functions of x and y:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\,\partial y}$$ which can also be written as f_{xy}

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\,\partial x}$$ which can also be written as f_{yx}

Fortunately, we have the simplifying fact that $\frac{\partial^2 f}{\partial x\,\partial y}=\frac{\partial^2 f}{\partial y\,\partial x}$, provided that all the derivatives exist and are continuous, i.e. the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ commute.

Other derivatives exist, e.g. $\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)=\frac{\partial^3 f}{\partial x^2\,\partial y}=f_{xxy}$ also $f_{xxy}=f_{xyx}=f_{yxx}$

Example

Show that $f_{xy}=f_{yx}$ for $f(x, y)=x^2\,e^{ay}\cos(by)$

Solution: from a previous example we have $f_x = 2\,x\,e^{a\,y}\cos(b\,y)$

$$\therefore f_{yx} = 2\,x\,a\,e^{a\,y}\cos(b\,y)-2\,x\,e^{a\,y}\,b\sin(b\,y)$$
$$= 2\,x\,e^{a\,y}\left(a\cos(b\,y)-b\sin(b\,y)\right)$$

also $$f_y = x^2\,e^{a\,y}\cos(b\,y)-x^2\,e^{a\,y}\,b\sin(b\,y)$$
$$\therefore f_{xy} = 2\,x\,a\,e^{a\,y}\cos(b\,y)-2\,x\,e^{a\,y}\,b\sin(b\,y)$$
$$= 2\,x\,e^{a\,y}\left(a\cos(b\,y)-b\sin(b\,y)\right)$$

and we find that $f_{xy}=f_{yx}$

Example

Show that $f(x, y)=\sin(ax)\exp(-by)$ satisfies $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ only for $a^2=b^2$.

This is Laplace's equation – a partial differential equation which is very important in mathematical physics. You will meet it again in second year courses.

Solution: $f(x,y) = \sin(a\,x)\exp(-b\,y)$

$$\therefore \frac{\partial f}{\partial x} = a\cos(a\,x)\exp(-b\,y) \qquad \frac{\partial f}{\partial y} = -\,b\sin(a\,x)\exp(-b\,y)$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = -\,a^2\sin(a\,x)\exp(-b\,y) \qquad \frac{\partial^2 f}{\partial y^2} = b^2\sin(a\,x)\exp(-b\,y)$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (b^2 - a^2)\sin(a\,x)\exp(-b\,y) = 0 \text{ when } a^2 = b^2$$

<u>Total differential</u>

Given $f(x, y)$, the amount by which the function changes when both variables change simultaneously is called the total differential and is defined as:

$$df = f(x+dx, y+dy) - f(x,y)$$

Let us express this in terms of partial derivatives.

Since $$\frac{\partial f}{\partial y} = \frac{f(x,y+dy) - f(x,y)}{dy}$$

then $$f(x,y+dy) - f(x,y) = \frac{\partial f}{\partial y}dy \qquad (1)$$

Similarly $$f(x+dx,y+dy) - f(x,y+dy) = \left(\frac{\partial f}{\partial x}\right)_{(x,\,y+dy)} dx$$

but $$\left(\frac{\partial f}{\partial x}\right)_{(x,\,y+dy)} = \left(\frac{\partial f}{\partial x}\right)_{(x,\,y)} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_{(x,\,y)} dy$$

$$\therefore f(x+dx,y+dy) - f(x,y+dy) = \left(\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial y\,\partial x}dy\right)dx \qquad (2)$$

Adding (1) and (2) gives $$f(x+dx,y+dy) - f(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial^2 f}{\partial x\,\partial y}dx\,dy$$

Neglecting second order small quantities then gives

$$\boxed{df = f(x+dx,y+dy) - f(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$$

The first term in the final expression is made from the partial derivative $\partial f/\partial x$, which gives the rate of change of f with x, multiplied by dx, the change in x. The product $\frac{\partial f}{\partial x}dx$ therefore gives the change in f due to the change dx in x. Likewise, the second term $\frac{\partial f}{\partial y}dy$ gives the change in f due to the change dy in y. It is seen that the total change in f (the total differential) is the sum of the changes due to shifts in x and y separately.

Example

Find the change in $f(x, y) = x\,e^{xy}$ when the values of x and y are changed from 2 to 2.02 and 1 to 1.05 respectively.

Solution:
$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$
$$= \left(e^{xy} + x\,y\,e^{xy}\right)\delta x + x^2\,e^{xy}\,\delta y = e^{xy}\left[(1 + x\,y)\delta x + x^2\,\delta y\right]$$

when $(x, y) = (2, 1)$: $\delta f = e^2\,(3\,\delta x + 4\,\delta y)$

when $(\delta x, \delta y) = (0.02, 0.05)$: $\delta f = e^2\,(0.06 + 0.20) = 0.26\,e^2 = \underline{1.92}$

The actual change is: $\delta f = f(2.02, 1.05) - f(2,1) = 16.68 - 14.78 = \underline{1.90}$

Implicit differentiation

If $f(x, y) = 0$ this determines y as a function of x or vice versa, i.e. y is an implicit function of x since it is not defined explicitly. There are two ways of determining dy/dx in this situation, the first of which you may have seen before:

1. differentiate terms in x as normal: $x^3 \to 3\,x^2$

differentiate terms in y as normal: $y^2 \to 2\,y\dfrac{dy}{dx}$

differentiate terms in xy using the product rule: $xy \to y + x\dfrac{dy}{dx}$

example: $2\,x^2\,y + \cos(x\,y) = 0$

differentiating gives
$$4\,x\,y + 2\,x^2\frac{dy}{dx} - \sin(x\,y)\left(y + x\frac{dy}{dx}\right) = 0$$

and an algebraic rearrangement gives
$$\frac{dy}{dx} = \frac{y\sin(x\,y) - 4\,x\,y}{2\,x^2 - x\sin(x\,y)}$$

2. since $f(x, y) = 0$ i.e. constant, then $df = 0$ so that
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \quad \text{giving} \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

example:
$$f(x, y) = 2\,x^2\,y + \cos(x\,y) = 0$$
$$\therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{4\,x\,y - y\sin(x\,y)}{2\,x^2 - x\sin(x\,y)}$$

which is an easier calculation than the first one.

Function of a function

For functions of a single variable, we already know that for $y(x)$ and $x(t)$ then
$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

This is sometimes called the chain rule. It can be extended to the case where we have a function of several variables.

Let $f(u)$ and $u(x, y)$ then: $f_x = \left(\frac{\partial f}{\partial x}\right)_y = \frac{df}{du}\left(\frac{\partial u}{\partial x}\right)_y$ and $f_y = \left(\frac{\partial f}{\partial y}\right)_x = \frac{df}{du}\left(\frac{\partial u}{\partial y}\right)_x$

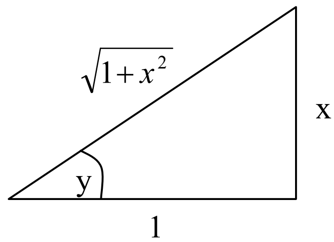
Example If $f(x, y) = \arctan\left(\frac{y}{x}\right)$ find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Before solving the problem, let us make sure we can differentiate the arctan function. If you remember the standard integral $\int \frac{dx}{1+x^2} = \arctan(x) + C$, then differentiating both sides gives immediately $\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$

Another way of doing it is to let $y = \arctan(x)$ then $x = \tan(y)$

$$\therefore \frac{dx}{dy} = \sec^2(y) = 1 + x^2$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$



Solution: Use this result to solve the original problem:

Let $u = y/x$ then we have $f(u) = \arctan(u)$ and f is a function of a single variable.

We also have $\frac{\partial u}{\partial x} = -\frac{y}{x^2}$ and $\frac{\partial u}{\partial y} = \frac{1}{x}$

Therefore $\left(\frac{\partial f}{\partial x}\right)_y = \frac{df}{du}\left(\frac{\partial u}{\partial x}\right)_y = \frac{d}{du}(\arctan(u))\left(\frac{\partial u}{\partial x}\right)_y = \frac{1}{1+u^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$

Also $\left(\frac{\partial f}{\partial y}\right)_x = \frac{d}{du}(\arctan(u))\left(\frac{\partial u}{\partial y}\right)_x = \frac{1}{1+u^2}\frac{1}{x} = \frac{x}{x^2+y^2}$

Example Given $f(x^2 - ay)$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Solution: We cannot differentiate the expression completely because the function f is not given. However, we can make some progress towards the derivatives as follows.

Let $u = x^2 - ay$ then the function becomes $f(u)$ and $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -a$

Then $\frac{\partial f}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = 2x\frac{df}{du}$ and $\frac{\partial f}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y} = -a\frac{df}{du}$

Example Show that $\phi(x,t) = f(x\text{-}vt)$ satisfies the equation $\frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \varphi}{\partial t^2}$ where f is an arbitrary differentiable function and v is a constant.

Solution: Using the same mathematical ideas as in the previous example, the power of calculus is demonstrated when we show that the equation is satisfied by an unknown function f. The equation, known as the wave equation, is an important one in mathematical physics as it describes the behaviour of many kinds of wave.

Let $y = x - vt$ then $\dfrac{\partial y}{\partial x} = 1$ and $\dfrac{\partial y}{\partial t} = -v$ also $\phi(x, t)$ becomes $f(y)$

$$\frac{\partial \varphi}{\partial x} = \frac{df}{dy}\frac{\partial y}{\partial x} = \frac{df}{dy} \qquad \frac{\partial \varphi}{\partial t} = \frac{df}{dy}\frac{\partial y}{\partial t} = -v\frac{df}{dy}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{d}{dy}\left(\frac{df}{dy}\right)\frac{\partial y}{\partial x} = \frac{d^2 f}{dy^2} \qquad \frac{\partial^2 \varphi}{\partial t^2} = -v\frac{d}{dy}\left(\frac{df}{dy}\right)\frac{\partial y}{\partial t} = v^2\frac{d^2 f}{dy^2}$$

$$\therefore \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \varphi}{\partial t^2}$$

Chain rule

If we have $f(x, y)$ given that $x(t)$ and $y(t)$, then f may be expressed as a function of t alone.

The complete derivative df/dt therefore exists and it can be expressed in terms of partial derivatives as follows.

The total differential of f is $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

It therefore follows that $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

i.e. two terms of the form previously given for a function of a single variable.

Example

If $f(x, y) = x e^{y}$ and $x = 2t$, $y = 1 - t^2$, determine df/dt two different ways.

Solution: 1) Express f as a function of t, then

$$\frac{df}{dt} = \frac{d}{dt}\left(2t e^{1-t^2}\right) = 2e^{1-t^2} + 2t(-2t)e^{1-t^2} = 2e^{1-t^2}(1-2t^2)$$

2) Use the chain rule

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = e^{y}\,2 + x e^{y}(-2t) = 2e^{1-t^2}(1-2t^2)$$

Chain rule again

A more general expression of the chain rule becomes necessary when we have $f(x, y)$ where $x(u, v)$ and $y(u, v)$.

Usually written as:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

and

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Using full symbols:

$$\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x\left(\frac{\partial y}{\partial u}\right)_v$$

$$\left(\frac{\partial f}{\partial v}\right)_u = \left(\frac{\partial f}{\partial x}\right)_y\left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x\left(\frac{\partial y}{\partial v}\right)_u$$

Differential operators

You have been using differential operators for some time even if you haven't called them by that name. For example, to differentiate $f(x)$ you apply the operator $\dfrac{d}{dx}$ to produce the result $\dfrac{df}{dx}$. This may be written as $\dfrac{d}{dx}(f) \to \dfrac{df}{dx}$.

Similarly, $\dfrac{d}{dx}\left(\dfrac{df}{dx}\right) \to \dfrac{d^2 f}{dx^2}$. You can apply the operator twice as in $\dfrac{d^2}{dx^2}(f) \to \dfrac{d^2 f}{dx^2}$.

For $f(x, y)$: $\qquad \dfrac{\partial}{\partial x}(f) \to \dfrac{\partial f}{\partial x}$ $\qquad$ and $\qquad \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) \to \dfrac{\partial^2 f}{\partial y\,\partial x}$

More complicated operators can arise for $f(x, y)$ with $x(u, v)$ and $y(u, v)$.

The chain rule gives $\qquad \dfrac{\partial f}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$

and this may be written in terms of differential operators as

$$\left(\frac{\partial}{\partial u}\right) f = \left(\frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y}\right) f$$

The left-hand operator is to be used when f is in terms of u and v; the right-hand operator is to be used when f is in terms of x and y. The operators perform the same operation and are therefore equivalent. This may be written as

$$\frac{\partial}{\partial u} \equiv \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y}$$

The right-hand operator tells you to differentiate partially wrt x and multiply by $\partial x/\partial u$ then add this to the partial derivative wrt y multiplied by $\partial y/\partial u$.

Example

If $f(x, y) = xy + 2y^3$ where $x = uv$ and $y = u - v$, find $\partial f/\partial u$ using two different operators.

Solution: **1.** Substitute for x and y in $f(x, y)$ and use the operator $\partial/\partial u$:

$$f(u,v) = u\,v(u-v) + 2(u-v)^3 \qquad \text{giving} \qquad \frac{\partial f}{\partial u} = v(u-v) + u\,v + 6(u-v)^2$$

2. Leave f in terms of x and y and use the operator $\dfrac{\partial x}{\partial u}\dfrac{\partial}{\partial x} + \dfrac{\partial y}{\partial u}\dfrac{\partial}{\partial y}$:

The partial derivatives in the operator are $\dfrac{\partial x}{\partial u} = v$ and $\dfrac{\partial y}{\partial u} = 1$

So $\dfrac{\partial}{\partial u} f = \left(\dfrac{\partial x}{\partial u}\dfrac{\partial}{\partial x} + \dfrac{\partial y}{\partial u}\dfrac{\partial}{\partial y}\right)(x\,y + 2\,y^3) = \left(v\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}\right)(x\,y + 2\,y^3)$

$$= v\frac{\partial}{\partial x}(x\,y + 2\,y^3) + \frac{\partial}{\partial y}(x\,y + 2\,y^3) = v(y) + x + 6\,y^2 = v(u-v) + u\,v + 6(u-v)^2$$

Example – determination of differential operators

Given that $x = u^2 - v^2$ and $y = 2uv$, express the operators $\partial/\partial x$ and $\partial/\partial y$ in terms of u and v.

Solution: The chain rule gives $\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y}$

The partial derivatives are: $\frac{\partial x}{\partial u} = 2u;\ \frac{\partial x}{\partial v} = -2v;\ \frac{\partial y}{\partial u} = 2v;\ \frac{\partial y}{\partial v} = 2u$

so the operators become:

$$\left.\begin{aligned}\frac{\partial}{\partial u} &= 2u\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial y}\\ \frac{\partial}{\partial v} &= -2v\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}\end{aligned}\right\}$$

solve these equations for $\partial/\partial x$ and $\partial/\partial y$

You can use any method you like to solve the equations, but the recommended method is to use Cramer's rule:

$$\frac{\partial/\partial x}{\begin{vmatrix}\partial/\partial u & 2v\\ \partial/\partial v & 2u\end{vmatrix}} = \frac{\partial/\partial y}{\begin{vmatrix}2u & \partial/\partial u\\ -2v & \partial/\partial v\end{vmatrix}} = \frac{1}{\begin{vmatrix}2u & 2v\\ -2v & 2u\end{vmatrix}}$$

therefore $$\frac{\partial}{\partial x} = \frac{2u\,\partial/\partial u - 2v\,\partial/\partial v}{4u^2 + 4v^2} = \frac{1}{2\left(u^2+v^2\right)}\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right)$$

and $$\frac{\partial}{\partial y} = \frac{2u\,\partial/\partial v + 2v\,\partial/\partial u}{4u^2 + 4v^2} = \frac{1}{2\left(u^2+v^2\right)}\left(v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right)$$

Cramer's rule

If you haven't seen Cramer's rule before, here it is as applied above. It is recommended as it needs less algebraic manipulation than other methods when applied to a pair of simultaneous equations. For the equations:

$$a_{11}\,x + a_{12}\,y = b_1$$
$$a_{21}\,x + a_{22}\,y = b_2$$

the solution is expressed in terms of determinants as:

$$\frac{x}{\begin{vmatrix}b_1 & a_{12}\\ b_2 & a_{22}\end{vmatrix}} = \frac{y}{\begin{vmatrix}a_{11} & b_1\\ a_{21} & b_2\end{vmatrix}} = \frac{1}{\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}}$$

The unknowns are x and y and the pattern of determinants is:

The last determinant, call it Δ, contains the array of left hand side coefficients.
The determinant for x (first unknown) is Δ with the first column replaced by the rhs.
The determinant for y (2nd unknown) is Δ with the 2nd column replaced by the rhs.

Comment

Can we avoid having to solve equations by doing the calculation in a different way? For example, for $f(u, v)$ with $u(x, y)$ and $v(x, y)$, the chain rule gives

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \quad \text{so the operators are} \quad \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}$$

giving $\partial/\partial x$ directly with an equivalent expression for $\partial/\partial y$. However, **don't be fooled**. The derivative $\partial u/\partial x$ above is <u>not</u> the reciprocal of $\partial x/\partial u$ in the previous example. Here, u is a function of x and y so the full symbol for the derivative is $\left(\frac{\partial u}{\partial x}\right)_y$, whereas in the example x is a function of u and v, i.e. the derivative is $\left(\frac{\partial x}{\partial u}\right)_v$.

To obtain $(\partial u/\partial x)_y$, the equations $x = u^2 - v^2$ and $y = 2uv$ will have to be solved for u and v to express them as functions of x and y. This is a considerably more difficult task than solving the equations in the example, so you can't win this way.

Change of variables – Cartesian to polar

Show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$ where f is a function of x and y (or the equivalent function of r and θ) and $x = r\cos(\theta)$, $y = r\sin(\theta)$, i.e. the expression is changed from Cartesian to polar coordinates. The expression itself is part of Laplace's equation.

From the chain rule:

$$\left.\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos(\theta)\frac{\partial f}{\partial x} + \sin(\theta)\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin(\theta)\frac{\partial f}{\partial x} + r\cos(\theta)\frac{\partial f}{\partial y}\end{aligned}\right\}$$

Solve these equations for $\partial f/\partial x$ and $\partial f/\partial y$ to give:

$$\frac{\partial f}{\partial x} = \cos(\theta)\frac{\partial f}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial f}{\partial \theta} \quad \text{and} \quad \frac{\partial f}{\partial y} = \sin(\theta)\frac{\partial f}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial f}{\partial \theta}$$

These expressions also provide the operators needed to obtain the second order derivatives:

$$\frac{\partial^2 f}{dx^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left(\cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(\cos(\theta)\frac{\partial f}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial f}{\partial \theta}\right)$$

Expand the operator:

$$= \cos(\theta)\frac{\partial}{\partial r}\left(\cos(\theta)\frac{\partial f}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial f}{\partial \theta}\right) - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\left(\cos(\theta)\frac{\partial f}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial f}{\partial \theta}\right)$$

Now differentiate, remembering to use the product rule where it applies:

$$= \cos^2(\theta)\frac{\partial^2 f}{\partial r^2} + \frac{\sin(\theta)\cos(\theta)}{r^2}\frac{\partial f}{\partial \theta} - \frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial^2 f}{\partial r\,\partial \theta}$$

$$+ \frac{\sin^2(\theta)}{r}\frac{\partial f}{\partial r} - \frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial^2 f}{\partial \theta\,\partial r} + \frac{\sin(\theta)\cos(\theta)}{r^2}\frac{\partial f}{\partial \theta} + \frac{\sin^2(\theta)}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

Collect like terms and put them in a logical order:

$$\frac{\partial^2 f}{\partial x^2} = \cos^2(\theta)\frac{\partial^2 f}{\partial r^2} - \frac{\sin(2\theta)}{r}\frac{\partial^2 f}{\partial r\,\partial\theta} + \frac{\sin^2(\theta)}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\sin^2(\theta)}{r}\frac{\partial f}{\partial r} + \frac{\sin(2\theta)}{r^2}\frac{\partial f}{\partial\theta}$$

A similar analysis gives the expression for $\partial^2 f/\partial y^2$:

$$\frac{\partial^2 f}{\partial y^2} = \sin^2(\theta)\frac{\partial^2 f}{\partial r^2} + \frac{\sin(2\theta)}{r}\frac{\partial^2 f}{\partial r\,\partial\theta} + \frac{\cos^2(\theta)}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\cos^2(\theta)}{r}\frac{\partial f}{\partial r} - \frac{\sin(2\theta)}{r^2}\frac{\partial f}{\partial\theta}$$

Adding them together gives the required result:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}$$

Taylor series for a function of two variables

For a function of a single variable, we have:

$$f(a+h) = f(a) + f'(a)\,h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \cdots + \frac{f^{(n)}(a)}{n!}h^n + \cdots$$

or alternatively:

$$f(x) = f(a) + f'(a)(x-a)\ \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

These two expressions are the same with $x = a + h$. Functions of several variables can be treated in a similar way. For two variables, define the differential operator $D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$. The Taylor expansion of $f(x, y)$ about the point (a, b) is then:

$$f(a+h, b+k) = f(a,b) + Df(a,b) + \frac{1}{2!}D^2 f(a,b) + \frac{1}{3!}D^3 f(a,b) + \cdots + \frac{1}{n!}D^n f(a,b) + \cdots$$

where $D^n f(a, b)$ means apply the differential operator n times to $f(x, y)$ and evaluate the result at the point (a, b).

Let us look at D^2: $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) = h^2\frac{\partial^2}{\partial x^2} + k\,h\frac{\partial^2}{\partial y\,\partial x} + h\,k\frac{\partial^2}{\partial x\,\partial y} + k^2\frac{\partial^2}{\partial y^2}$

$$= h^2\frac{\partial^2}{\partial x^2} + 2\,h\,k\frac{\partial^2}{\partial x\,\partial y} + k^2\frac{\partial^2}{\partial y^2}$$

This is the same mathematical form as a binomial expansion.

Writing out all the terms in the Taylor expansion up to order 3 gives:

$$f(a+h, b+k) = f(a,b) + f_x(a,b)\,h + f_y(a,b)\,k + \frac{1}{2!}\left[f_{xx}(a,b)\,h^2 + 2\,f_{xy}(a,b)\,hk + f_{yy}(a,b)\,k^2\right]$$

$$+ \frac{1}{3!}\left[f_{xxx}(a,b)\,h^3 + 3\,f_{xxy}(a,b)\,h^2\,k + 3\,f_{xyy}(a,b)\,h\,k^2 + f_{yyy}(a,b)\,k^3\right] + \cdots$$

The alternative form is obtained by putting $x = a + h,\ y = b + k$:

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2!}(x-a)^2$$

$$+ f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2!}(y-b)^2 + \cdots$$

Example

Expand $f(x, y) = \exp(x \sin(y))$ as a power series about the point $(1, \pi/2)$ to terms of the second degree. Hence obtain an approximate value for $f(1.1, \pi/3)$.

$f = \exp(x \sin(y))$	$f(1, \pi/2) = e$
$f_x = \sin(y) \exp(x \sin(y))$	$f_x(1, \pi/2) = e$
$f_y = x \cos(y) \exp(x \sin(y))$	$f_y(1, \pi/2) = 0$
$f_{xx} = \sin^2(y) \exp(x \sin(y))$	$f_{xx}(1, \pi/2) = e$
$f_{xy} = \cos(y) \exp(x \sin(y)) + x \cos(y) \sin(y) \exp(x \sin(y))$	$f_{xy}(1, \pi/2) = 0$
$f_{yy} = -x \sin(y) \exp(x \sin(y)) + x^2 \cos^2(y) \exp(x \sin(y))$	$f_{yy}(1, \pi/2) = -e$

The Taylor series about $(1, \pi/2)$ is:

$$f(x,y) = f(1,\pi/2) + f_x(1,\pi/2)(x-1) + f_y(1,\pi/2)(y-\pi/2) + \frac{1}{2} f_{xx}(1,\pi/2)(x-1)^2$$

$$+ f_{xy}(1,\pi/2)(x-1)(y-\pi/2) + \frac{1}{2} f_{yy}(1,\pi/2)(y-\pi/2)^2$$

Putting in the values of the derivatives gives:

$$\exp(x\sin(y)) = e + (x-1)e + (y-\pi/2)0 + \frac{1}{2}(x-1)^2 e + (x-1)(y-\pi/2)0 - \frac{1}{2}(y-\pi/2)^2 e$$

$$= e\left[1 + x - 1 + \frac{1}{2}(x-1)^2 - \frac{1}{2}(y-\pi/2)^2\right] = \frac{e}{2}\left[x^2 + 1 - (y-\pi/2)^2\right]$$

For $x = 1.1$ and $y = \pi/3$, the approximate value of $f(1.1, \pi/3) = \frac{e}{2}\left[2.21 - \frac{\pi^2}{36}\right] = 2.63$

The correct value of $f(1.1, \pi/3) = \exp\left(1.1\sin(\pi/3)\right) = 2.60$

Useful property

For functions of a single variable:

If $f(x)$ can be written as $P(x)Q(x)$ then
Taylor expansion of $f(x)$ = [Taylor expansion of $P(x)$] [Taylor expansion of $Q(x)$]

For a function of two variables:
If $f(x,y)$ can be written as $P(x)Q(y)$ then
Taylor expansion of $f(x,y)$ = [Taylor expansion of $P(x)$] [Taylor expansion of $Q(y)$]

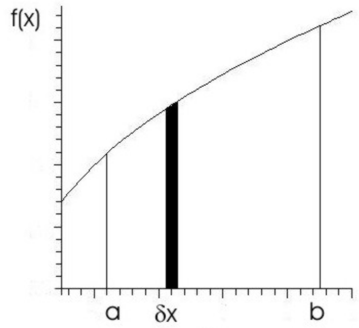
Integration

Recall that the area under a curve is given by integration:

Element of area = $f(x)\,\delta x$

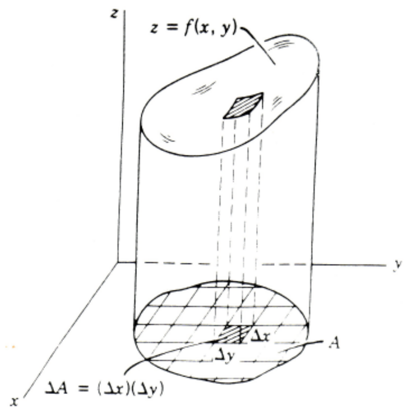
$$\text{Total area} = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x$$

In the limit as $\delta x \to 0$, total area $= \int_a^b f(x)\,dx$



Double integrals

Let us extend this idea to give an expression for the volume under a surface. The function $z = f(x, y)$ gives the surface and the volume between this and a given region in the xy-plane is the volume to be calculated. The volume associated with each element of area in the xy-plane, $\delta x\,\delta y$, is $f(x, y)\,\delta x\,\delta y$ and the sum of these gives the desired volume. However, this is done in a very systematic manner.



Let the volume be bounded by:

Top $z = f(x, y)$
Bottom xy-plane
Sides curve ABCD

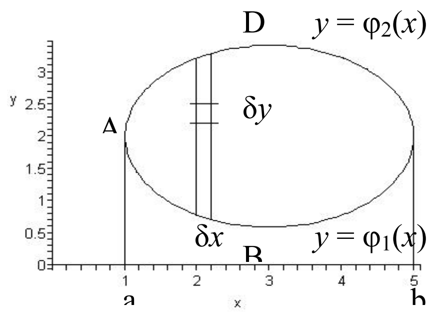
Let the curve ABC be $y = \phi_1(x)$
Let the curve ADC be $y = \phi_2(x)$

Now take an elementary slice of constant x.

Element of volume = $f(x, y)\,\delta x\,\delta y$

Therefore, volume of elementary slice

$$= \lim_{\delta y \to 0} \sum_{y=\phi_1(x)}^{\phi_2(x)} f(x,y)\,\delta x\,\delta y = \left\{ \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right\} \delta x$$



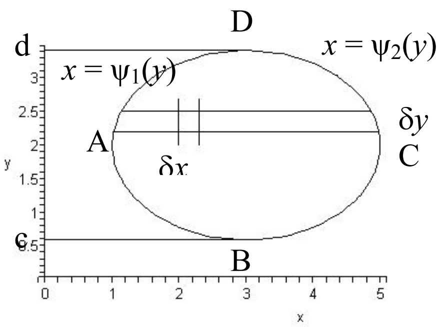
$$\text{So total volume under surface} = \lim_{\delta x \to 0} \sum_{x=a}^{b} \left\{ \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right\} \delta x = \int_a^b \left\{ \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right\} dx$$

In this double integral, note that the first integration is over y where x is treated as a constant. Double integrals are not normally written in this way, but are more commonly expressed without brackets as:

$$\int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \quad \text{or} \quad \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

In the above analysis, it is possible to interchange the roles of x and y to obtain a second double integral, equal in value to the first. Taking elementary slices of constant y leads to the double integral:

Volume under surface = $\int_{c}^{d} dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx$



where the volume is bounded on the left by $x = \psi_1(y)$ and on the right by $x = \psi_2(y)$.

We can therefore equate the two double integrals:

$$\int_{a}^{b} dx \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy = \int_{c}^{d} dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx$$

and we have **changed the order of integration**.

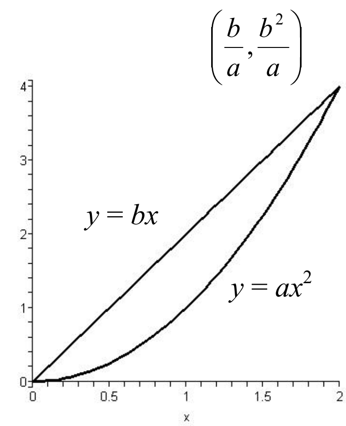
On the left, we integrate over *y* first, holding *x* constant, then integrate over *x*.
On the right, integrate over *x* first, holding *y* constant, then integrate over *y*.

Note that the limits on the two double integrals are very different. However, they both describe the **same field of integration**.

Changing the order of integration

You can only safely deduce the limits on a double integral by drawing the field of integration. Take the following integral as an example.

$$\int_{0}^{b/a} dx \int_{ax^2}^{bx} f(x,y)\,dy$$

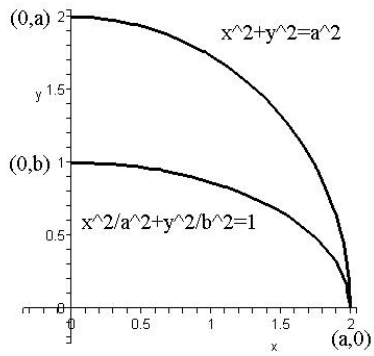


The limits on the integration over *y* show that the field of integration is bounded by the lines $y = ax^2$ and $y = bx$. Therefore, the field of integration is as shown in the diagram, with *x* going from 0 to b/a, as given by the limits on the integration over *x*.

On integrating over x first, the diagram shows that *x* goes from the line $x = y/b$ to $x = \sqrt{y/a}$. These therefore form the limits on the integral and the range of *y* is seen to be $0 < y < b^2/a$. We therefore have:

$$\int_{0}^{b/a} dx \int_{ax^2}^{bx} f(x,y)\,dy = \int_{0}^{b^2/a} dy \int_{y/b}^{\sqrt{y/a}} f(x,y)\,dx$$

In this second example, the double integral has to be split into the sum of two integrals to accommodate the different lower limits when integrating over *x* first.



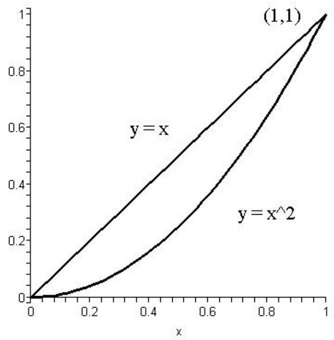
$$\int_{0}^{a} dx \int_{b\left(1-\frac{x^2}{a^2}\right)^{1/2}}^{(a^2-x^2)^{1/2}} f(x,y)\,dy \quad \text{when } b < a$$

$$= \int_{b}^{a} dy \int_{0}^{(a^2-y^2)^{1/2}} f(x,y)\,dx + \int_{0}^{b} dy \int_{a\left(1-\frac{y^2}{b^2}\right)^{1/2}}^{(a^2-y^2)^{1/2}} f(x,y)\,dx$$

It is important that the field of integration should be the same before and after changing the order of integration.

<u>Examples</u> – evaluating double integrals

1. Evaluate $\iint (2x^2 + y)\, dx\, dy$ over the area bounded by $y = x$ and $y = x^2$. Verify that the same result is obtained when the order of integration is reversed.



Solution: The limits on the integrals have to be obtained from a diagram of the field of integration. Integrating over y first:

$$\int_0^1 dx \int_{x^2}^{x} \left(2x^2 + y\right) dy = \int_0^1 \left[2x^2 y + \frac{1}{2} y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(2x^3 + \frac{1}{2}x^2 - 2x^4 - \frac{1}{2}x^4\right) dx = \left[\frac{1}{2}x^4 + \frac{1}{6}x^3 - \frac{5}{2}\frac{1}{5}x^5\right]_0^1 = \frac{1}{6}$$

Integrating over x first:

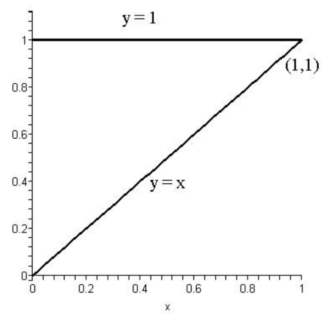
$$\int_0^1 dy \int_{y}^{y^{1/2}} (2x^2 + y)\, dx = \int_0^1 \left[\frac{2}{3}x^3 + x\, y\right]_{y}^{y^{1/2}} dy$$

$$= \int_0^1 \left(\frac{2}{3}y^{3/2} + y^{3/2} - \frac{2}{3}y^3 - y^2\right) dy = \left[\frac{5}{3}\frac{2}{5}y^{5/2} - \frac{1}{6}y^4 - \frac{1}{3}y^3\right]_0^1 = \frac{1}{6}$$

2. Evaluate $\int_0^1 dx \int_x^1 \sin\left(\frac{\pi y^2}{2}\right) dy$

Solution: There is no easy way of performing the integration over y, so consider changing the order of integration. This can only be done by plotting out the field of integration first. The integral becomes

$$\int_0^1 dy \int_0^y \sin\left(\frac{\pi y^2}{2}\right) dx$$



Note that $\sin\left(\frac{\pi y^2}{2}\right)$ is constant in the integration over x so it can be taken out of that integral. The integration over x is now trivial:

$$\int_0^1 \sin\left(\frac{\pi y^2}{2}\right) dy \int_0^y dx = \int_0^1 \sin\left(\frac{\pi y^2}{2}\right) [x]_0^y\, dy = \int_0^1 y \sin\left(\frac{\pi y^2}{2}\right) dy$$

and the integration over y now becomes possible because the integrand has changed.

Use the substitution $u = \frac{\pi y^2}{2}$ then $du = \pi\, y\, dy$ and the integral becomes:

$$\int_0^{\pi/2} \frac{1}{\pi}\sin(u)\,du = \frac{1}{\pi}\left[-\cos(u)\right]_0^{\pi/2} = \frac{1}{\pi}$$

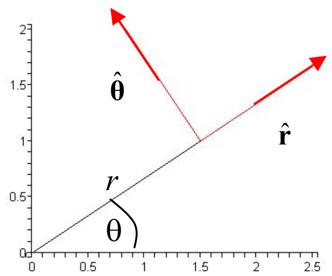
Separation of variables

In the special case where the limits of integration are constant (not functions of the variables) and the integrand $f(x, y)$ is of the form $F(x)G(y)$, we have:

$$\int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \Rightarrow \int_a^b dx \int_c^d F(x)\,G(y)\,dy = \int_a^b F(x)\,dx \int_c^d G(y)\,dy$$

This is the product of two single integrals that can be evaluated independently of each other.

Double integrals in polar coordinates

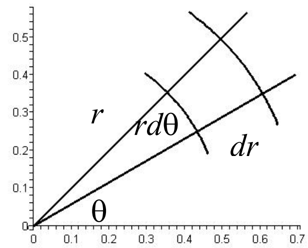


$x = r\cos\theta$; $y = r\sin\theta$

Unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are defined to lie in the directions of increasing r and θ respectively.

Changing from Cartesians to polars in a double integral is an example of substituting for both variables simultaneously.

$dx\,dy$ is an element of area in the xy-plane.



The equivalent element of area in the $r\theta$-plane is:

$(r\,d\theta)\,dr \;=\; r\,dr\,d\theta$

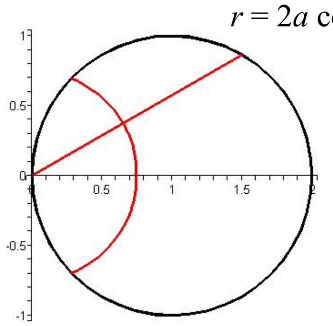
i.e. element of area $=$ $dx\,dy$ $=$ $r\,dr\,d\theta$

Note that $dx\,dy$ is not replaced by $dr\,d\theta$ – the dimensions are wrong!

$$\iint_A f(x,y)\,dx\,dy \;\Rightarrow\; \iint_A f(r,\theta)\,r\,dr\,d\theta$$

Change of order of integration

$r = 2a\cos(\theta)$



$$\int_{-\pi/2}^{\pi/2} d\theta \int_0^{2a\cos\theta} f(r,\theta)\,dr = \int_0^{2a} dr \int_{-\arccos(r/2a)}^{+\arccos(r/2a)} f(r,\theta)\,d\theta$$

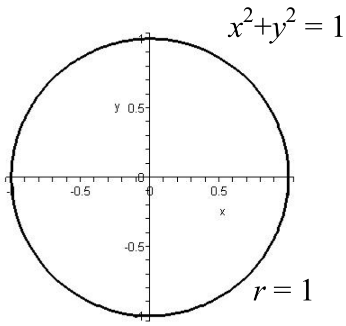
The equation $r = 2a\cos(\theta)$ gives a circle of radius r centred on $(a, 0)$.

Curved line in circle: integrate over θ at constant r.
Straight line in circle: integrate over r at constant θ.

Example

Evaluate $\iint_R \left(1-\sqrt{x^2+y^2}\right)dx\,dy$ where R is the region bounded by $x^2+y^2=1$

Solution: Converting to polar coordinates will simplify both the integrand and the description of the field of integration.



$x = r\cos\theta \qquad y = r\sin\theta \qquad dx\,dy = r\,dr\,d\theta$

$$\int_0^{2\pi} d\theta \int_0^1 (1-r)\,r\,dr = 2\pi\left[\frac{1}{2}r^2 - \frac{1}{3}r^3\right]_0^1 = \frac{\pi}{3}$$

Example

Evaluate $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x^2\,dx\,dy}{\left(1+\sqrt{x^2+y^2}\right)^5}$

Solution: Transform to polar coordinates to simplify the integrand:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad dx\,dy = r\,dr\,d\theta$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x^2\,dx\,dy}{\left(1+\sqrt{x^2+y^2}\right)^5} = \int_0^{2\pi} d\theta \int_0^{\infty}\frac{r^3\cos^2\theta\,dr}{(1+r)^5} = \int_0^{2\pi}\cos^2\theta\,d\theta\int_0^{\infty}\frac{r^3\,dr}{(1+r)^5}$$

There is separation of variables, so this is a product of two single integrals.

Use the identity $\cos 2\theta = 2\cos^2\theta - 1$ and let $1 + r = u$ then $dr = du$

$$\frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta\int_1^{\infty}\frac{(u-1)^3}{u^5}\,du = \frac{1}{2}2\pi\int_1^{\infty}\left(\frac{1}{u^2}-\frac{3}{u^3}+\frac{3}{u^4}-\frac{1}{u^5}\right)du$$

$$= \pi\left[-\frac{1}{u}+\frac{3}{2u^2}-\frac{3}{3u^3}+\frac{1}{4u^4}\right]_1^{\infty} = \pi\left(1-\frac{3}{2}+1-\frac{1}{4}\right) = \frac{\pi}{4}$$

Triple integrals

The mathematical ideas that give us double integrals can easily be extended to triple integrals:

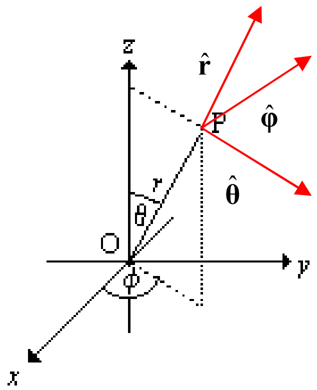
Integrand = $f(x, y, z)$
Field of integration = volume in 3D space
Element of volume = $dx\,dy\,dz = dV$

Repeated integral: $\displaystyle\int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} dy \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\,dz$

In order to deal with triple integrals, we need to look at 3D coordinate systems.

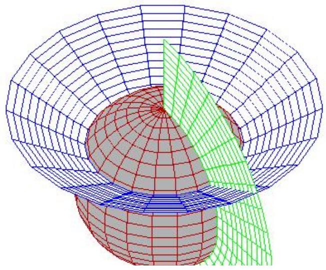
Spherical polar coordinates

A point P at (x, y, z) in Cartesian coordinates is also at (r, θ, ϕ) in spherical polars where



$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$

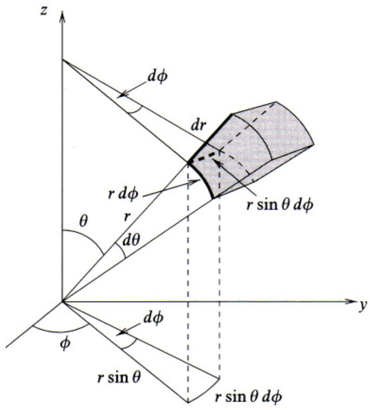
At P, unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ are defined to lie in the directions of increasing r, θ and ϕ respectively. These vectors form a right-handed orthogonal coordinate system at that point.



A surface of constant r is a sphere
A surface of constant θ is a cone
A surface of constant ϕ is a semi-infinite plane

These surfaces intersect at the point (r, θ, ϕ)

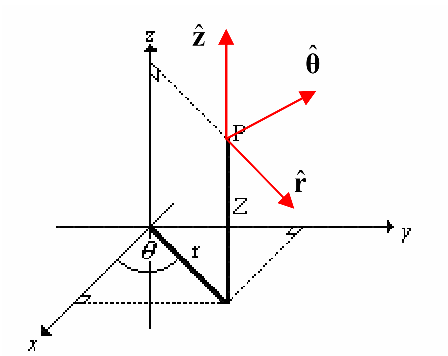
The element of volume is most easily obtained geometrically:



The box in the diagram has sides of dr, $r\,d\theta$ and $r \sin\theta\, d\phi$

This makes the element of volume $= dV$
$= dx\,dy\,dz = dr\,(r\,d\theta)\,(r \sin\theta\, d\phi) = r^2 \sin\theta\, dr\, d\theta\, d\phi$

Cylindrical polar coordinates



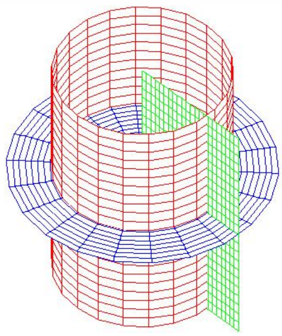
A point P at (x, y, z) in Cartesian coordinates is also at (r, θ, z) in cylindrical polars where

$x = r \cos\theta$

$y = r \sin\theta$

$z = z$

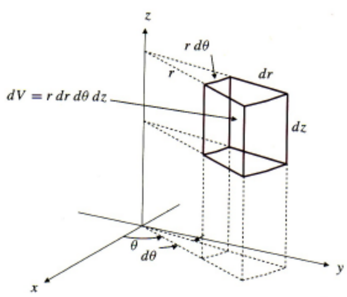
At P, unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$ in the directions of increasing r, θ, z form an orthogonal right-handed system at P.



A surface of constant r is a cylinder
A surface of constant θ is a semi-infinite plane
A surface of constant z is an infinite plane

These surfaces intersect at the point (r, θ, z).

The element of volume is obtained geometrically:

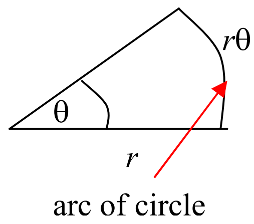


The box in the diagram has sides of dr, $r\,d\theta$ and dz

This makes the element of volume $= dV$
$= dx\,dy\,dz = dr\,(r\,d\theta)\,dz = r\,dr\,d\theta\,dz$

Solid angle

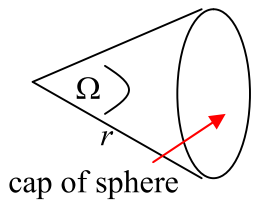
Now that we're into 3D coordinate systems, let us look at the measurement of angles in three-dimensional space. It will be best to start with angles you are familiar with in two-dimensional space:



The diagram shows an arc of a circle of radius r. The length of the arc is $r\theta$. The angle θ is the ratio of the length of the arc to the radius,

i.e. $\theta = \dfrac{length\ of\ arc}{radius} = \dfrac{r\theta}{r}$ radians.

If the arc is lengthened to complete the circle, the angle becomes $\dfrac{circumference\ of\ circle}{radius} = \dfrac{2\pi r}{r} = 2\pi$ radians.



In 3D, angles are called **solid angles**, but they are defined in a similar way to angles in 2D. The diagram shows part of a sphere of radius r, which subtends a solid angle Ω at its centre.

The angle Ω is defined as $\dfrac{area\ of\ spherical\ surface}{(radius\ of\ sphere)^2}$ steradians.

If the surface is extended to a complete sphere, the solid angle at the centre becomes $\dfrac{surface\ area\ of\ the\ sphere}{(radius\ of\ the\ sphere)^2} = \dfrac{4\pi r^2}{r^2} = 4\pi$ steradians.

Solid angles are used in physics, for example, to describe the 3D angle into which a source of radiation may radiate.

Example of a triple integral

A simple illustration of the use of a triple integral is to calculate the volume of a sphere of radius a.

Solution: Use spherical polar coordinates and integrate the element of volume over the sphere.

$$V = \iiint_{sphere} dV = \int_{r=0}^{a} dr \int_{\theta=0}^{\pi} d\theta \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\phi = \int_0^a r^2\, dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{1}{3}a^3 \,.2\,.\,2\pi = \frac{4}{3}\pi a^3$$

Make sure you understand the assignment of the limits:

- The integration over r is along a line from the origin to the surface of the sphere.
- The integration over θ rotates this line about the origin to sweep out the area of a semicircle.
- The integration over ϕ rotates the semicircle about the z-axis to sweep out the volume of the sphere.

The limits on the following triple integral also give the field of integration as a sphere:

$$\int_0^a r^2\, dr \int_0^{2\pi} \sin\theta\, d\theta \int_0^{\pi} d\phi = \frac{1}{3}a^3 .0.\pi = 0$$

Can the volume of the sphere really be zero? In spherical polars, the element of volume, dV, is $r^2 \sin\theta \, dr d\theta d\phi$. In the first integral, the range of θ is from 0 to π so dV is always positive. However, in the second integral, dV is negative over half its range, cancelling out the positive contribution to the integral and ending up with zero. If you intend to have negative volume, then the second integral is perfectly correct, but you need to be aware of what you are doing.

Dimensions in integrals – examples of applications in physics

If $\iint f(x,y)\,dx\,dy$ represents the volume under a surface, what does $\iiint f(x,y,z)\,dx\,dy\,dz$ represent? The double integral only represents a volume if $f(x,y)$, dx and dy all have the dimensions of length. The dimension of the integrand is then (length)^3, which is a volume. We have already used a *triple* integral to determine the volume of a sphere – go back and check that the dimensions are correct.

Not everything is measured in metres and we need to consider the dimensions of the quantities in the integral when applying it to physics. The thing to note is that the symbols dx, dy etc. are not just labels reminding you of the variables over which you perform the integration, but they are also physical quantities with dimensions. Let us look at some examples of how to construct various integrals for applications in physics.

1. A region of space contains an electric charge density of $\rho(x, y, z)$ coulombs/m^3. What is the total charge in a particular volume V?

Charge in element of volume $= \rho(x, y, z)\, dx\, dy\, dz$

Therefore, total charge in volume V $= \iiint_V \rho(x,y,z)\,dx\,dy\,dz$ coulombs.

2. Volume of field of integration $= \iiint_V dV$ m^3.

3. The speed of a particle changes with time as $v(t)$. Find the distance travelled in time T.

Distance travelled in time $dt = v(t)\, dt$

Therefore total distance travelled in time $T = \int_0^T v(t)\,dt$ metres.

4. The density of a thin sheet of material varies as $\rho(x, y)$ kg/m^2. Find its total mass.

Mass of element of area $= \rho(x, y)\, dx\, dy$

Therefore total mass of sheet $= M = \iint_R \rho(x,y)\,dx\,dy$ kg, where R defines the shape of the sheet.

5. The mean density of the sheet in the previous example is obtained by dividing the total mass by the area, i.e. mean density $= \frac{1}{A}\iint_R \rho(x,y)\,dx\,dy$ kg/m^2, where A is the area of the sheet.

Similar integrals will determine the average value of a function in 1D or 3D space:

Average of $f(x)$ between $x = a$ and $x = b$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$

Average of $f(x, y, z)$ within the volume $V = \frac{1}{V}\iiint\limits_V f(x,y,z)\,dx\,dy\,dz$

6. Another important use of integrals is to determine the centre of mass of an object. Using the same thin sheet is in example 4, the mass at (x, y) is $\rho(x, y)\,dx\,dy$ kg so its moment about the y-axis is $x\,\rho(x, y)\,dx\,dy$ kg m. The total moment about the y-axis is therefore $\iint\limits_R x\,\rho(x,y)\,dx\,dy$ and this must be equal to $M\,\bar{x}$ where M is the total mass and $\bar{x}$ is the x-coordinate of the centre of mass. We therefore have for the coordinates of the centre of mass:

$$\bar{x} = \frac{1}{M}\iint\limits_R x\,\rho(x,y)\,dx\,dy \quad \text{and} \quad \bar{y} = \frac{1}{M}\iint\limits_R y\,\rho(x,y)\,dx\,dy$$

7. A slightly more adventurous use of a multiple integral is in the calculation of moment of inertia.
The moment of inertia of a point mass m a distance r from the axis of rotation is mr^2.
For a body of density $\rho(x, y, z)$ kg/m^3, the point mass at (x, y, z) is $\rho(x, y, z)\,dx\,dy\,dz$
If z is the rotation axis, the distance from the axis of the point mass is $\sqrt{x^2+y^2}$
The moment of inertia of the point mass is therefore $(x^2+y^2)\rho(x, y, z)\,dx\,dy\,dz$
Therefore, the moment of inertia of the complete body = $\iiint\limits_V \left(x^2+y^2\right)\rho(x,y,z)\,dx\,dy\,dz$

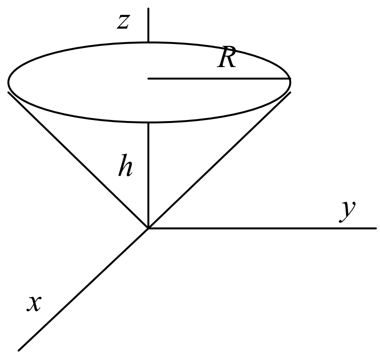
Example

Find the position of the centroid of a uniform solid cone of height h and base radius R.

Solution:
The centroid is the centre of mass of a body of uniform density.
By symmetry, the centroid lies on the cone axis, so only the z-coordinate is required.

Using the integrals in the previous examples, the z-coordinate of the centroid is given by:

$$\bar{z} = \frac{1}{V}\iiint\limits_{cone} z\,dV$$



There is a choice of coordinate system, but cylindrical polars will be the easiest.

Element of volume $= dV = r\,dr\,d\theta\,dz$

The curved surface of the cone is given by $r = \frac{R}{h}z$ so the triple integral is

$$V\,\bar{z} = \int_0^{2\pi} d\theta \int_0^h z\,dz \int_0^{\frac{R}{h}z} r\,dr = 2\pi\int_0^h z\,dz\left[\frac{1}{2}r^2\right]_0^{\frac{R}{h}z} = \pi\int_0^h \frac{R^2}{h^2}z^3\,dz = \frac{\pi R^2}{h^2}\left[\frac{1}{4}z^4\right]_0^h = \frac{1}{4}\pi R^2 h^2$$

But the volume of the cone is given by $V = \frac{1}{3}\pi R^2 h$ so that $\bar{z} = \frac{3}{4}\frac{\pi R^2 h^2}{\pi R^2 h} = \frac{3}{4}h$

Volume of a cone

If you had forgotten (or never known) the formula for the volume of a cone used in the previous example, we can derive it here. We have already deduced the limits on a triple integral that describe a cone, so we can immediately use them here:

$$V = \iiint_{cone} dV = \iiint_{cone} r\,dr\,d\theta\,dz = \int_0^{2\pi} d\theta \int_0^h dz \int_0^{\frac{Rz}{h}} r\,dr = 2\pi \int_0^h \frac{1}{2}\frac{R^2 z^2}{h^2}dz = \frac{\pi R^2}{h^2}\frac{1}{3}\left[z^3\right]_0^h = \frac{1}{3}\pi R^2 h$$

Example

a) Determine the total mass and mean density of a body occupying the positive octant (where x, y and z are all positive) bounded by $x^2 + y^2 + z^2 = a^2$ and whose density is $\rho(x, y, z) = kxyz$ kg/m^3.

b) What are the physical dimensions of the constant k? Using this result, confirm that the physical dimensions of your answers to part a) are correct.

Solution

a) Mass of element of volume $= \rho(x, y, z)\,dV$ kg

Therefore, total mass $= k\iiint_V x\,y\,z\,dV$ kg

Use spherical polar coordinates:

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$; $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\therefore \text{ total mass} = k\int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} r^3\sin^2\theta\cos\theta\sin\varphi\cos\varphi\, r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= k\int_0^a r^5\,dr\int_0^{\pi/2}\sin^3\theta\cos\theta\,d\theta\int_0^{\pi/2}\sin\varphi\cos\varphi\,d\varphi$$

let $u = \sin\theta$ then $du = \cos\theta\,d\theta$ also use the identity $2\sin\phi\cos\phi = \sin(2\phi)$

$$= k\frac{a^6}{6}\int_0^1 u^3\,du\,\frac{1}{2}\int_0^{\pi/2}\sin(2\varphi)\,d\varphi = k\frac{a^6}{6}\frac{1}{4}\frac{1}{2}\left[-\frac{\cos(2\varphi)}{2}\right]_0^{\pi/2} = \frac{k\,a^6}{48}\text{kg}$$

$$\text{Mean density} = \frac{\text{total mass}}{\text{volume}} = \frac{k\,a^6}{48}\frac{8}{\frac{4}{3}\pi a^3} = \frac{k\,a^3}{8\pi}\text{ kg m}^{-3}$$

b) The dimensions of x, y and z are all metres.

Since $kxyz$ is a density, its dimensions must be kg m^{-3}, written as $[kxyz]$ = kg m^{-3}

This makes $[k]$ m^3 = kg m^{-3}, so that $[k]$ = kg m^{-6}

The mass of the object is $\frac{k\,a^6}{48}$ kg. From the above, we have $[k\,a^6]$ = kg m^{-6} m^6 = kg

The mean density of the object is $\frac{k\,a^3}{8\pi}$ kg m^{-3} and $[k\,a^3]$ = kg m^{-6} m^3 = kg m^{-3}.