Physics of Music

An optional course given to first year physics students

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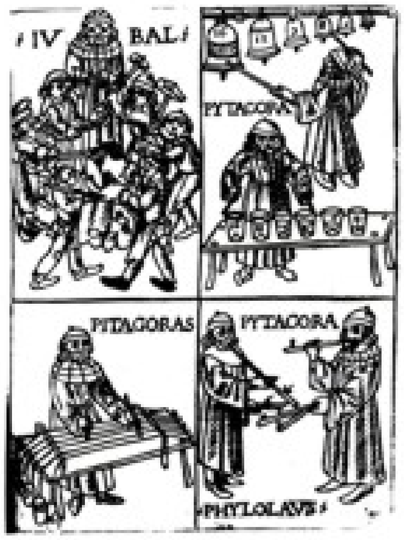
http://www-users.york.ac.uk/~pm1/PMweb/Physics_of_Music_notes.pdf

Introduction – from Pythagoras to Newton



The first recorded discovery in musical acoustics was made by Pythagoras (572 – 497 BC) when he found a basic relationship between musical harmony and mathematics. This discovery is acknowledged in a book on musical theory published in 1492. The picture shows Pythagoras with his helper Philolaus performing experiments with strings, pipes, bells and beakers. The panel at the top left is of Jubal, the Biblical father of music. He is mentioned in Genesis and in Handel's oratorio Joshua – "O had I Jubal's lyre or Miriam's tuneful voice."

Pythagoras's discovery is easily replicated with a stretched string. He divided the string into two lengths that were in simple whole-number ratios and found the two parts gave harmonious chords when sounded together:

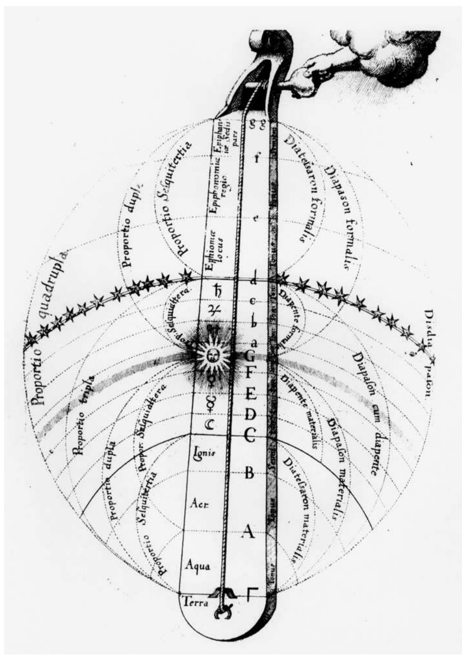
1 : 2 octave   2 : 3 perfect fifth   3 : 4 perfect fourth

i.e. the intervals that sound most pleasing to our ears correspond to divisions of the string in the ratio of small whole numbers.

For Pythagoras, this discovery had mystical rather than scientific or mathematical significance. The relationship between nature and number was so powerful that it persuaded Pythagoras and his followers that not only the sounds of nature but all her characteristics must be simple numbers that express harmonies. Nature must be in harmony with herself and therefore all natural dimensions and regularities must sound pleasing when translated into the sounds of a stretched string. The motto of Pythagoras and his school of followers was "Number rules the universe."

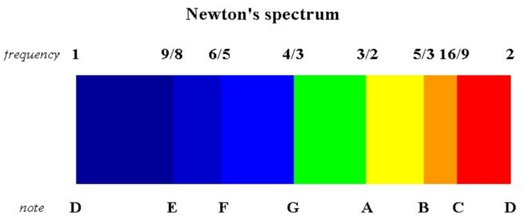
This idea that nature operated on a musical pattern pervaded enlightened thought for the next 2,000 years. It influenced not only science but architecture as well, especially during the Renaissance – the proportions of buildings were determined by harmonic ratios. It was still a strong influence at the time of Kepler in the early 17th century. He looked for a musical pattern in planetary motion and this led him to his 3rd law that:

$$\frac{(\text{period of revolution})^2}{(\text{average distance from the Sun})^3} = \text{constant}$$



An illustration from a book of 1617 (contemporary with Kepler) shows how the universe worked. It is a monochord on which the musical scale appears along with the four classical elements – earth, air, fire and water – together with the bodies in the Solar System as it was known at the time and the stars. Also marked are the musical intervals of Pythagoras and combinations of them. At the top of the picture, God's hand appears out of a cloud, tuning up the monochord of the universe. That's how everything worked in the early 17th century.

Newton's spectrum



It is widely held that there are seven colours in the rainbow. Why is this when it is virtually a continuous spectrum? Before the time of Newton, the rainbow was described in terms of five colours – red, yellow, green, blue, violet. However, Newton performed experiments to determine where the colours came from by looking for a musical pattern. He marked on the wall of his bedroom the string lengths required to give all the notes in an octave and used a glass prism to project the Sun's spectrum onto it. The lengths of string coincided with the five colours but, to complete the picture, he named two extra colours, orange and indigo, that coincided with the semitones E→F and B→C. Thus, there are seven colours of the rainbow to match the notes of a musical octave. He wasn't to know that the notes higher in pitch and therefore shorter in wavelength correspond to the longer wavelength colours.

Musical gravity

Newton is famous for his discovery of the law of gravity. While not claiming that he discovered it by looking for a musical pattern, there is evidence that music played a part in his thinking about the problem. Consider the following.

Let the distance from a planet to the Sun be represented by the length of a string.

Let the mass of the planet be represented by the tension in the string.

The formula giving the frequency of vibration of the string is: $f = \frac{1}{2l}\sqrt{\frac{T}{\rho}}$

where l = length of string T = tension ρ = linear density (mass / length)

Note that $f \propto \frac{1}{l}$ and $f \propto \sqrt{T}$.

Eliminating f gives $\sqrt{T} \propto \frac{1}{l}$ or $T \propto \frac{1}{l^2}$, which is the condition required to keep the frequency constant.

T is a force and l is a distance, so we have force $\propto \frac{1}{\text{distance}^2}$

This is the inverse square law which is embodied in the law of gravity:

$$F = G\frac{M\,m}{r^2}$$

It is this formula more than anything else that changed science for good. It enabled scientists to calculate the orbits of planets, predict solar eclipses with unprecedented accuracy and explain the behaviour of tides. As a consequence of this, we no longer look for a musical pattern in nature, we look for mathematical patterns instead. However, the search for musical patterns worked well for over 2,000 years.

Preliminaries

What is a musical sound?

In order to make a sound we need an oscillating or vibrating system. Vibrations in the air transmit the sound and the source of the sound could be the vibration of a string (violin), metal bar (glockenspiel), membrane (drum), reed (clarinet), electrical voltage (synthesiser) or the air itself (flute).

Displaying the vibrations associated with various sounds on an oscilloscope, which gives displacement as a function of time, allows us to see some of their characteristics:

Demo: Display wave forms of noise, speech, tuning fork, organ pipe, flute.

It appears that the more musical the sound, i.e. a sound that has a recognisable pitch, the more regular is the vibration. **A musical sound therefore has a definite pitch caused by periodic vibrations.** This is the definition given by Helmholtz ("On the sensations of tone" 1885). The study of periodic vibrations is a good starting point for our journey into musical acoustics.

It is also easily demonstrated that the sensation of pitch is associated with the frequency of the wave and that loudness is associated with the amplitude of the wave.

Period of vibration

Regularity of vibration gives constant frequency. However, why should frequency remain constant for a vibrating system such as a metal plate, string or tuning fork even as it loses or gains energy? Why don't the vibrations become slower as they die out?

Demo: The pitch of a tuning fork remains constant as it loses energy.

Dimensional analysis will give us the condition required for constant frequency. For a tuning fork, the period will depend upon restoring force (*F*), displacement (*x*) and mass (*m*).

Assume that $\text{period} \propto F^{\alpha}\, x^{\beta}\, m^{\gamma}$

The dimensions are: $T \propto \left(M\,LT^{-2}\right)^{\alpha} \left(L\right)^{\beta} \left(M\right)^{\gamma}$

Equating powers gives:

M	$0 = \alpha + \gamma$
L	$0 = \alpha + \beta$
T	$1 = -2\alpha$

These are satisfied by $\alpha = -½$ and $\beta = \gamma = ½$ so that: $\text{period} \propto \sqrt{\dfrac{xm}{F}}$

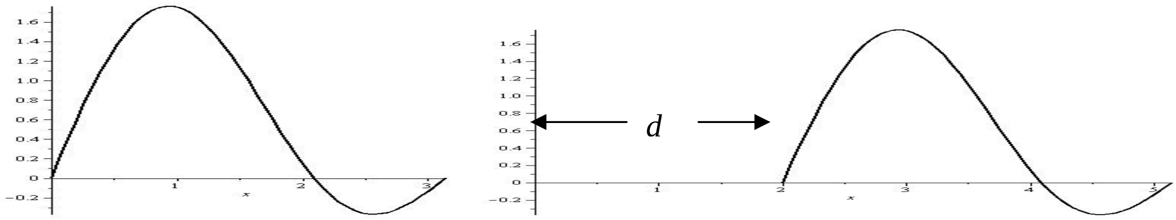
Mass is expected to remain constant during vibration so, for a constant period, we must have:

$$\frac{x}{F} = \text{constant, i.e. restoring force} \propto \text{displacement}$$

This condition is true for any perfectly elastic substance and true for real substances for small displacements. It gives rise to **simple harmonic motion**. Therefore, we expect most vibrating systems to have a constant period provided the amplitude of vibration is not too large.

Mathematical description of a wave

Waves move, so we will consider the propagation of a wave in the positive *x* direction:



At time = 0, the displacement of the wave can be described by the function $\phi(x)$.

At time = t, the displacement is $\phi(x\text{-}d)$, where the same function is used with a shift of origin, i.e. the wave has not changed in shape, it has only moved a distance d along x.

If the wave travels at a constant speed, v, the distance travelled in time t is $d = v\,t$

The wave is therefore of the form $\phi(x\text{-}vt)$ and the physics built into this is that it travels with **constant profile** at a **constant speed**.

A simple periodic wave can be described mathematically by a wavy function such as a sine or a cosine. The displacement as a function of position and time can then be represented by $\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$ where the scale factor $\frac{2\pi}{\lambda}$ converts distance to radians, with λ as the wavelength.

It is conventional to express this in terms of frequency, f, so we use $v = \lambda f$ and replace $2\pi f$ by the angular frequency ω. This makes $\frac{2\pi v}{\lambda} = 2\pi\ f\ = \omega$.

Likewise, $\frac{2\pi}{\lambda}$ is replaced by the angular wave number k, so that the displacement of the wave as a function of position and time is $\phi(x,t) = a\sin(kx - \omega t)$ where a is the amplitude.

Equation of motion of an oscillator

The choice of sine or cosine to describe the shape of a simple periodic wave is justified by the equation of motion of the oscillator. The symbols used are:

x = displacement b = damping factor / unit mass
t = time ω^2 = restoring force / unit mass / unit displacement

for unit mass: $$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega^2 x = 0$$

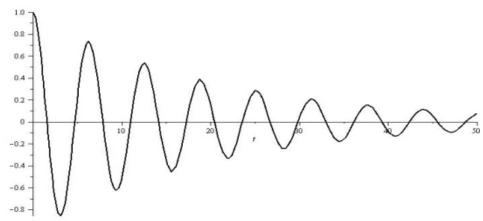
i.e. inertial force + damping force + restoring force = 0

The damping force is proportional to velocity and the restoring force is proportional to displacement. The latter gives SHM. With no external forces acting on the oscillator, the right-hand-side of the equation is 0.

This is a second order differential equation with constant coefficients and its solution is:

$$x = \exp(-bt/2)\left(A\cos(\upsilon t) + B\sin(\upsilon t)\right)$$

where the angular frequency υ is given by $\upsilon^2 = \omega^2 - b^2/4$



A plot of this gives damped simple harmonic motion.

With b = 0, i.e. no damping, the solution becomes $x = A\cos(\omega t) + B\sin(\omega t)$ which is SHM.

Waves on a string

We will now examine the behaviour of waves that we can see – waves on a rubber cord or waves on a stretched string.

Standing waves

Demo: Pluck a rubber cord under tension to show a travelling wave being reflected from each end. The period is long enough to be easily estimated. Set up a standing wave in the first mode of vibration and show it has the same period.

This Maple animation shows the standing wave. Click on the plot to display the icons, select the cyclic mode to make the motion continuous, then start the animation.

```
restart: with(plots):
mode:=1: animate(plot, [sin(mode*Pi*x)*cos(2*Pi*t), x=0..1], t = 0..1);
```

The standing wave is so-called because there is no obvious movement of the wave along the cord in either direction. It is clearly related to the travelling wave as the two have the same period. The following mathematical analysis shows that the standing wave is the sum of two travelling waves moving in opposite directions. The travelling waves have to be reflected from the ends of the cord otherwise the standing wave cannot exist.

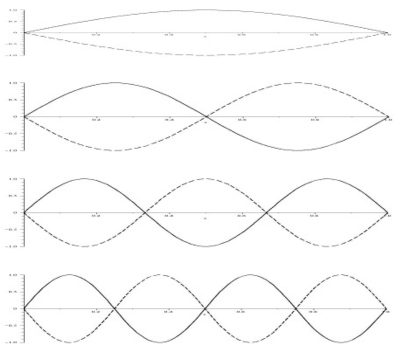
Using the mathematical description of a wave just developed, the first travelling wave is $\sin(kx - \omega t)$ and the second one is $\sin(kx + \omega t)$ which is made to move in the opposite direction by changing the sign of t.

Their sum is: $$\sin(kx - \omega t) + \sin(kx + \omega t) = 2\sin(kx)\cos(\omega t)$$

where the sum has been re-expressed using a trig identity. The sine function gives the shape of the wave along the cord and the cosine gives its variation with time. It is this formula which is implemented in the Maple animation, where it can be seen that the wavelength is twice the length of the cord.

Modes of vibration

This is not the only standing wave which can be set up. Since the speed of the wave along the cord is fixed, the frequency of the standing wave is also fixed unless the wavelength is exactly equal to the length of the cord. In this case, a standing wave with two loops on it can be set up and its frequency is exactly twice the frequency of the original wave.



Demo: Set up several modes of vibration on the cord and demonstrate the relationship between their frequencies.

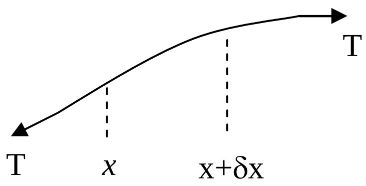
The pictures show some different standing waves and these are all examples of **modes of vibration**. The only standing waves that can be set up are those whose wavelengths fit exactly along the length of the cord, i.e. with wavelengths of $(2\,l\,/\,n)$ where l is the length of the cord and n is a positive integer. n is referred to as the mode number. The Maple animation can also demonstrate these by altering the value of **mode**.

From the fundamental wave relationship $v = \lambda f$, it can be seen that as λ takes on the values $(2\,l\,/\,n)$, the frequencies must have the values $(n\,f)\,/\,(2\,l)$, so the frequencies are in the ratios 1 : 2 : 3 : 4 etc. These are precisely the ratios required to give the Pythagorean musical intervals of octave, perfect fifth and perfect fourth.

Demo: Melde's experiment demonstrates standing waves on a string and the frequencies can be obtained from the signal generator. They are seen to be in the ratios 1 : 2 : 3 : 4 as predicted above.

A formula for frequency

We have been considering transverse waves on a string under tension and now we need to predict the frequency of those waves. The diagram shows a short segment of the string and the tension is pulling in slightly different directions at each end. This gives a resultant vertical force which acts as the restoring force for the vibrations. The acceleration of the string therefore depends upon tension and linear density (mass / length).



Dimensional analysis can be used to obtain the speed (v) of the wave as a function of tension (T) and linear density (ρ).

Assume that $$v \propto T^{\alpha} \rho^{\beta}$$

The dimensions are: $$L T^{-1} = \left(M L T^{-2}\right)^{\alpha} \left(M L^{-1}\right)^{\beta}$$

Equating powers gives:

M	$0 = \alpha + \beta$
L	$1 = \alpha - \beta$
T	$-1 = -2\alpha$

These are satisfied by $\alpha = ½$ and $\beta = -½$ so that $v \propto \sqrt{\frac{T}{\rho}}$

A deeper mathematical analysis shows that the constant of proportionality is 1. For periodic waves, $v = \lambda f$ and for standing waves on a string $\lambda = 2\,l\,/\,n$ where n is the mode number. The above formula for speed can then be expressed in terms of frequency as

$$f = \frac{n}{2l}\sqrt{\frac{T}{\rho}}$$

This analysis assumes the string is perfectly flexible, which means the restoring force comes only from the tension. A stiff string adds to the restoring force by its own internal elasticity so, for piano and guitar strings, the formula has to be changed slightly as we shall see.

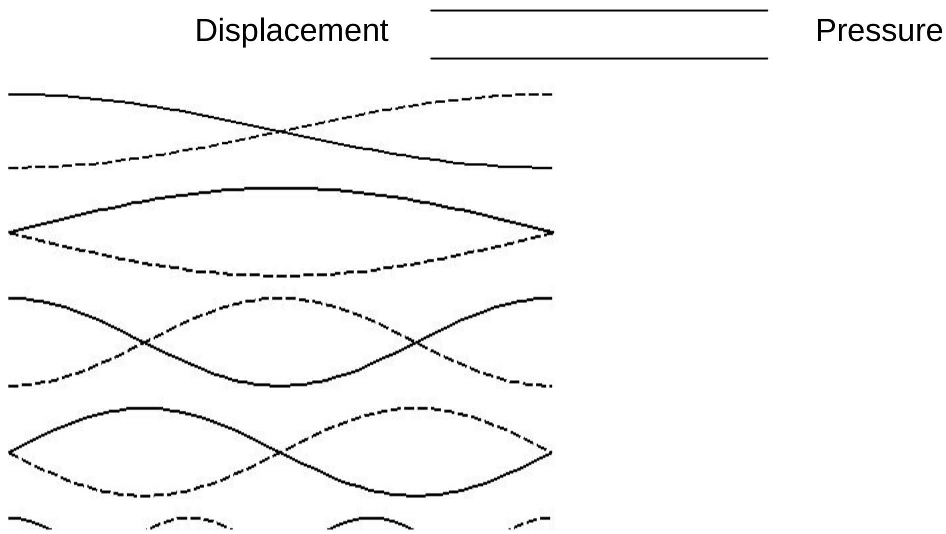
Demo: Measure l, T and ρ in Melde's experiment and calculate the frequency of the waves. Compare with the frequency on the signal generator.

Waves in a column of air

Having examined the behaviour of waves that we can see, let us now investigate waves that we can hear, i.e. waves in air.

Modes of vibration – open pipes

Sound waves in a gas are longitudinal. Standing waves can be set up in an air column, which show almost exactly the same pattern of behaviour as transverse waves on a string. This is shown in the following diagrams which display the patterns of displacement and pressure in a pipe.



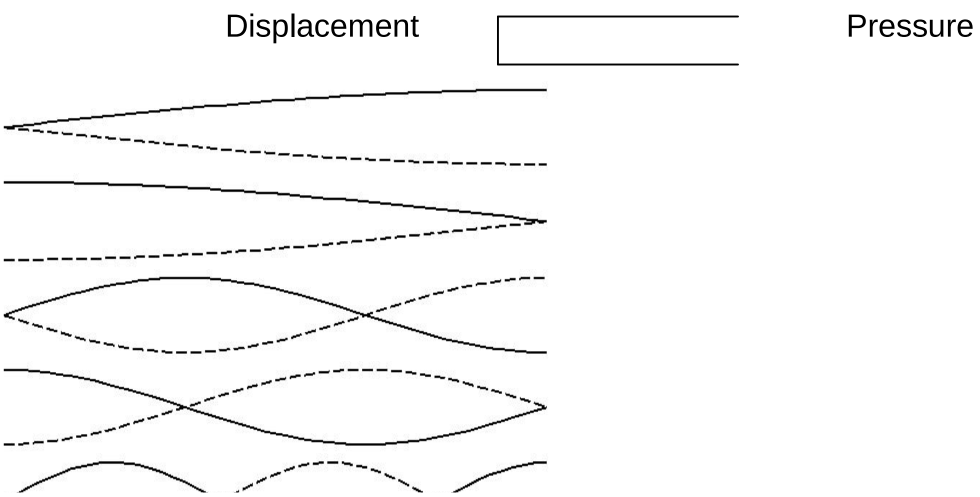
Note that:

- There is a displacement antinode at the open end of the pipe because that is where the air is most free to move.
- There is a pressure node at the open end of the pipe because that is where the pressure is least able to change – it is held at atmospheric pressure.
- Pressure nodes and antinodes are at the positions of displacement antinodes and nodes respectively.
- Each standing wave is a superposition of two identical travelling waves moving in opposite directions and reflected from the ends of the pipe.
- The wavelength corresponding to each mode is $\lambda = \frac{2l}{n}$ where l is the length of the pipe and n is the mode number.
- The frequency of each mode is $f = \frac{vn}{2l}$ where v is the speed of sound. Hence, the frequencies occur in the ratios 1 : 2 : 3 : 4 …, i.e. the ratios required to reproduce the Pythagorean intervals.

Demo: Obtain modes of vibration in a drain pipe by blowing across the end.

Modes of vibration – closed pipes

A pipe may be closed at one end and still support standing waves in different modes. This is also used in musical instruments, although the equivalent for strings is never used because it is impractical. The modes of a closed pipe are shown in the following diagrams.



Note that:

- The displacement must be zero at the closed end.
- Pressure variations are at a maximum at the closed end where the air is most confined.
- The wavelength corresponding to each mode is $\lambda = \dfrac{4l}{2n-1}$ where l is the length of the pipe and n is the mode number.
- The frequency of each mode is $f = \dfrac{v(2n-1)}{4l}$, so the frequencies are in the ratios 1 : 3 : 5 : 7 …, i.e. a closed pipe produces only the odd numbered harmonics.

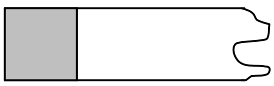
In the first mode (n = 1), the wavelength is 4*l*. Compare this with the first mode of an open pipe where $\lambda = 2l$. Therefore, a closed pipe will give half the frequency of an open pipe of the same length. Alternatively, a closed pipe need only be half the length of an open pipe to give the same frequency.

Demo: Stopped organ pipe. Compare the sound with and without the stopper,

Speed of longitudinal waves in solids and gases

The same physics applies to longitudinal waves in solids, liquids and gases. (Liquids and gases will not support transverse waves, but solids will.) Because the waves are longitudinal, they are propagated by squashing and relaxing the medium, so the speed of sound depends upon density and modulus of elasticity.

Consider a solid rod of Young's modulus E and density ρ. The end, of area A, is subject to a sudden force F which sends a compression wave along the rod.



Compressed part of rod

After time δt, the compression has advanced a distance $v\,\delta t$, where v is the speed of the compression wave.

The resulting shortening of the rod is δx.

Young's modulus is defined as $E = \dfrac{\text{force / area}}{\text{fractional change in length}} = \dfrac{F\ /\ A}{\delta x\ /\ v\,\delta t} \Rightarrow \dfrac{\delta x}{\delta t} = \dfrac{F\,v}{A\,E}$

But $\dfrac{\delta x}{\delta t}$ is the speed of movement of the rod.

Therefore, change in momentum $= A v \delta t \, \rho \frac{\delta x}{\delta t}$

This makes rate of change of momentum = $A v \rho \frac{\delta x}{\delta t} = A v \rho \frac{F v}{A E} = \frac{v^2 \rho F}{E}$

Newton's 2nd law of motion states that rate of change of momentum = applied force

$\therefore \frac{v^2 \rho F}{E} = F$ and a rearrangement gives the speed of longitudinal waves in a solid rod as:

$$v = \sqrt{\frac{E}{\rho}}$$

This has been derived with a solid in mind, but for a gas, the equivalent elasticity is:

$$E = -\frac{\text{increase in pressure}}{\text{fractional increase in volume}}$$

The negative sign is necessary because an increase in pressure results in a *decrease* of volume. For rapid changes in pressure and volume, the adiabatic gas law is the appropriate one to use, i.e. $P V^\gamma$ = constant, where γ depends upon the molecular structure of the gas.

The reason for using the adiabatic gas law is because changes take place on a millisecond time scale and heat must move a distance of half a wavelength (about 1 m) in that time for thermal equilibrium. This cannot take place, so adiabatic conditions prevail.

Differentiate the adiabatic gas law wrt V: $\frac{dP}{dV} V^\gamma + \gamma P V^{\gamma - 1} = 0 \Rightarrow \frac{dP}{dV} V + \gamma P = 0$

This can be rearranged to give: $-\frac{dP}{dV/V} = -\frac{\text{increase in pressure}}{\text{fractional increase in volume}} = E = \gamma P$

Substitute this into the formula for a solid to obtain the speed of longitudinal waves in a gas:

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

Examples

For air at standard temperature and pressure: $\gamma = 1.400$; $P = 101325$ N/m^2; $\rho = 1.293$ kg/m^3

Putting these into the formula gives: **v = 331.4 ms^{-1}**; experimental value is 331.36±0.08m/s

For helium, $\gamma = 5 / 3$; $\rho = 0.180$ kg/m^3 giving **v = 969 m/s**

Temperature variation

Since the speed of sound depends upon the density of the gas and the density changes with temperature (at constant pressure), the speed of sound will be a function of temperature:

$\frac{PV}{T} = const.$, so for constant P and $\rho \propto \frac{1}{V}$ we must have ρT = constant

This leads to $\rho_T = \frac{273}{T + 273} \rho_0$ where T is now in °C and ρ_T is the density at T°C

The speed of sound at T°C is therefore $v_T = v_0 \sqrt{\frac{T + 273}{273}}$ = 343.3 m/s at 20°C.

Musicians who play wind instruments are well aware of the change of speed of sound with temperature as they must warm their instrument up to playing temperature before trying to tune it.

Demos: For aluminium, $E = 7.03\times10^{10}$ N/m^2 and $\rho = 2.7\times10^3$ kg/m^3. This makes the speed of sound = 5102 m/s. Obtain longitudinal vibrations in an aluminium rod and measure the frequency using a signal generator. For the rod provided, l = 1.49 m so that $\lambda = 2\times1.49$ m. The expected frequency is therefore $\frac{5102}{2\times1.49} = 1712Hz$.

Stimulate the modes of the drain pipe using a loudspeaker and read off the frequencies from the signal generator. Do a similar calculation to that above. Needs to use end correction (= 0.58 r, but changes with frequency).

End correction

Because the pressure variations of the standing wave do not fall to zero exactly at the end of the pipe but a little beyond it, the effective length of the pipe is greater than the actual length. The extra bit to be added on is known as the **end correction**.

For an open end, add 0.58r to the length where r is the radius of the (circular) cross-section. For two open ends, two end corrections must be applied. There is no end correction for a closed end.

The end correction changes slightly with frequency, so it is different for each mode of vibration.

Resonance amplification

All the sounds we have heard up to now are examples of **resonance**. This occurs during forced vibrations when the forcing frequency is such as to produce a large response from the vibrating system and the condition of resonance is reached when the response is at a maximum. This condition is necessary because most vibrators produce very weak tones requiring some form of amplification. To understand resonance properly, we need to look at the mathematics of forced vibration.

The equation of motion for unit mass of a forced, damped harmonic oscillator is:

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega^2 x = f\cos(nt)$$

where x = displacement b = damping factor /mass f = external force / mass
t = time ω = angular frequency of natural vibrations n = angular frequency of f

i.e. inertial force + damping force + restoring force = externally applied force

Forces intrinsic to the oscillator are on the left and external forces are on the right.

The complete solution of the equation is:

$$x = \exp(-bt/2)\left(A\cos(\upsilon t) + B\sin(\upsilon t)\right) + \frac{f\cos(nt-\delta)}{\left[(\omega^2-n^2)^2 + b^2n^2\right]^{1/2}}$$

$$\text{where} \qquad \upsilon^2 = \omega^2 - b^2/4 \quad \text{and} \quad \tan(\delta) = \frac{bn}{\omega^2 - n^2}$$

The first term in the solution is subject to damping and will quickly die out. This means the steady state of the oscillator is represented by the second term only. The following analysis therefore uses only the second term. Of most interest is the average power of the oscillating system as this relates to the intensity of sound radiated.

$$\text{power = rate of doing work = force}\times\text{velocity} = f\cos(nt)\frac{dx}{dt} = f\cos(nt)\frac{-f\,n\sin(nt-\delta)}{\left[(\omega^2-n^2)^2 + b^2n^2\right]^{1/2}}$$

The power is a function of time, so we require the time averaged power, P.

$$P = -\frac{f^2 n}{\left[(\omega^2-n^2)^2 + b^2n^2\right]^{1/2}}\frac{1}{2\pi}\int_0^{2\pi}\cos(nt)\sin(nt-\delta)\,dt = \frac{1}{2}\frac{f^2\,bn^2}{(\omega^2-n^2)^2 + b^2n^2}$$

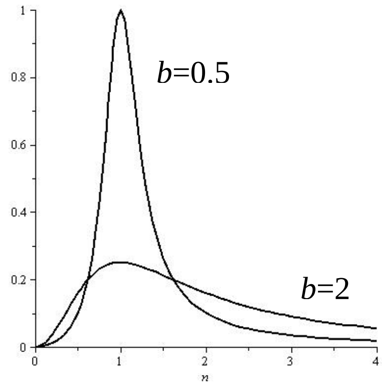
which makes use of $\sin(\delta) = \dfrac{bn}{\left[(\omega^2-n^2)^2 + b^2n^2\right]^{1/2}}$

A plot of P as a function of n is known as the resonance curve of the oscillator. Such a plot is shown on the next page for two different values of b with $f = 1$ and $\omega = 1$. Power resonance occurs at the maximum of the curve so we will determine the height of the maximum, its position and the width of the peak.

Maximum power occurs when $dP/dn = 0$: $\dfrac{dP}{dn} = \dfrac{f^2\,bn\,(\omega^4 - n^4)}{\left[(\omega^2-n^2)^2 + b^2n^2\right]^2}$

$= 0$ when $\omega = n$

Power resonance therefore occurs at ω = *n*, i.e. when the frequency of the external force matches the frequency of natural vibrations.



Putting this condition into the expression for *P* gives:

$$P_{\max} = \frac{f^2}{2b}$$

The maximum power depends upon the square of the applied force, so you get a good increase in power for a moderate increase in force. Also, power is inversely proportional to the damping factor. If this is small, the maximum power may be very large indeed.

The width of the peak is measured as the width at half the maximum height, i.e. the distance (in frequency) between the half power points. This is calculated from:

$$\frac{P}{P_{\max}} = \frac{b^2 n^2}{(\omega^2 - n^2)^2 + b^2 n^2} = \frac{1}{2}$$

This is a quartic equation giving 4 roots. It is satisfied by $\omega^2 - n^2 = \pm b n$

So that *n* at the half power points is $n = \dfrac{\mp b \pm \sqrt{b^2 + 4\omega^2}}{2}$

Picking out only the positive frequencies gives $n = \sqrt{\omega^2 + \dfrac{b^2}{4}} \pm \dfrac{b}{2}$

The difference between them gives the width of the resonance peak as *b*.

It can be seen from this that, as the damping decreases, the power response at resonance increases and can become very large for a lightly damped system. Also, a good response is obtained when the applied frequency is within ½ *b* of the resonance frequency. However, as *b* decreases, the frequencies must match more and more closely. In the extreme as *b* tends to zero, the power can be huge, but only when the frequencies match almost perfectly. On the other hand when *b* is large, the resonance response is much smaller, but it occurs over a wider frequency range.

Stories of opera singers shattering glass windows are probably not true, but it is certainly possible to shatter things using resonance. A friend of mine once smashed an expensive glass object in someone’s house with a note from his French horn. Resonances in turbines, motors, generators etc. are to be avoided because of the possibility of mechanical failure from very large vibrations.

One of my best demos was to shatter a lab beaker with sound. It could be done quite easily provided the sound from the loudspeaker was accurately tuned to the resonance frequency of the beaker. However, with improvements in their manufacture, lab beakers are now much tougher and it is the loudspeaker which breaks first. The demo is best carried out with an expensive wine glass. If you give it a “ping” with your finger and it rings for a long time, it is clearly lightly damped and so can be shattered by sound of the right frequency.

Demo: A simpler demo is to use a tuning fork. The tuning fork rings for a long time when sounded and so is very lightly damped. Its vibrations can be stimulated from a loudspeaker, but only if the frequency is within about 1 Hz of the frequency of the fork. Detuning the signal generator by more than 2 Hz results in no vibrations of the fork at all.

This can be contrasted with the response of the wooden box on which the fork is mounted. Its resonance frequency is about 415 Hz (the fork is 440 Hz), but it gives a good response over a range of more than 30 Hz. This shows that air is a heavily damped system, which should be obvious if you think of a flute – it doesn't ring like a bell, but stops sounding as soon as you stop blowing it.

Aside 1: The box on which the tuning fork is mounted does not resonate at the same frequency as the fork, otherwise it would radiate sound too rapidly and not sustain the vibrations for a long enough time.

Aside 2: The tuning circuit in a broadcast receiver (TV, radio, mobile phone, etc.) is a resonant circuit which behaves in the same way as the oscillator described in this lecture. It ought to be very lightly damped so the resonance peak is narrow, allowing it to respond to the frequency of only one broadcasting station and effectively cutting out the signals from all other stations broadcasting on other frequencies. The circuit can be tuned to another station by changing its resonant frequency. This is carried out by altering the circuit values, usually the capacitance.

We already know that a string or air column has a number of resonances, so the resonance curve of a musical instrument can have many peaks. Nevertheless, the resonances still behave in the way described here. Most instruments will have resonance peaks at frequencies in the ratios 1 : 2 : 3 : 4 … which form a harmonic series.

There is no law of physics which says an instrument can use only one resonance at a time. In general, it will vibrate using all resonances simultaneously at the appropriate frequency giving what is called a **complex tone.** We can therefore analyse the sound and find how much of each harmonic is present in the complex tone.

Wave analysis

We have just seen that a musical instrument has a number of resonances. There is nothing to stop it from using all the resonances simultaneously and, indeed, that is what normally happens. The sound produced by almost any instrument is therefore a complex tone consisting of contributions from a number of resonances simultaneously. Each instrument has its own resonance curve and "vibration recipe" which is partly what gives it its distinctive sound. In this lecture we will analyse the sound from a number of instruments to determine how much of each resonance is present. Since the resonances are expected to form a harmonic series, this is usually known as "harmonic analysis".

Vocabulary

There is a redundancy of words describing the component parts of a complex tone, so let us look at some of them to avoid future confusion.

Mode Pattern of vibration giving rise to a single frequency.
Partial Contribution to the complete wave of a single mode of vibration.
Fundamental Partial with the lowest frequency.
Overtone Same as partial, but refers to partials with frequencies higher than the fundamental.
Harmonic Refers to partials when their frequencies are in the simple ratios 1 : 2 : 3 : 4 …, i.e. the harmonic series. The first harmonic is the same as the fundamental.

Fourier series

The standing wave corresponding to the fundamental of a complex tone is:

$$\phi = a \sin(kx)\cos(\omega t) \quad \text{where } k = 2\pi / \lambda \text{ and } \omega = 2\pi f.$$

For the nth mode: $$\phi_n = a_n \sin(nkx)\cos(n\omega t + \alpha_n)$$

There is no reason why the waves from each mode should be in phase with each other, so the phase angle α_n is introduced.

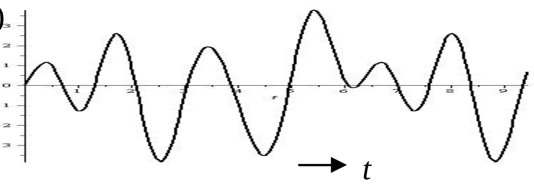
A complex tone is: $$\phi(x,t) = \sum_{n=1}^{\infty} \phi_n = \sum_{n=1}^{\infty} a_n \sin(nkx)\cos(n\omega t + \alpha_n)$$

Consider only the time variations: $$\phi(t) = \sum_{n=1}^{\infty} a_n \cos(n\omega t + \alpha_n)$$

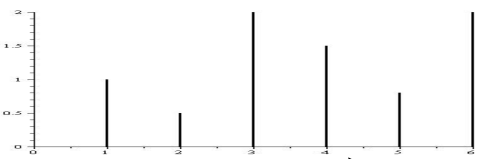
Writing it out: $$\phi(t) = a_1\cos(\omega t + \alpha_1) + a_2\cos(2\omega t + \alpha_2) + a_3\cos(3\omega t + \alpha_3) + \cdots$$

which is a sum of simple waves with frequencies of ω, 2ω, 3ω, etc., i.e. harmonics. Mathematically, this is known as a Fourier series and the coefficients a_1, a_2, a_3 … give the amount of each harmonic present in the wave. The wave can then be plotted as a function of time, $\phi(t)$: or as a function of frequency, a_n:

$\phi(t)$



t



n

Fourier synthesis

Before going on to harmonic analysis, we will have a brief look at Fourier synthesis. Almost any function can be represented as a sum of harmonics and this includes the sound of musical instruments.

Demos: Use the Fourier synthesiser to demonstrate harmonics and their intervals.

- Build up a wave form by adding harmonics – the addition of each harmonic alters the character of the sound. (A square wave has coefficients $a_{2m-1} = \frac{(-1)^m}{2m-1}$ for odd harmonics only.)
- Change relative phases of harmonics, which alters the wave shape, and show this doesn't alter the sound – the ear is sensitive to harmonic content and not the actual wave shape.
- Set up the harmonic analysis of an instrument and show it doesn't sound like that instrument. There is a lot more to the sound of an instrument than the steady state of a single note.

Harmonic analysis

Demos: Use the Fourier analyser to find the harmonic content of a variety of instruments. Show both the wave form and the sound spectrum. Compare open and closed organ pipes. Look at policeman's whistle – gives two sine waves. The hosepipe gives a rich spectrum which changes enormously with pitch. Look at a flute, violin and clarinet if available.

What we hope to find:

- Harmonic content changes with loudness and pitch.
- Wind instruments, e.g. flute, trumpet, normally give few harmonics.
- The clarinet spectrum is rich for low notes, but contains fewer harmonics for high notes.
- A low note on the clarinet has very weak or absent even harmonics – it behaves like a closed pipe.
- Stringed instruments give a rich harmonic spectrum.
- The fundamental (1st harmonic) is not necessarily the strongest harmonic present.

Percussion instruments

We have now done most of the physics common to all musical instruments, such as the behaviour of waves and modes of vibration. We will now consider particular instruments and look at their individual characteristics. The instruments considered will be mainly orchestral.

Vibrating bars

The longitudinal vibrations of bars have already been mentioned. However, although they make a good sound, they are not the basis of main-stream instruments and so will not be considered further.

The transverse vibrations of bars *are* used in a number of instruments such as the xylophone, marimba, tubular bells, celesta.

Lord Rayleigh (Theory of sound, 1877) determined the frequencies of the partials of transverse vibrations of a bar with free ends as:

$$f = \frac{\pi v \mathrm{K}}{8 l^2} m^2$$

where v = speed of sound = $\sqrt{E/\rho}$; E = Young's modulus; ρ = density.

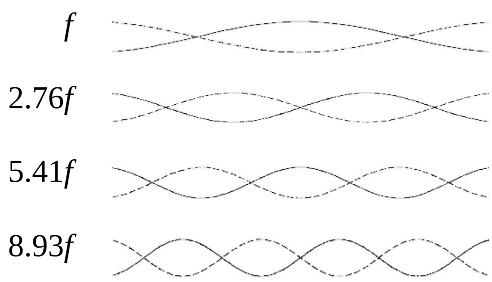
$\mathrm{K} = t/\sqrt{12}$ for a bar of thickness t; $= ½\sqrt{r_1^2 + r_2^2}$ for a tube of radii r_1 and r_2.

l = length of bar or tube.

m is very close in value to the odd integers 3, 5, 7, 9 … for the different modes.

The bar is not under tension and only its own elasticity provides the restoring force for the vibrations.

The modes with their frequencies are:

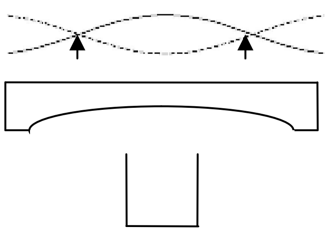


The frequencies clearly do not form a harmonic series. A bar is therefore expected to give a rather harsh musical sound.

Xylophone

The bars are made of a hard wood, such as African rose wood, mounted to encourage the vibrations of the fundamental and damp out the inharmonic partials.

Mounted underneath each bar is a resonator tuned to the fundamental, which enhances the sound and helps to sustain the vibrations.



The cross-section of each bar is shaped as shown. It is thinner in the middle where the first mode has its maximum curvature and the second mode its minimum curvature. The thinner bar therefore supplies less restoring force to the first mode, leaving the second mode almost unaffected. This will reduce the first mode frequency so it can be tuned to give a frequency ratio of 1 : 3 with the second mode, i.e. two members of a harmonic series. The resonator will also respond at 3*f*. The shaping of a vibrator to tune partials to particular frequencies occurs in a number of instruments; this is our first example.

Demo: Two keys of a toy glockenspiel an octave apart should have a length ratio of $1 : \sqrt{2}$. The bottom C is of length 17.10 cm; the top C is 12.05 cm. The ratio is 1.42. ($\sqrt{2} = 1.41$)

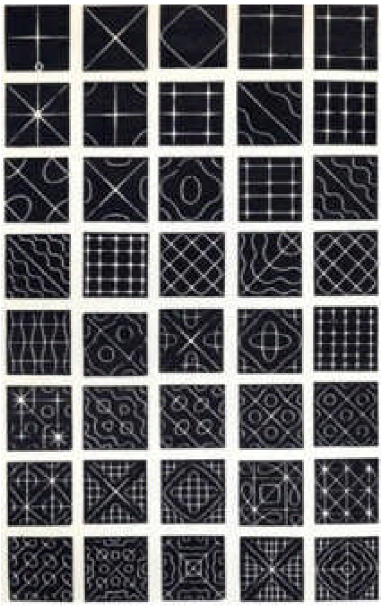
Notice the dimple drilled in the middle of the key to tune it – making it thinner in the middle reduces the frequency, making it shorter increases the frequency.

Vibrating plates and membranes

A plate uses its own elasticity to provide the restoring force for the vibrations; a membrane is flexible and must be under tension to vibrate.

Instruments which use plates and membranes are gongs and drums. The body of a violin, guitar or the sound board of a piano all consist of plates which are important for vibrations, but these aren't the primary vibrators.

One of the first people to study vibrations of plates was Ernst F. F. Chladni (1756 – 1824). His book "Entdeckungen über die Theorie des Klanges" (Discoveries on the Theory of Sound) was published in 1787. The experiment for which he is most famous is known as Chladni's plate. A horizontal metal plate is sprinkled with sand and made to vibrate either by a violin bow or using a mechanical vibrator. The sand is shaken towards the nodes of the standing wave, revealing the pattern of vibration. Patterns obtained by Chladni himself are shown in the picture.



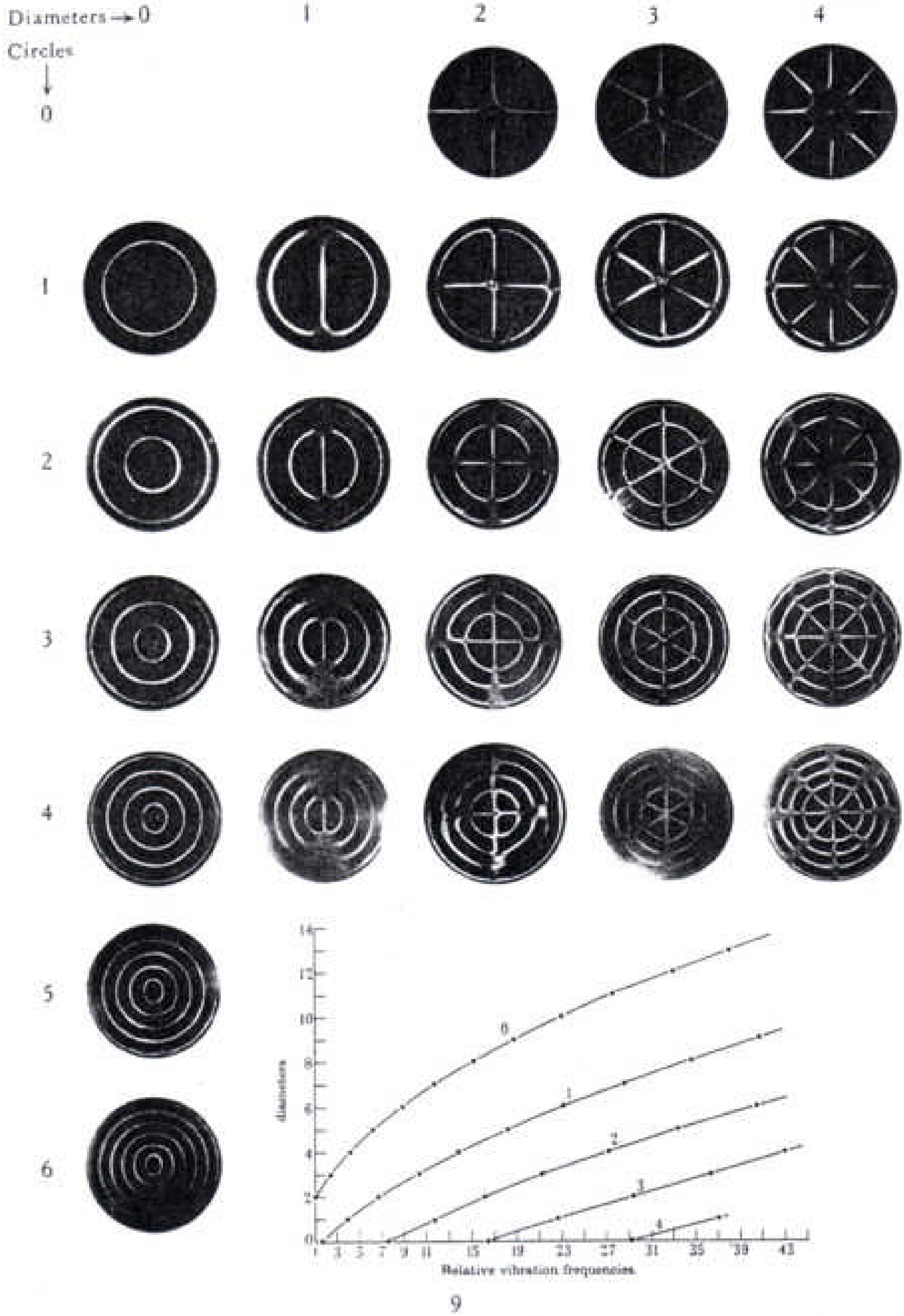
Demo: Chladni's plate. Obtain several patterns including at least one at high frequency. Note that the higher the frequency, the more intricate is the pattern of vibration. The resonances do not form a harmonic series. Chladni's plate also features on some YouTube videos. See, for example: http://tinyurl.com/c7g57an.

Aside: Chladni was invited to repeat his experiments on sound to the learned societies and royal courts around Europe. Napoleon, in particular, was delighted by the demonstrations and is reported to have exclaimed in wonder, "He makes sounds visible!" Napoleon paid for the translation of Chladni's book into French and offered a prize of one kilogram of gold to anyone who could explain mathematically the sand patterns that formed on Chladni's plate. It was won by the French mathematician Sophie Germain.

Vibrations of a circular membrane

A circular membrane is of interest because it is used in drums. The modes of vibration are denoted by two indices *mn*, where *m* is the number of nodal diameters and *n* is the number of nodal rings. The following pictures are of Chladni patterns of a circular plate, but they have nodal rings and diameters just the same. A drum membrane is normally held round its perimeter, which will always be a nodal ring. The relative frequencies of the modes are given in the graph, from which it is clear that they do not form a harmonic series. Drums are therefore not expected to have a definite pitch when struck and are used for rhythm rather than melody.

PLATE 4



Diameters → 0
1
2
3
4
Circles
0
1
2
3
4
5
6
diameters
Relative vibration frequencies

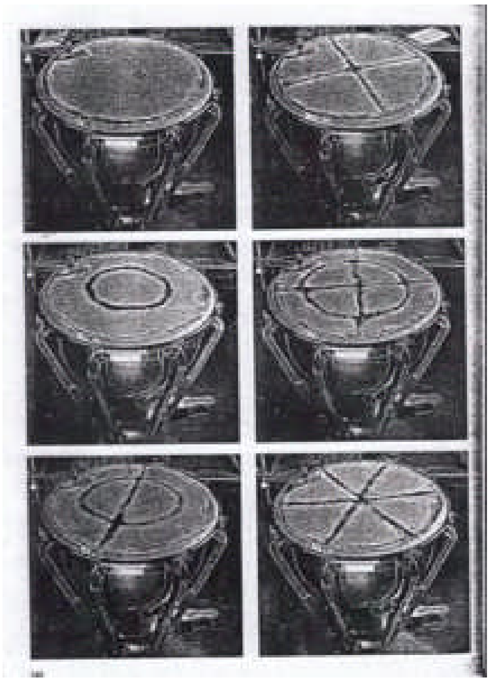
9

Kettledrum

A kettledrum consists of a hemispherical copper shell covered by a membrane of calf skin or mylar (0.2 mm thick).

Most drums are used to give rhythm, but the kettledrum has a definite pitch. Why is this so when the frequencies of the modes do not form a harmonic series?

Vibrations of the drum head can be examined using Chladni's method and some of the patterns obtained are shown in the picture. The mechanical vibrator can be seen at the top left-hand corner of each picture. This is the most accurate method of determining the frequency of each mode.



Modes with m = 0 (no nodal diameters – the circular or symmetric modes) are heavily damped, especially 01, because they change the volume of air in the kettle. They do work by increasing and decreasing the pressure and so very quickly die out. Hitting the drum in the middle to excite these modes just produces a dull thud. A mode with m = 1 divides the drum head into two, such that when one half is descending the other half is ascending and the volume of air in the kettle is preserved. The mode is therefore relatively lightly damped.

The drum head is normally struck at a point ¼ of the way in from edge to centre. This excites the asymmetric modes for which m > 0 (at least one nodal diameter) and it is the 11 mode which corresponds to the pitch of the drum.

Analysis of the sound from a kettledrum gave the following frequency ratios identified with the modes:

m n	11	21	31	41	51
f_{mn}	1	1.5	2	2.44	2.90

These were the most prominent modes. Others like 01, 02, 12, 22, 03 had much smaller amplitudes. The modes in the table have frequencies that are close to a harmonic series with frequency ratios 2 : 3 : 4 : 5 : 6. Hence the definite pitch. However, the pitch perceived is that of the 11 mode, which is the 2nd harmonic. Normally it is the pitch of the 1st harmonic which is perceived, even if the 1st harmonic is absent. In this case, the rapid decay of the sound gives rise to the perceived pitch. A computer generated sound which is more sustaining lowers the perceived pitch by an octave as expected.

The major role of the kettle is to act as a baffle to isolate the top surface from the bottom and hence improve the radiation of the sound. Also, coupling between the motion of the air in the kettle and vibrations of the membrane affects the rate of decay of the sound.

A change in the volume of the kettle has no effect on the modes except for 11. Lowering the volume of the kettle decreases the frequency of this mode, thus destroying the harmonic relationships seen in the table.

The violin

Everyone recognises a violin when they see one, but look at the picture showing it in pieces. The sound post is not normally seen since it is inside the body, wedged (not glued) between the top plate and the back plate, near the foot of the bridge. Note also the bass bar, glued to the top plate inside the body, parallel with the strings. This reinforces the top plate to withstand the combined tension in the strings, which amounts to about 24 kg and results in a downward force on the bridge of about 9 kg.

It is clear that the primary vibrators are the four strings which are tuned a perfect fifth apart. Each is made to vibrate by bowing and the pitch of the note is changed by stopping the string, i.e. by changing its vibrating length, using the fingers.

The violin will be compared with the viola and cello, so let us look at the tuning of each instrument:

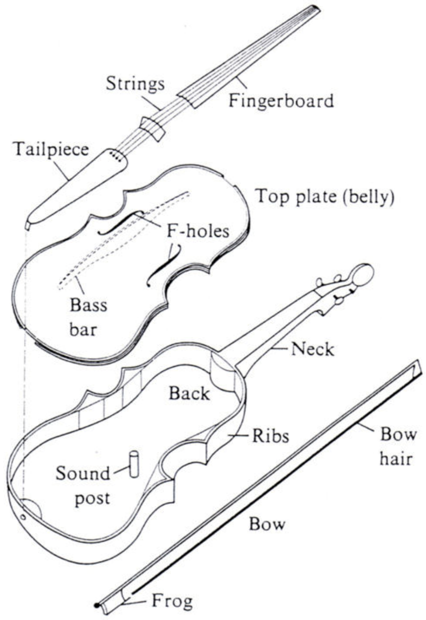


Fig. 10.1. Parts of a violin (from Rossing, 1982).

Instrument:	Cello	Viola	Violin
Strings tuned to:	CGDA	CGDA	GDAE
Pitch:	C is two octaves below middle C	C is one octave below middle C	D is the note above middle C
Range:	3 octaves + 3rd	3 octaves + 3rd	3 octaves + 6th

In each case, the strings are tuned a perfect fifth apart. It can be seen that the viola plays a perfect fifth below the violin and the cello plays an octave below the viola.

The bowed string

The motion of the bowed string is not what you would expect without a lot of thought. Let us consider first of all a simple but inadequate explanation of how the bow sets the string vibrating. The hairs on the bow are coated with rosin to increase the coefficient of friction with the string. The bow draws the string to one side until the friction is overcome by the increasing restoring force. The string then slides back quickly under the bow, since dynamic friction is less than static, until it catches on the bow again. Once more it is drawn slowly to one side and the cycle repeats itself.

This description of bowing relies upon friction to make the string slide and then to stick to the bow. Friction is notoriously unreliable and, if this were the only mechanism, a horrible scraping sound would result. To get a musical note we need a constant and well-defined period of vibration, i.e. we need something that defines precisely the point at which the string slips and when it sticks again.

The following Maple animation shows the motion of the string:

```
restart: with(plots):
y:=(x,t)->sum(sin(n*Pi*x)*sin(2*n*Pi*t)/n^2, n=1..50): nframes:=100:
```

```
animate(y(x,t), x=0..1, t=0..1-1/nframes, colour=black, frames=nframes);
```

It can be seen that the shape of the string is always in two straight-line segments. The kink between the segments travels along the string at the speed of sound and is reflected from each end. Notice that the reflection inverts the wave. With the kink moving in an anti-clockwise direction, the bow will be near the left hand end of the string moving upwards. Note the transverse motion of the string at this point. The bow moves the string slowly upwards and, at the arrival of the kink, the string slips and moves quickly downwards. Upon the return of the kink, the string sticks to the bow once more and is moved slowly upwards. The arrival of the kink therefore provides the signal for sticking and slipping, which gives rise to perfectly periodic motion, as required. When the direction of the bow is reversed, the kink goes round the other way. A slow motion video of a bowed string can be seen at http://tinyurl.com/c69v6dd. Notice that when the bowing stops, the vibration of the string becomes a stationary wave in the first mode.

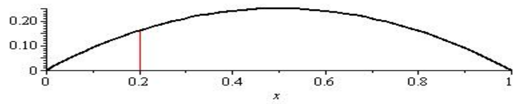
Bowing speed

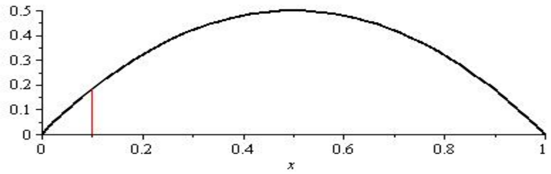
The time during which the bow draws the string to one side is fixed for a fixed position of the bow. It is the time taken for the wave to travel from the bow to the end of the string and back again. It follows that the faster the bow moves, the greater will be the amplitude of vibration. Therefore, the speed of the bow controls the volume of sound.

Position of bow

There can be no harmonic on the string which has a node at the position of the bow. Since the bow is usually very near the bridge, this only affects high harmonics.

The major effect of altering the bow position is to change the amplitude of vibration and hence the loudness of the note. The envelope of vibration of the string is a parabola as shown in the diagrams below. The displacement at the position of the bow is given by the vertical line, which is approximately the same in both diagrams. However, halving the distance between the bow and the bridge doubles the displacement at the centre of the string. Moving the bow towards the bridge therefore results in a louder note.





Bowing pressure

The combined effect of bow pressure and bow position is shown in the diagram. Note the logarithmic scale on both axes. ‘Sul tasto’ means the bow is near the fingerboard. ‘Sul ponticello’ means the bow is near the bridge. Clearly, only a limited area of the diagram produces good violin tone.

With too much bowing pressure, the arrival of the kink at the bow cannot overcome the

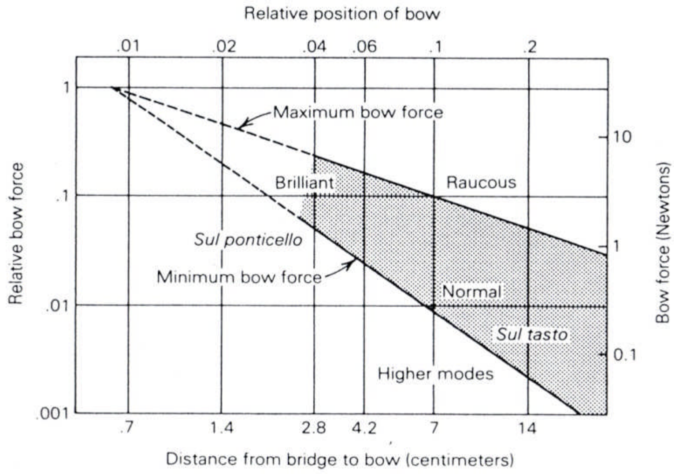
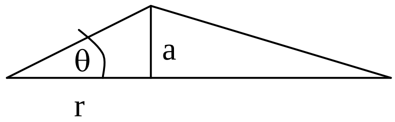


Diagram after Schelleng (1974) showing different bowing regimes for a cello A string with bow speed $V = 20$ cm/sec. Only in the shaded region between the two slanting lines can normal string vibrations be maintained.

friction and the string remains stuck to the bow. Friction is then the only mechanism to release the string and a raucous scraping sound results.

With too little pressure, the restoring force overcomes friction before the arrival of the kink and this initiates a second kink. The result is a higher mode of vibration of the string and the fundamental is not excited. This gives the violin an ethereal sound, sometimes used for special musical effect.

As the bow moves closer to the bridge, the restoring force increases for a given displacement and more bow pressure is required to maintain good tone. From the diagram: $\textit{restoring force} \propto \sin\theta \approx \tan\theta = \frac{a}{r}$



i.e. maximum pressure is roughly proportional to *1/distance from bridge*. Minimum pressure is roughly proportional to *1/(distance from bridge)*^{2}. The use of more pressure sharpens the curvature of the kink resulting in stronger higher harmonics, giving a more brilliant sound. Try the string animation using fewer harmonics and note the more rounded appearance of the kink.

Impedance matching

A string vibrating between rigid supports will transmit very little sound into the air. We have already seen this with Melde's experiment. In order to hear the violin, all the air in the room has to be set vibrating. The amount of air moved by the string is very little indeed and is certainly not enough to fill a room with sound – it is like stirring a swimming pool with a toothpick. It should be noted, however, that the amplitude of vibration is large enough to be clearly visible. The displacement amplitude of a sound wave in air which can be detected by the ear is incredibly small – something like 2×10^{-12}m can be detected, which is a small fraction of an atomic diameter! What is therefore required is a device to transform *large amplitude, small volume* into *small amplitude, large volume* the total sound energy in each case being about the same. This is similar to an electrical transformer which will convert *large current, small voltage* to *small current, large voltage*.

The device in acoustics which does this is known as a matching transformer – it matches the vibrations of the primary oscillator (the string in the case of a violin) to vibrations in the open air so that the sound of the instrument may be heard. In the violin, this is done by the violin body. Without the body, the violin would be inaudible, like an electric guitar without amplification. Most instruments require a transformer to match the primary vibrator to the atmosphere.

Demo: a tuning fork held in the hand is very quiet, but when it touches a table top, it is quite audible.

Flow of energy

The energy in the vibrating string is transmitted to the front plate through the bridge. Only a small amount of energy should be transmitted in this way as most of it should be reflected back along the string to sustain the waves on the string. Vibrations of the front plate are transmitted to the back plate through the sound post, ribs and air cavity. All of these will have different resonant frequencies. Ideally, what is required is a good response at all frequencies. This is impossible if the resonances are sharp as in a lightly damped system, but resonances of both air and wood are well damped (a block of wood doesn't ring like a bell). The violin body therefore gives broad band resonance amplification. If some resonances are much stronger than others, some notes will sound strongly while other parts of the range will be comparatively dead. Such a violin would be of poor quality. We will examine this shortly.

A problem suffered by some cellos is known as the "wolf note", so-called because it is a note that sounds bad and cannot be controlled. This occurs when there is a sharp resonance at a particular frequency. Because the resonance is sharp, it can absorb a lot of power, to the point that it drains the string of energy which then stops vibrating. It is restarted by the bow but, because the string is too well matched to the body, little reflection of the waves from the bridge takes place. This makes it extremely difficult to control the sound and it results in the wolf note. The cure is to increase the damping of the resonance and shift the frequency to one not used in normal playing. Usually, wool wrapped around the tail of the string achieves this.

Demo: Show transfer of energy between matched oscillators using coupled pendulums.

Wood and cavity resonances

There are two main types of resonances on a violin:
The fundamental vibration of the body – the 'wood resonance'.
The fundamental vibration of the air inside the body – the 'cavity resonance'.

These are investigated using loudness curves. The player is instructed to play as loudly as possible on each note over the whole range of the instrument and the loudness of each note is recorded. When the pitch of the note corresponds to a main resonance, it sounds louder than the notes beside it and the main resonances can usually be seen as peaks on the loudness curve. The sound of the cavity resonance is communicated to the air through the f-holes in the top plate. The shape of the holes is decorative; it is the area of the holes which affects the resonance and is therefore important.

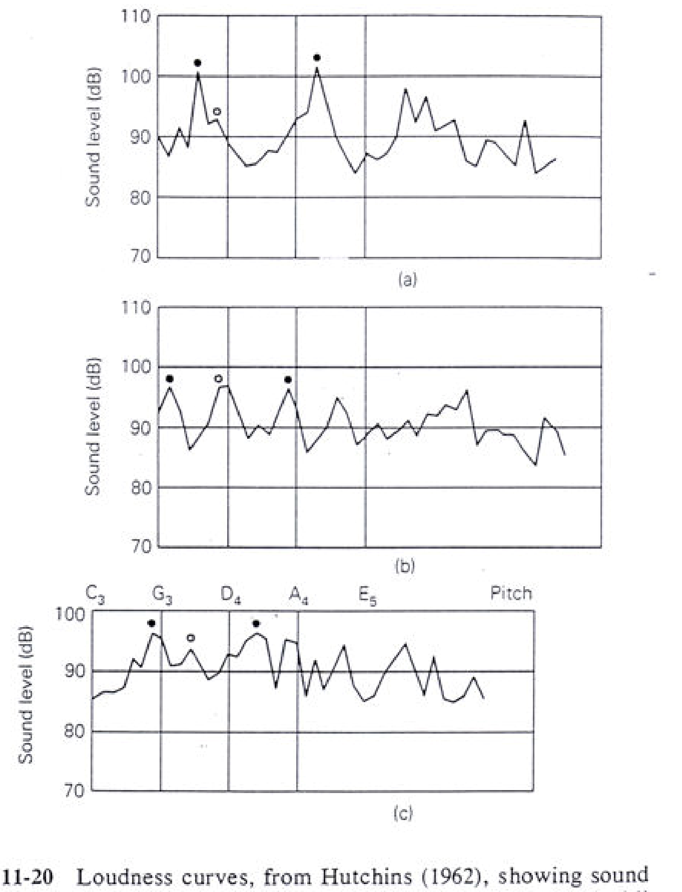


Figure 11-20 Loudness curves, from Hutchins (1962), showing sound levels produced by playing each note as loudly as possible. Vertical lines mark open-string pitches. (a) A poor violin from the early eighteenth century. (b) A good Stradivarius of similar age. (c) A good viola. Open circles indicate air resonance; solid dots indicate wood resonances. (By permission of *Scientific American*)

From practical experience, a good violin will have its main wood resonance very close to the A string (the third string). There is always a related resonance on a sub-harmonic an octave below this. In addition, the main cavity resonance should be close to D, i.e. near the second string. Violins whose main resonances deviate significantly from these frequencies will almost certainly sound inferior.

The top graph shows the loudness curve of a poor quality violin. The wood resonances are too prominent and too far away from the optimum frequency. The cavity resonance is hardly to be seen. The middle graph is taken from a good violin for which the overall response is much more even than the one above. The main wood resonance and the cavity resonance are both close to the frequencies of the middle strings. The third graph is from a good quality viola. Note the positions of the resonances. They are considerably shifted from the optimum positions, suggesting that the viola acoustics are not as good as those for a violin. This will be examined more closely later.

Demo: Video of an underwater violinist. Speed of sound in water at 20^{o}C is 1483ms^{-1}. This shifts the cavity resonances by a large amount and considerably alters the sound. In addition, the wood resonances are almost completely damped out.

Acoustics of violin plates

How can the wood resonances be correctly placed as the violin is being constructed? At what frequencies should the isolated top and back plates resonate to produce a top quality violin? These are the big questions of violin making. There are no definitive answers, but enormous skill and experience helps. The resonances of the top and back plates can be examined separately using the Chladni method or possibly by holographic interference. However, these are not easily related to the resonances of the complete instrument. There are recipes for the frequencies of different modes and how the two plates should be related, but much more investigation needs to be done.

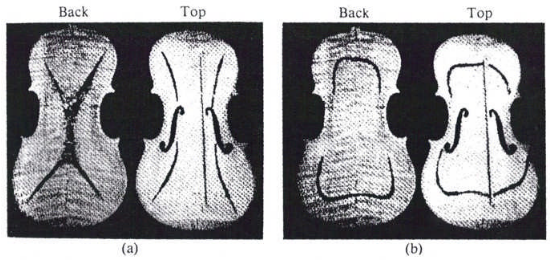


Fig. 3.18. Chladni patterns showing two modes of vibration in the top and back of a viola (Hutchins, 1977).

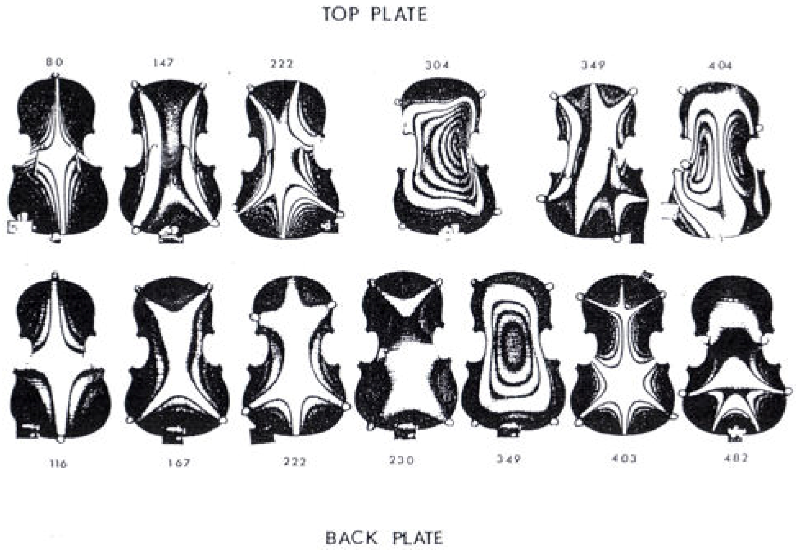


Fig. 10.15. Time-average holographic interferograms of a free violin top plate and back plate (Hutchins et al., 1971).

Individual plate resonances can be adjusted once the Chladni pattern is known. Making a plate thinner where the maximum bending occurs for a particular mode will lower the frequency of that mode. Other modes which have a node at that place will be almost unaffected. By careful shaping of the plate, all the modes can be adjusted to give the desired frequencies.

Comparison with the viola and cello

It is recognised that the violin is a fully developed instrument, leaving little room for future improvement. The best violins are now worth millions of pounds – well beyond the means of musicians who could make good use of them. However, it is also recognised that the viola and cello are acoustically inferior to the violin. Why is this and what can be done about it?

Surprisingly, there are no standard measurements for the size of the violin, viola or cello. Sizes are decided by each individual maker, but the following are acceptable averages:

	Length of body	Rib thickness	Theoretical length	Theoretical thickness
Violin	35.0cm	4.4cm	35.0cm	4.4cm
Viola	40.0cm	4.7cm	52.5cm	6.6cm
Cello	72.5cm	13.6cm	105.0cm	13.2cm

Taking the violin as our pattern, what sizes should the viola and cello be? Since the viola plays a perfect fifth below the violin, it should be 50% larger than the violin, in its linear dimensions, to keep the resonances in the same relative places. (A perfect fifth has frequency ratio 2 : 3.) Similarly, the cello plays an octave lower than the viola, so it should be twice as large as a viola or exactly three times the size of the violin to possess the same acoustical characteristics. These theoretical sizes have been included in the above table and they clearly don't match the sizes of the real instruments.

The effect of this is clearly seen in the loudness curve of the viola. Since the viola is too small, the resonances are all at higher frequencies than are desirable for a good quality instrument. However, a viola of the theoretical size is far too big to be played under the chin like a violin – it would have to be played vertically like a cello instead. There is no reason to restrict a viola to being played under the chin, so why not be adventurous and design a new viola of the correct size, modelled on the violin, which is played like a cello? Why stop there? Why not design a complete stringed instrument family using the violin as the pattern for all the members? This has already been done and we can now take a quick look at the result, which is the string octet.

The string octet

A group of physicists and violin makers have applied scientific principles to violin making and have produced an octet of stringed instruments using the violin as the pattern. The new viola, for example is the correct theoretical size and it has to be played vertically like a cello. Notice that the new cello is bigger than the standard orchestral cello and a new instrument now fills the gap between the two.

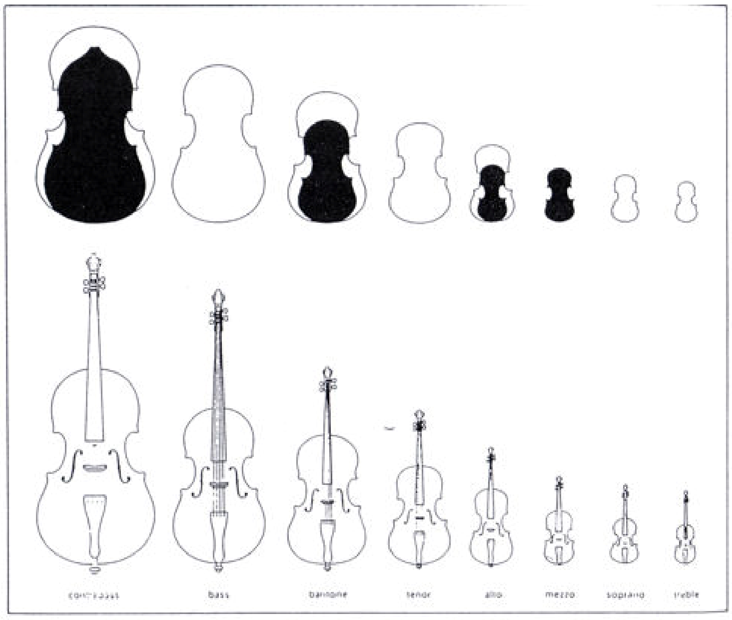


Figure 3.22 Sizes of the instruments of the Catgut Octet. The black shapes represent the outlines of the conventional orchestral strings on the same scale.

Players and composers are also in the project, so music can be written for the new instruments and they are played by professional musicians. Reports of recitals they have given show that the new instruments are very successful and a considerable improvement over the standard viola, cello and double bass.

The secret of the Stradivarius violin?

Violins made by Antonio Stradivari (1644-1737) are claimed to be the best ever made and can cost millions of pounds if you want to buy one. Over the centuries, violin makers have strived to determine Stradivari's secret, but none has succeeded. However, some modern violins are almost as good as a Strad., as determined by listening tests in which professional violinists have failed to identify the Stradivarius violin when compared with a modern one. Ideas of how Stradivari made such good violins are slowly emerging and I have found a recent one which is entirely plausible. Definitive tests have yet to be made and the results are certain to be interesting.

It appears that Stradivari made his violins from wood which had grown during the 'Little Ice Age' 1645 – 1750. The long winters and cold summers produced wood with very narrow growth rings and which was very even in its properties. Its elasticity is normal, but its density is low, which results in the speed of sound in the wood being higher than normal. Such wood is not readily available nowadays.

A wood scientist has discovered a species of fungus that degrades wood in such a way that it has the properties of the 'little ice age' wood that Stradivari used. Violins made of this treated wood seem to have tonal qualities indistinguishable from a genuine Strad. The tests carried out have been too simple and not statistically valid to draw firm conclusions, but the idea is being pursued and more valid tests are to be made. See http://esciencenews.com/articles/2012/09/08/treatment.with.fungi.makes.a.modern.violin.sound.a.stradiavarius for a news item on this topic.

In a recent trial, five violins were tested: one was a Stradivarius, two were made of treated wood (one had been treated for 9 months and the other for 6 months) and two were made of untreated wood. Of 180 listeners (not specially selected), 90 felt that the tone of the 9 months treated violin was best, the Stradivarius came second with 39 votes and the other treated violin came third. 113 listeners thought that the 9 months treated violin was the Strad.

The piano

Look at the picture showing the construction of the piano and note the sound board. This is an important part of the piano that many people are not aware of. The strings are stretched over a bridge attached to the sound board, similar to the strings of a violin, so the sound board is the matching transformer of the piano. It is this which radiates the sound and the piano would be a very quiet instrument without it. However, most of the interesting research into the piano has been done on the strings, so we concentrate mainly on those.

Some piano facts will get us going:

- The piano has 88 keys which give the notes from bottom A at 27.5 Hz to top C at 4224 Hz, a range of over 7 octaves. This is the greatest range of any instrument apart from the pipe organ.
- The sound board is made of wood, which has a huge number of overlapping resonances and so will resonate almost equally well at any frequency.
- The strings are struck by hammers operated from the keyboard. When the key is released, a damper falls onto the string and stops it from vibrating.
- The strings are made of steel and, in a concert grand, each is under a tension of about 90 kg.
- The total tension in all the strings is about 30,000 kg. This huge force is supported by a cast iron frame weighing about 200 kg – this is the main reason why pianos are so heavy.

Early pianos were nowhere near as heavy as this and were lightly strung, requiring only a wooden frame to support the tension on the strings. There is a big difference in the sound they make as is obvious from the following recordings:

Demo: Recorded sound from a piano of 1720 followed by the sound of a modern piano.

What is the physics behind this difference in sound and why do modern pianos require such a huge string tension?

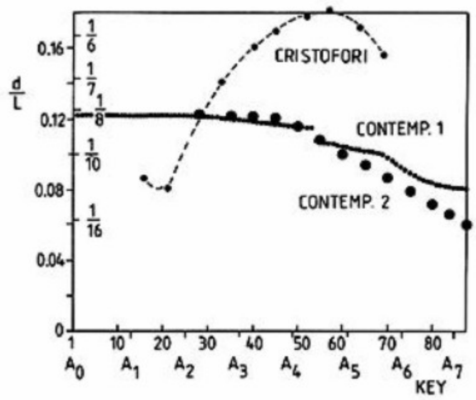
The first piano

The piano was developed from the harpsichord at the beginning of the 18th century. The harpsichord is a keyboard instrument in which the keys operate quills that pluck the strings. This limits its dynamic range. That is, it cannot change the volume of sound unless it is fitted with two keyboards and even then it only has two different levels of loudness. Musicians wanted a more flexible instrument and, if possible, a more powerful one.

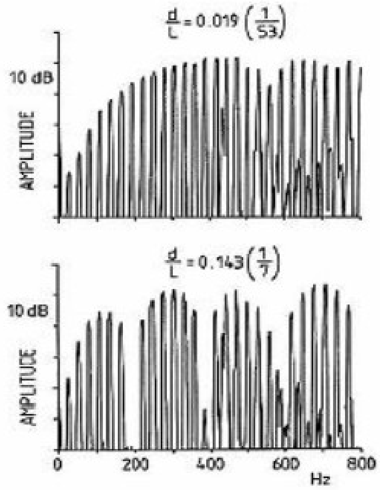
Both of these problems were solved by the Italian harpsichord maker Bartolomeo Cristofori when, in 1709, he built the first hammer-action keyboard instrument. The action of the harpsichord plucked the strings, which limited it to fairly light strings. On the other hand, a hammer action could cope with heavier strings quite easily. This immediately gave it a louder sound and the instrument could play both softly and loudly using a light or a heavy touch on the keyboard. It has therefore become known as a **pianoforte** (Italian for soft loud). Cristofori described it as ‘arpicimbalo che fa il piano e il forte’ – harpsichord which goes soft and loud.

Contact ratio

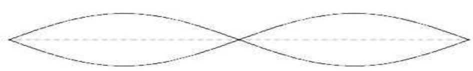
The pianoforte has undergone a steady development ever since, with one or two breakthroughs on the way. However, it took 100 years for piano makers to realise that the sound was affected by the string contact ratio. If *d* is the distance from the end of the string to where the hammer strikes and *L* is the length of the string, then *d/L* is the contact ratio. The graph compares the contact ratio for a Cristofori piano with two modern instruments. There is no grand theory which dictates what the contact ratio should be, it is simply adjusted to give the most pleasing sound.



The effect on the sound is shown in two contrasting harmonic analyses. The upper one is the analysis of the sound obtained using an extremely small contact ratio (1/53), where it can be seen that the low harmonics are very weak, but all harmonics are present. The other graph is for a contact ratio of exactly 1/7. This leads to every 7th harmonic being either absent or very weak.



In general, striking the string at 1/*n* of its length will suppress every nth harmonic, because that is the position of one of its nodes. The diagram below shows a string vibrating in its first harmonic, which is strongly excited by hitting the string half way along. The next diagram shows the second harmonic, which has a node at this point, and therefore cannot be excited in the same way.



Metal frame

With the increase in size of the orchestra during the classical period, pianists required more power from their instruments to balance the greater sound of the orchestra. The total energy that a string can have is ultimately a function of its total mass. A more powerful piano must therefore have more massive strings, i.e. an increase in linear density. This, in turn, requires greater tension if the strings are to vibrate at the same frequencies as before. Eventually, a limit to the total tension is reached when the wooden frame is no longer strong enough to withstand the forces involved.

It was clear that a more powerful instrument required a metal frame. Unsuccessful experiments with metal frames were conducted from about 1800 until 1855 when Henry

Steinway (a German-born American) produced a piano with a cast iron frame that proved to be satisfactory. It has served as the model for all subsequent piano frames.

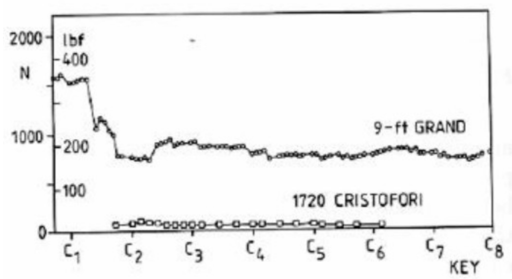
High string tension

A metal frame suddenly allowed even heavier strings to be used, requiring even more tension, but also giving a louder sound. However, the tension is high not only because the strings are massive but because they also produce a more brilliant sound. The equation of motion of the damped harmonic oscillator, $\ddot{x}+b\,\dot{x}+\omega^2 x = 0$, shows that the restoring force, $\omega^2 x$, is related to the frequency of vibration, ω. Since the restoring force is supplied by the tension, a higher tension means the string can support higher frequency vibrations. Higher harmonics give a more brilliant sound, which most people prefer.

String thickness

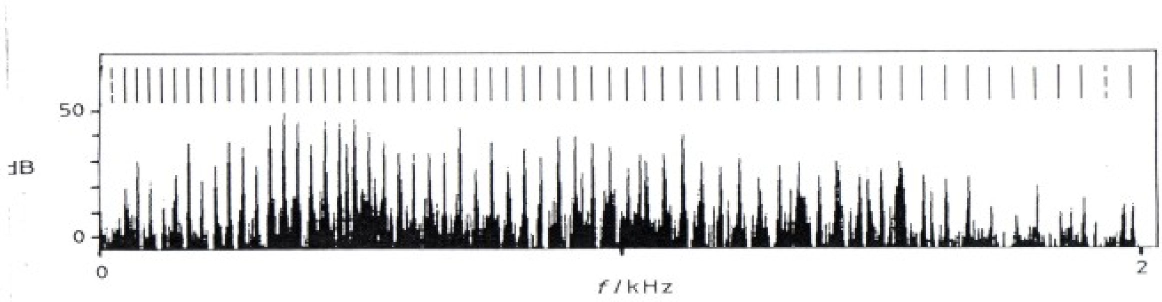
The top note of my piano at home (4224 Hz) has a string length of 6.2 cm. If the same string is used for the bottom note (27.5 Hz), it would have a length of $\frac{4224}{27.5}\times 0.062 = 9.5\,\text{m}$ at the same tension. This is quite impractical, so thicker strings need to be used for the lower notes. In fact, six different thicknesses of string are normally used throughout the piano, which keeps the lengths within a practical range.

The graph compares the string tension in a Cristofori piano with a modern one. It not only shows that modern piano strings have about 12 times the tension of those of early pianos, but that the tension remains roughly constant over the whole range of frequencies so as not to distort the frame.



Inharmonicity

Thicker strings come with a disadvantage. Because they are made of steel, they have their own internal elasticity (stiffness) which adds to the restoring force supplied by the tension. In the extreme, if the strings are so thick that they behave as metal bars, no tension at all would be required and we would have an instrument with a sound like a glockenspiel. This is definitely not good piano tone, as the partials of a glockenspiel do not form a harmonic series. Stiff piano strings therefore result in inharmonic partials and this effect is enhanced the thicker the strings are. The effect is also enhanced with increasing frequency because of the greater curvature of the string. Look at the frequency spectrum of a low note from a Broadwood grand piano. It is made up of about 60 partials, but note also that the spacing of the partials increases with increasing frequency, i.e. they are not harmonic and the inharmonicity increases with frequency.



Too much inharmonicity in the string is obviously bad, but how much is too much? Do we need to get rid of it altogether, or is a small amount desirable? Experiments can be conducted using synthesised piano sound in which the amount of inharmonicity can be adjusted. In the opinion of most people, a small amount of inharmonicity is desirable, as it adds a slight "edge" to the tone, making the piano more interesting to listen to. However, the inharmonicity which occurs naturally is more than is desirable, so the task of the piano engineer is to reduce it.

Inharmonicity can be quantified by the **inharmonicity parameter**, *B*, defined as

$$B = \frac{\pi^3 E r^4}{4 T l^2}$$

where E = Young's modulus $\qquad r$ = radius of cross-section of string (assumed circular)
T = tension $\qquad l$ = length of string

The fundamental frequency of a perfectly flexible string is $\qquad f_0 = \frac{1}{2l}\sqrt{\frac{T}{\rho}}$

This is modified by stiffness, so the frequency of the *n*th partial is $\quad f_n = n f_0 \left(1 + B n^2\right)^{1/2}$

Note that $B \propto r^4$ and $B \propto \frac{1}{l^2}$. The strings most affected are therefore the low notes where *r* is large and the high notes where *l* is small.

It can also be seen that the value of B is reduced with high string tension. This is because with a high tension, there is proportionately more restoring force provided by the tension than by internal elasticity.

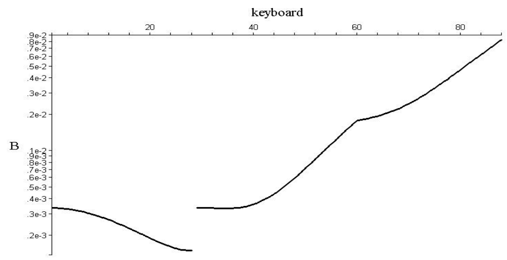
Starting with the high notes, the value of *B* is decreased by using thin strings, i.e. by reducing *r*. This clearly also reduces the vibrating mass, but this is restored by having more than one string per note. The details depend upon the manufacturer, but typical for a grand piano is:

- The top 60 notes have 3 strings each.
- The next 18 notes have 2 strings each.
- The lowest 10 notes have 1 string each.

The strings of the lowest notes have to be very massive to reduce their lengths. This would make them too stiff, so they are made with a relatively thin steel core wrapped in copper wire. The wrapping restores their mass without adding to the stiffness. They therefore have the stiffness of the internal steel core. The lowest 28 notes will typically use wound strings. Bass guitars will normally use wound strings for the same reason.

Demo: Display a wound piano string.

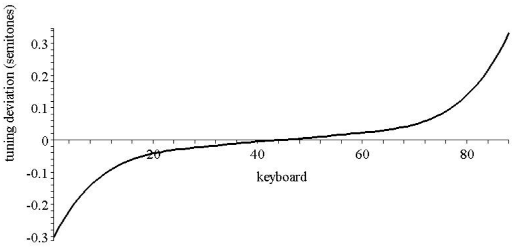
The value of *B* can vary enormously throughout the piano as seen in the graph. Note the logarithmic scale on the *B* axis. The graph refers to a cheap upright piano where the inharmonicity is expected to be greater than for a grand piano.



For B = 0.001:

- The 16th partial will be a whole tone sharper than the 16th harmonic.
- The 25th partial will be two tones sharper than the 25th harmonic.

Stretch tuning



The effect of inharmonicity on the tuning of the piano is to stretch the octaves. They are slightly wider apart than the expected 2 : 1 frequency ratio. Instead of tuning the fundamentals accurately together and leaving the higher partials out of tune, a better sound is produced (more in tune) if the interval is widened slightly to tune the higher partials together. The effect is greatest in the lower and higher octaves of the piano and is least in the middle. The graph shows the tuning of octave intervals averaged over a number of professionally tuned grand pianos.

Woodwind instruments

Most woodwind instruments use a single vibrating column of air, with finger holes, e.g. flute, oboe. A few instruments, which work in the same way as woodwinds, use a different air column for each note, e.g. organ, Pan pipes.

The most common way of characterising woodwinds is by the method of initiating the vibrations:

Edge tone – flute, recorder, flue organ pipes, Pan pipes.
Single reed – clarinet, saxophone, reed organ pipes.
Double reed – oboe, cor anglais, bassoon.

If a single column of air is used, the pitch is changed by altering the effective length of the pipe. This is done by covering or uncovering holes in the pipe using fingers or mechanical aids. In addition, the range of the instrument is increased by making use of higher modes of vibration. However, the positions of the finger holes cannot be changed, so the same set of holes has to be used for each mode. This greatly restricts the shape the air column can take, since the positions of nodes and antinodes for each mode are quite different for all shapes of pipe except for the cylinder and the cone. Only these shapes are therefore used in orchestral instruments. If only a single mode needs to be used, the pipe can be any shape.

We can therefore characterise the instruments according to their shape:

Cylindrical - flute, clarinet.
Conical - oboe, cor anglais, bassoon, saxophone.

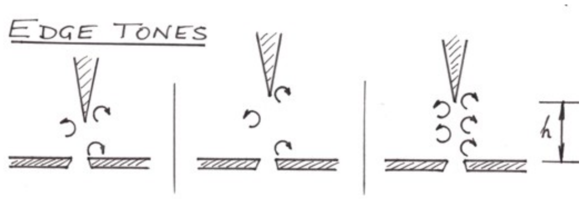
Organ pipes can be any shape because each one is designed to produce only one note.

The flute

A flute consists of a cylindrical column of air whose effective length can be altered using finger holes. Its range is 3 octaves. The piccolo works in the same way as a flute, but it plays an octave higher. One of the first things we need to know is how to get the air column vibrating. This is done using an edge tone.

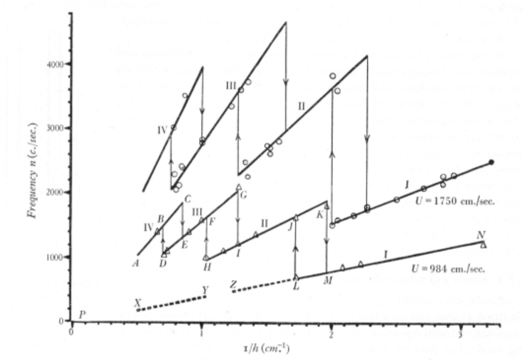
Edge tone

The edge tone is so-called because it is produced by blowing a stream of air against an edge. When air is blown out of a slit, the stream of air sheds vortices on either side and these are separated by the edge as shown. This produces periodic pressure variations in the air which give a tone of a definite pitch. It is the means of initiating vibrations in the flute, recorder, flue organ pipes and all whistles.



The pitch of the edge tone can be varied in two ways. As seen in the diagram, increasing the distance between the slit and the edge stretches out the train of vortices and the pitch is lowered because of the increased wavelength. There is a limit to the stretching and a point is reached when an extra vortex is fitted into the gap between slit and edge. This results in a sudden raising of the pitch. The opposite happens when the gap is narrowed, but there is a significant amount of hysteresis. This is shown in the next diagram.

The second way of altering the pitch is to change the speed of the air through the slit. A higher speed gives a higher pitch and this is also seen on the graph. The relationship between wind speed, *v*, frequency of edge tone, *f*, and edge to slit distance, *h*, is to a good approximation $\frac{v}{h} = 2f$



<u>Demo</u>: edge tone generator – alter edge to slit distance and wind speed.

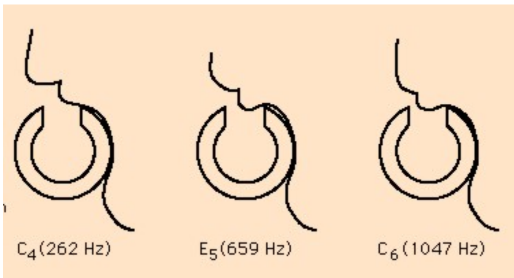
<u>Playing a note</u>

In a flute, the slit for the air is formed by the player’s mouth and the air stream is directed against the edge of the hole (called the embouchure hole) at one end of the flute. With the pitch of the edge tone being very variable, some mechanism is required to control it in order to play in tune. This is done by the column of air, which is the rest of the flute, and it will respond only at its resonant frequency. The reflected wave from the bore then feeds back to the edge tone and controls its frequency. By altering the distance between edge and lips and by controlling the speed of the air, a skilful player matches the frequency of the edge tone to that given by the bore. This produces the best flute tone. The bore is seen to be simply a resonator. The edge tone is in the stream of vortices shed by the embouchure hole, so no matching transformer is needed to get the sound out of the flute – you have already heard the edge tone generated in the demo and that was fairly loud!

<u>Playing a tune</u>

The length of the air column can be changed by opening or closing finger holes. The holes in a flute are wide enough so the air column is effectively terminated at the first open hole. Starting with all the holes closed gives the lowest note (middle C) and opening the holes in turn gives one octave of the scale of C. However, this has used up all the finger holes.

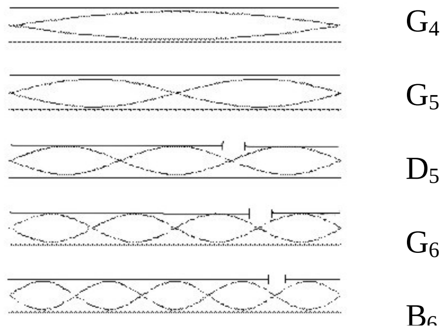
<u>Modes of vibration</u>



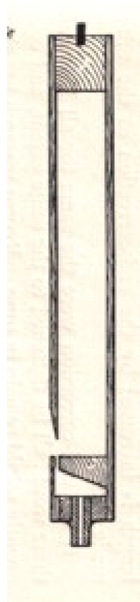
In order to increase the range of the flute, higher modes of vibration are used. The second mode is stimulated by a higher frequency edge tone, obtained by the player either blowing a bit harder or moving his/her lips closer to the edge. Often, a combination of the two is used. Since the flute behaves as an open pipe, the frequency of the second mode is twice that of the first. This means that the notes produced by the second mode are an octave higher than those of the first. Note that there are sufficient finger holes to give all the notes in the octave gap between modes. The range of the flute is three octaves and the third octave is obtained by using still higher modes. To help obtain the higher modes, extra holes are sometimes opened at the positions of displacement antinodes as seen in the diagram. In every case, however, the correct mode is selected by the appropriate frequency of the edge tone.

Demos:

- Play scale of C in lower octave to show changing length of resonator.
- Use flute head joint with short length of hose to demonstrate changes of mode of vibration.
- Play two octaves of C to show combination of modes and resonator lengths.
- Play G_4, G_5, D_5, G_6 by changing mode only.
- Play the bottom note (C_4) and stop the end of the flute – it gives G_4 as the second mode of a stopped pipe.



Flue organ pipes



Flue pipes use the edge tone to stimulate vibrations in the resonator. Since each pipe produces only a single note, the frequency of the edge tone can be matched exactly to the frequency of the pipe. In addition, the pipe may be of any shape and pipes of different shapes are used to produce different quality sounds. The sound comes from the mouth of the pipe (not the end) and, as with a flute, no matching transformer is needed. An open pipe 8 feet long gives a note two octaves below middle C which is the bottom note on an organ keyboard. However, 16 foot pipes are common and 32 foot pipes are to be found on cathedral organs.

The three major families of pipes are:

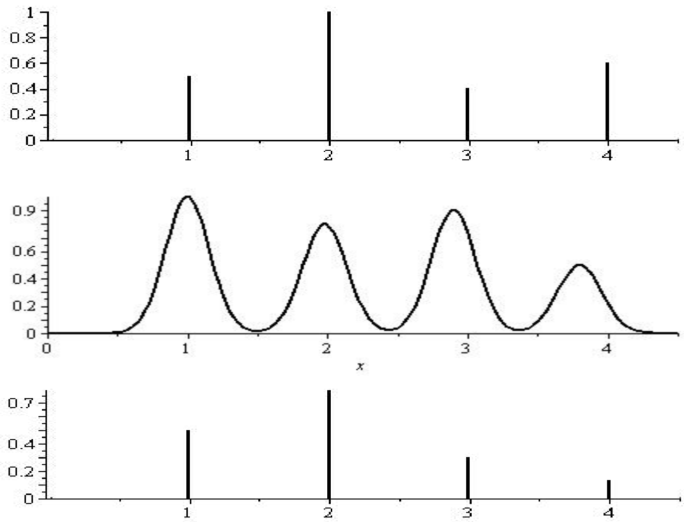
Flute - wide pipe giving few high harmonics.
Diapason - standard organ tone, intermediate between flute and string.
String - narrow pipe, high harmonics prominent.

The width of the pipe clearly has an effect on the harmonic content of the sound produced. This is a consequence of the dependence of the end correction on frequency as well as the radius of the pipe. The following set of purely fictitious figures illustrates this where the frequency ratios of the first 4 partials are calculated:

	Wide pipe		Narrow pipe	
Wavelength	$\varepsilon = 6$	Frequency	$\varepsilon = 2$	Frequency
$2(l + \varepsilon)$	412.0	1.00	404.0	1.00
$(l + 1.2\,\varepsilon)$	207.2	1.99	202.4	2.00
$2(l + 1.4\,\varepsilon)/3$	138.9	2.97	135.2	2.99
$(l + 1.6\,\varepsilon)/2$	104.8	3.93	101.6	3.98

The length of the pipe is l = 200cm; ε is the end correction for the fundamental. The figures exaggerate the change of end correction with frequency, but it is seen that the partials of a narrow pipe are much closer to a harmonic series than those of a wide pipe. Since the edge tone is purely harmonic, it cannot excite modes of vibration which depart significantly from a harmonic series. This limits the harmonic content of wide pipes more than narrow ones.

Below is a diagram showing the frequency spectrum of the edge tone (strictly harmonic), the resonance curve of the pipe (higher modes get further away from harmonic frequencies) and the resultant sound spectrum (product of the edge tone spectrum and the resonance curve) where the higher harmonics are only weakly excited.

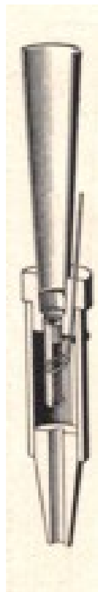


Edge tone spectrum. — Harmonic series

Resonance curve. — Resonance peaks not corresponding to harmonics.

Sound spectrum. — Higher resonances not excited by edge tone.

Reed organ pipes

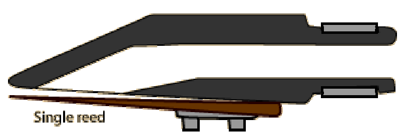


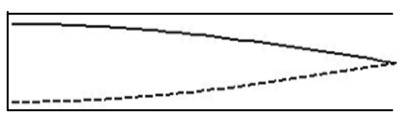
Reeds are normally made of brass as in a harmonium, accordion or Highland bagpipes. The reed is made to vibrate by the **Bernoulli effect** which, in the case of air, means the faster the air flows, the lower is its pressure. The reed is therefore a pressure-operated device.

Demo: Blow across a sheet of paper and it rises.

As air moves past the reed into the resonator, the pressure is lowered and the reed moves to restrict the air flow. This allows the pressure to rise and the reed springs back allowing the air flow to increase again. The process repeats on a regular cycle, i.e. the reed vibrates, and this stimulates a standing wave of sound in the pipe. The reed is tuned to the resonator by changing its vibrating length using the tuning wire. The reed produces a vast range of overtones and the pipe acts as an acoustic filter, resonating at some frequencies and not others. The sound is radiated from the end of the pipe.

The clarinet

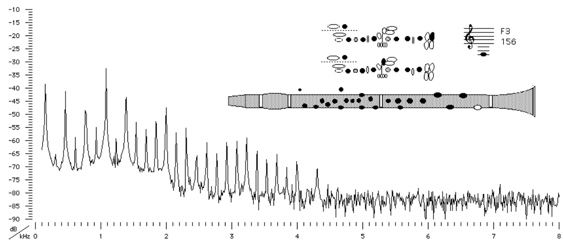




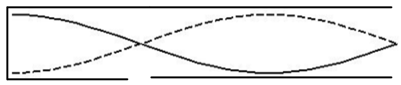
The clarinet has a single reed, made of cane, which operates in the same way as the reed in an organ pipe. The resonator is cylindrical, apart from a small bell at the end. It is about the same length as a flute, but plays an octave lower ($D_3 \rightarrow B_6$b). Why is this?

We have already seen that a cylindrical pipe, closed at one end, resonates at half the frequency of an open pipe. It appears, therefore, that the clarinet behaves as a closed pipe. This is because the reed is a pressure-operated device and so must be situated where the pressure variations are at a maximum, i.e. at the closed end of the pipe. The first mode of vibration must therefore be as shown in the diagram.

This affects the harmonic content of the sound of a clarinet, since a closed pipe only gives the odd numbered harmonics. We saw this when we did wave analysis. The sound spectrum of the clarinet is shown in the diagram, where it can be seen that, for a low note, the even harmonics are weak or absent.

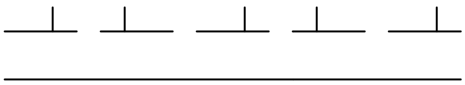


This interpretation is reinforced by the fact that when the clarinet changes mode, the pitch goes up by a 12th (octave + 5th), which is three times the frequency of the fundamental. The diagram shows the pressure variations of the second mode, at a wavelength 1/3 of that of the fundamental. To obtain this mode, the player must make the reed vibrate at the higher frequency either by increasing the wind speed or by using a shortened length of reed. Also, to assist this mode of vibration, the player opens a small hole in the clarinet which is at the pressure node. A skilled player will always match the pitch produced by the reed to the resonant frequency of the bore.

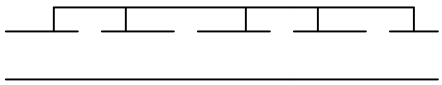


An interval of a 12th upon changing mode means there must be enough finger holes to provide all the notes in between. This is more than the finger holes provided on a flute, which changes pitch by an octave upon going to the second mode. It is also more than the player has fingers, so mechanical devices have to be provided which open or close several holes at once.

The small bell at the end of the instrument serves to match the vibrations in the bore to vibrations in the open air, especially for the low notes when most of the finger holes are closed. However, when some holes are open, they radiate sound and can act as an acoustic filter. That is, only a selected set of frequencies will be radiated from the holes, the rest being radiated from the bell. (More will be said about the acoustics of a bell when brass instruments are considered.) The diagrams below illustrate the action of acoustic filters. In the case of a clarinet, the open finger holes tend to make the clarinet a high-pass filter, so the high frequencies are radiated from the bell and the low frequencies from the finger holes.



High-pass filter
- clarinet or oboe

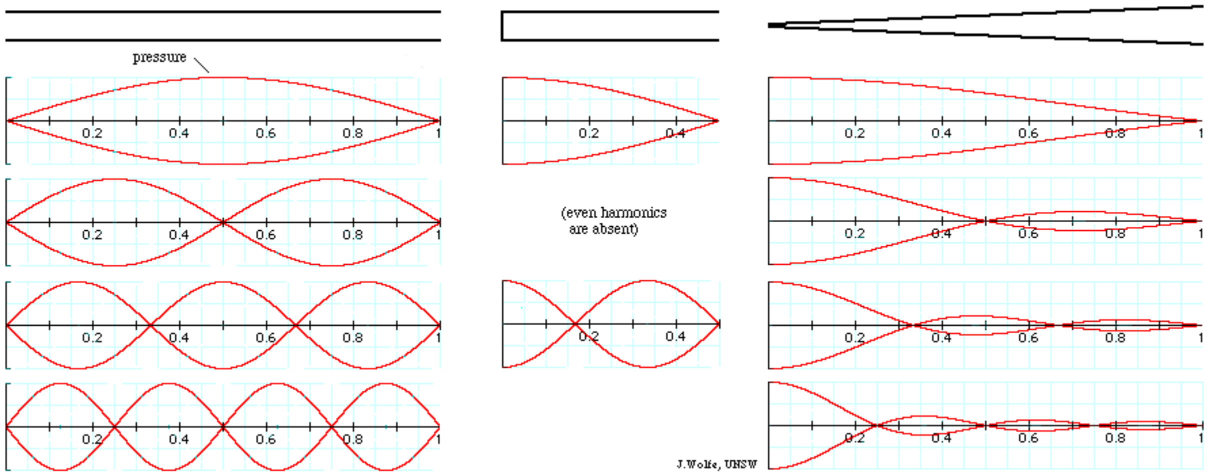


Low-pass filter
- silencer on car exhaust

The oboe

The oboe uses a double reed (see the picture) as the primary vibrator and the reed must be situated near a pressure antinode, as in a clarinet. Unlike the clarinet, the oboe has a conical resonator. It is also about the same physical size as a flute or a clarinet and it plays at roughly the same pitch as the flute ($B_3b \rightarrow F_6$). Why is this when it must act as a pipe closed at one end?

The answer is that the oboe has a conical bore and this changes the physics of the modes of vibration. The mathematics needed to describe these modes is not particularly difficult, but it is a bit beyond the level of this course. A pictorial description should suffice. As the wave front moves along the cone it must either expand or contract depending upon the direction of movement. It is, in fact, part of a spherical wave, i.e. a wave radiating from a point, as opposed to the plane waves encountered in a cylindrical pipe. This gives rise to the modes of vibration shown in the diagram, which includes the modes of vibration of open and closed cylinders for comparison. The three resonators pictured all give the same fundamental frequency.



Note that a conical resonator gives modes of vibration with frequencies in the ratios 1 : 2 : 3 : 4 as does an open cylinder. All harmonics can therefore be present in the sound of the instrument. It also means that a change from first to second mode makes the pitch go up an octave, as in a flute, so the oboe requires fewer finger holes than a clarinet.

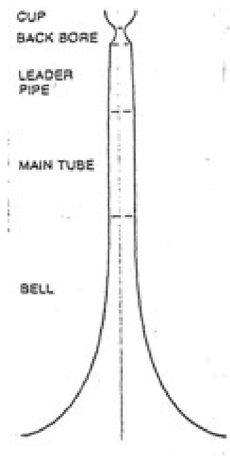
The cor anglais ($E_3 \rightarrow B_5$b), bassoon (B_1b $\rightarrow D_5$) and double bassoon (B_0b $\rightarrow G_3$) are larger versions of the oboe and play at correspondingly lower pitches.

The saxophone

The saxophone is a hybrid instrument between a clarinet and an oboe. It was invented by a Belgian instrument maker called Adolphe Sax, hence its name. It uses a clarinet mouthpiece and reed, but has a conical bore like an oboe. However, the angle of the cone of a saxophone is greater than that of the oboe. It therefore plays at the same pitch as the oboe for the same size of instrument, overblows at the octave and all harmonics can be present in the sound. This accounts for the major difference in sound between a saxophone and a clarinet.

Brass instruments

There are many brass instruments and they all use the same physics. We will therefore consider mainly a generic brass instrument and some specific instruments will get a brief mention.



A brass instrument is simply a metal pipe with a flared bell at one end. The player's lips form a reed and they are held against a mouthpiece at the other end in a position which allows their vibrations to be controlled. Since there are no finger holes, the shape of the pipe is not restricted and the exact shape differs from one instrument to another. Very often it is coiled to make the instrument easier to handle.

With woodwind instruments, we have seen that the shape of the bore (cylindrical or conical) has a marked effect on their behaviour. So, what sort of behaviour should we aim for in a brass instrument and how shall we determine the shape which gives this? An essential requirement is that the instrument should play in tune. That is, changes of mode should give notes which are on the musical scale, i.e. the resonances should form a harmonic series.

Change of shape

To illustrate how a change of shape can alter the frequency of a resonance, consider changing a closed cylindrical pipe to a cone by gradually making the closed end narrower.

We have already seen that a clarinet plays roughly an octave lower than an oboe even though the two instruments are about the same physical size. The frequency of the first mode of the cone is therefore twice the frequency of the first mode of the closed pipe for the same length of resonator. This means that:

- making the bore narrower at a pressure antinode raises the frequency.
- making the bore wider at a pressure antinode lowers the frequency.

Alternatively:

- making the bore narrower at a pressure node lowers the frequency.
- making the bore wider at a pressure node raises the frequency.

This now enables us to alter the frequencies of chosen modes in different ways. Let us look at the first two modes of vibration of a closed cylinder:

If the cylinder is made narrower 1/3 of the way from the closed end, the frequency of the second mode will be lowered as this coincides with a pressure node. However, this is close to the pressure antinode of the first mode, so the frequency of this mode will be raised slightly. Therefore, by carefully choosing the positions where the bore should be widened or narrowed, selected modes can be raised or lowered in pitch to tune the modes to a harmonic series. The exact shape of a brass instrument is therefore fundamental to its playing in tune.

Modes of vibration

The changes of shape just described are usually reserved for small adjustments of the resonances to avoid ending with an instrument that looks rather weird. We still need to look at the overall pattern of resonances in order to start with an instrument that is close to being in tune.

Since the lips of the player form the reed in a brass instrument, there must be a pressure antinode at the mouthpiece. Also, there must be a pressure node near the open end. However, all brass instruments give the complete harmonic series and not just the odd harmonics as in a clarinet. A rough illustration of how this is so is given by the following. The frequencies are entirely invented, but they behave in the same way as a real brass instrument.

Start with a cylinder closed at one end with a fundamental frequency of 100Hz. The resonances give the odd numbered harmonics:

100 300 500 700 900 1100 1300

Add a short extra length for the mouthpiece – reduce the frequencies by 5%:

95 285 475 665 855 1045 1235

But the mouthpiece bore is narrow and it is at a pressure antinode, which raises the pitch of the resonances. Since the narrow section is short, it will be a fraction of the wavelength of even the highest harmonics. However, it will be a larger proportion of the wavelength of the high harmonics than the low, so the high frequencies will be affected the most:

100 295 485 675 870 1060 1250

Now flare the other end into a bell. There is no increase in length, but it is at a pressure node which raises the pitch of all modes. Since the widening occurs over a fairly long length of pipe, the low frequencies are altered most and the high frequencies are altered very little. This is because for a short wavelength, the widening will occur near a pressure antinode as well as a pressure node:

170 355 535 715 900 1080 1260

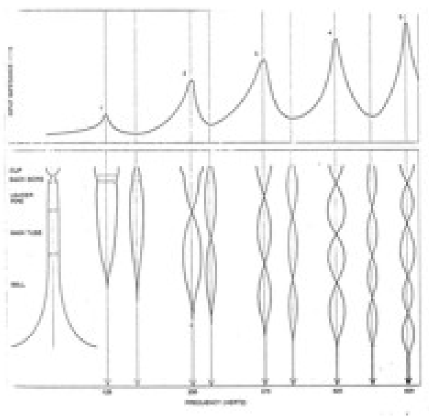
If these form a complete harmonic series, the harmonic numbers must be:

1 2 3 4 5 6 7

In which case, the fundamental must have a frequency of:

170 178 178 179 180 180 180

Even with these rough numbers, the instrument is nearly in tune, starting with the second harmonic. The fundamental (first harmonic) is badly out of tune and this mimics the behaviour of most brass instruments. Note that the complete harmonic series is present even though there is a pressure antinode at one end. These resonances are shown in the diagram. The upper graph is essentially the resonance curve of the instrument and the lower graphs show the standing wave for each mode of vibration. (Ignore the standing waves corresponding to the resonance curve minima.) Note that the effective length of the horn is a function of frequency and that the



positions of the nodes and antinodes are different for each mode.

Bugle

The simplest of the brass instruments is the bugle. The notes it plays are obtained simply by changing mode, so the frequencies are in the ratios 2 : 3 : 4 : 5 : 6 (the fundamental is out of tune and is not used). If the second mode is tuned to C, the other notes it plays will be:

2	3	4	5	6
C	G	C	E	G

It can be seen that the intervals are:

2 : 3 5th 3 : 4 4th 4 : 5 major 3rd 5 : 6 minor 3rd

Trumpet

The lowest note of all brass instruments is based on the second mode. For special musical effect, the out of tune fundamental is sometimes used, but that is rare. A change of mode from the second to the third requires that means be provided to fill in all the notes in the interval of a perfect 5th. Taking the trumpet as the typical example, three valves are provided to switch in extra lengths of tubing that will lower the pitch by various amounts as follows:

interval	semitone	tone	minor 3rd	major 3rd	4th	diminished 5th
valves	2	1	1+2	2+3	1+3	1+2+3

Don't worry – you don't need to remember details like this for the exam.

Trombone

The trombone is unique in that it does not use valves but has a slide instead. The bore of the instrument is adjusted in length using the slide, which of course must be a cylindrical length of tubing.

The flared bell

Apart from the shape of the bell being important for the tuning of the instrument, it has a second important function, that of a matching transformer. It radiates the sound and matches the vibrations in the bore to vibrations in the open air.

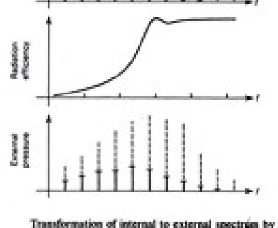
Demo: Modes of vibration can be demonstrated on a hose pipe, but it is rather quiet. Fitting a funnel to the end makes it very much louder.

Because no attention has been paid to the shape of the hose pipe instrument, it is not expected to play in tune. The resonant frequencies do not necessarily form a harmonic series, so changes of mode will not give recognised musical intervals. Nevertheless, it can be used to play music out of tune as demonstrated by:

Demo: Dennis Brain playing Leopold Mozart's Alpenhorn Concerto on a hose pipe at a Hoffnung Concert.

Reflection characteristics

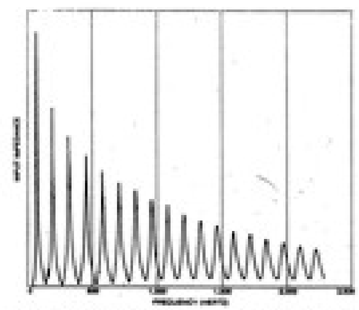
The bell also has the big effect of altering the reflection of sound at the end of the instrument. The proportion of sound reflected varies strongly with frequency, as seen in the diagram. The upper graph gives the spectrum of the sound in the bore. The middle diagram shows the radiation efficiency of the bell

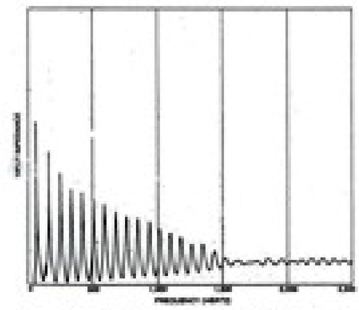


Transformation of internal to external spectrum by the radiation efficiency for B♭ on a trumpet; Solid bars indicate mp; dashed extensions, ff.

and the lower diagram gives the spectrum of sound radiated, which is the product of the two upper graphs.

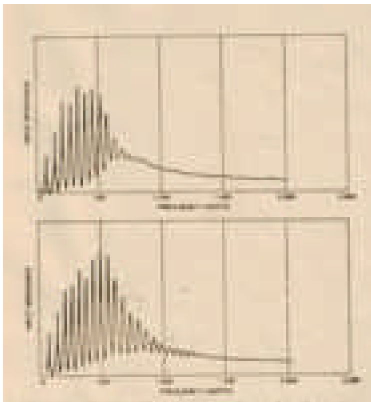
The effect of the bell is also seen in the next graph which shows the resonance curve of a cylindrical pipe. Adding a trumpet bell to the pipe moves the resonances closer together because the pipe has been lengthened. However, of particular note is that the resonances fade out at a lower frequency because the bell radiates sound more effectively at higher frequencies and there is less reflection to sustain the standing wave. This limits the number of modes that can be used and also limits the harmonic content of the sound produced. At very high frequencies the horn simply acts as a megaphone and any high frequency sound produced comes directly from the player's lips and not from a standing wave.





French horn

French horns originally had no valves and players developed the technique of putting a hand into the bell to alter the reflection characteristics so higher modes could be used. The upper graph shows what is essentially the resonance curve of a French horn without the player's hand in position. There are virtually no resonance peaks above 750 Hz so notes like G_5 (783 Hz) cannot be played. Notes in the octave below G_5 would be very weak for lack of strong feedback. The lower curve shows the effect of placing the hand into the bell, where it can be seen that the improved reflection gives rise to more resonance peaks. This not only increases the range of the instrument, but improves its tone. The higher harmonics give notes that are closer together, so almost a complete scale can be played as seen in the following table:



Harmonic	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Note	C	C	G	C	E	G		C	D	E		G			B	C

The harmonics left blank do not correspond to notes on the musical scale. For example, the 7^{th} harmonic is 0.3 of a semitone flat compared with A#. Some of the remaining notes can be obtained by changing the effective length of the horn by altering the position of the hand in the bell. However the quality of the tone was not always good and it was a significant advance when the horn was fitted with valves.

Ophicleide

There are no satisfactory brass instruments with finger holes. Some early trumpets had a small finger hole, but this was just to help stabilise high modes of vibration. The ophicleide is a brass instrument which has about a dozen woodwind-type finger holes. It was the last such instrument in main-stream music and was used till the middle of the 19th century. The fingering is awkward and many notes it produces are of doubtful quality.

Since the whole bore of a brass instrument – from mouthpiece to bell – is essential for the tuning and the sound of the instrument, terminating the standing wave using finger holes before it reaches the bell is bound to be unsatisfactory. This is why the clarinet has such a small bell, making it less essential to the sound, as much of the sound is radiated from the finger holes. The ophicleide, similarly, has a small bell, though the opening is fairly large as it is at the end of a conical resonator – it has to be conical or cylindrical if it has finger holes. Adding finger holes to an instrument with a large bell, like a trumpet or French horn, would be disastrous to both the tuning and the quality of sound. In addition, since the bore is neither conical nor cylindrical, the nodes of the standing waves will be in different positions for different modes of vibration, making the tuning of the notes totally impossible.

7th harmonic

Note that the 7th harmonic is the first not to give a note on the normal scale. It is 0.3 semitones flatter than A#. If notes are played based on this resonance, they will sound badly out of tune with other instruments, so it is not normally used. However, composers have sometimes specifically asked for this resonance to be used for special effect. Vaughan Williams uses it in one of his symphonies and so does Benjamin Britten in the opening movement of his Serenade for Tenor, Horn and Strings.

Transients

The beginning of a note played on a musical instrument is always different from the continuation. This is because the vibrations have to be established where there were no vibrations to begin with. Examples are:

- Vibrations in the body of a violin are driven by the string.
- Vibrations of a piano soundboard are driven by vibrations of the strings.
- The standing wave in the bore of a flute is driven by the edge tone.
- The standing wave in the bore of a clarinet is driven by vibrations of the reed.

There are many similar examples but, in each case, establishing forced vibrations in an air column or setting up vibrations in plates or membranes requires a different kind of movement from the steady state. This initiation of the note is known as the **starting transient**. It normally lasts only for a fraction of a second, but it is a characteristic part of the sound of the instrument. The instrument would sound different without it and it would be more difficult to identify from the sound alone.

We have already seen the equation of motion of a forced, damped harmonic oscillator:

$$\ddot{x} + b\dot{x} + \omega^2 x = f\cos(nt)$$

and its solution is:

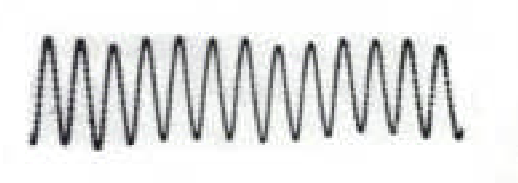
$$x = \exp(-bt/2)\left(A\cos(\upsilon t) + B\sin(\upsilon t)\right) + \frac{f\cos(nt-\delta)}{\left[(\omega^2 - n^2)^2 + b^2 n^2\right]^{1/2}}$$

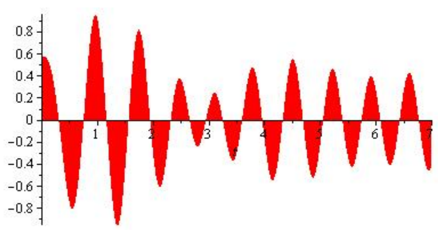
$$\text{where} \quad \upsilon^2 = \omega^2 - b^2/4 \qquad \text{and} \qquad \tan(\delta) = \frac{bn}{\omega^2 - n^2}$$

The first term in the solution is subject to damping and quickly dies out leaving the steady state oscillations given by the second term. The transient at the beginning is given by both terms together. The following Maple commands produce a plot of x as a function of t showing the starting transient followed by the steady state:

```
b:=1.5: omega:=38: f:=500: n:=17: phi:=-90:
phi:=evalf(Pi*phi/180): nu:=sqrt(omega^2-b^2/4):
plot(exp(-b*t/2)*cos(nu*t+phi)+f*((omega^2-n^2)*cos(n*t)+b*n*sin*n*t))/((omega^2-
n^2)^2+b^2*n^2), t=0..8);
```

The transient vibrations of a guitar string are shown in the left-hand picture and the steady state vibrations a short time later are on the right.





The simple mathematical model above can also reproduce the guitar transient as shown by the next graph. This is obtained from the above Maple with the parameters:

```
b:=1: omega:=7: f:=15: n:=9: phi:=0:
```

Demo: Dulciana organ pipe has quite a marked transient.

We can also play back a recording at slow speed so the transient lasts a significant period of time:

Demo: 1. Staccato notes on a flute followed by playback at 1/8 speed.
2. Staccato notes on an oboe followed by playback at 1/8 speed.

To investigate the importance of transients, we can change them on a recording to see what difference it makes.

Demo: 1. Oboe scale passage played normally.
2. Oboe scale passage with transient replaced by a sharp exponential rise.
3. Oboe scale passage with instantaneous initiation and termination of the note.

Transients are very characteristic of each instrument and are one of the clues that allow us to recognise which instrument it is. In a small group of instruments playing a chord, if one drops out, it can be very difficult to identify which one. On the other hand, if one joins in, the identification is very much easier, especially if it has a marked transient.

Temporal evolution

There is much more to a note than its transient. The whole evolution of the note is also important and is also characteristic of the particular instrument. The different stages of the evolution can be labelled **transient**, **steady state** and **decay**. These are also referred to as **attack**, **sustain** and **release**. In the following demo, the recording of each note is played backwards so that it starts with the decay, sounding like an exponential rise, and the transient comes at the end:

Demo: Scale passage on a cello played normally, then repeated with the time reversed.

Reversing the temporal evolution brings your attention to the different stages of each note and alters the characteristic sound of the cello, even though nothing else has been changed. Instead of separate notes, the following demo is a time reversal of a cello playing legato, i.e. the notes join up without a break in between:

Demo: Time reversal of a passage of music on a cello.

There are places in this recording where it sounds as if two cellos are playing. This is because the notes overlap when the decay of a previous note is still taking place as the next note is initiated by the bow. Played backwards, the decay is heard at the same time as the steady state of the adjacent note, whereas played forwards, this goes unnoticed. With two notes sounding simultaneously, your brain interprets this as two instruments playing.

The instrument which has a most marked temporal evolution is the piano. It has no steady state and goes directly from transient to decay. A time-reversed piano therefore sounds most un-piano-like and could be mistaken for an organ:

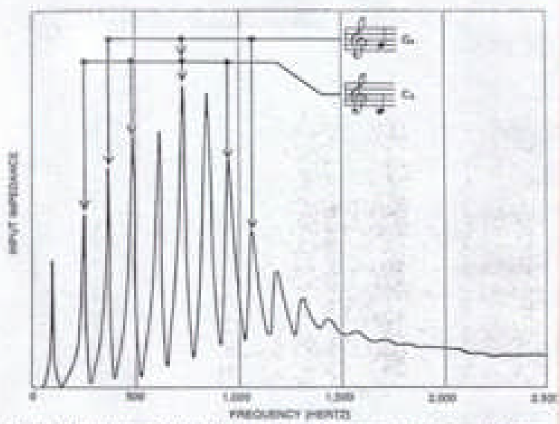
Demo: Piano passage played normally followed by the time reversal of a longer passage.

Formants

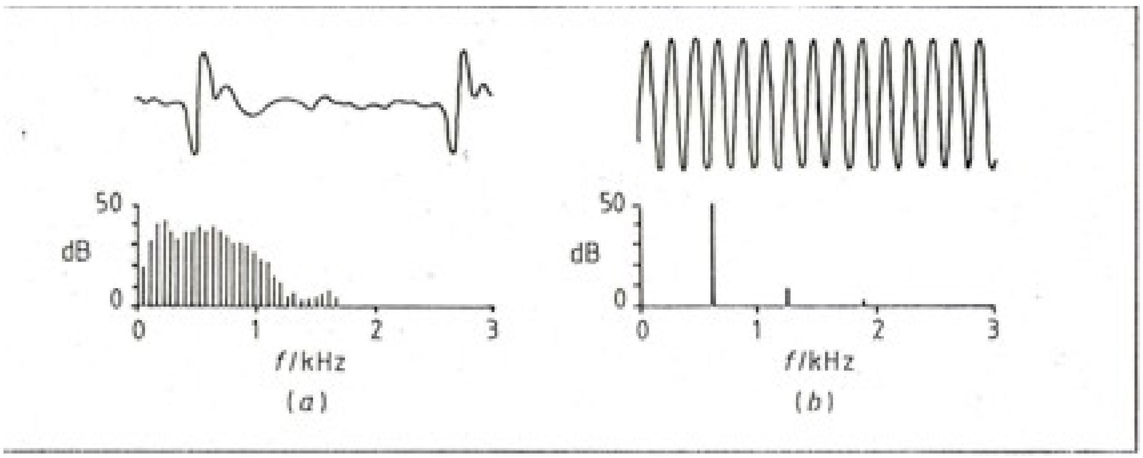
Another important instrumental characteristic is its formant. This is the change in harmonic content with loudness and pitch. Not all notes will have the same harmonic content and the character of the notes changes with pitch. This is illustrated in the diagram below which shows what is essentially the resonance curve of a trumpet.

When the note C_4 is sounded, four resonance peaks collaborate to set up a steady oscillation. They are the second, fourth, sixth and eighth peaks on the resonance curve. The sixth is the highest, which is the third harmonic of the note. When G_4 is sounded, three resonance peaks contribute to the note, the third, sixth and ninth on the curve. As before, it is the sixth peak which is the highest, but this time it is the second harmonic of the note. Clearly, the harmonic content changes with pitch.

It should be clear from this that, for wind instruments, the higher the pitch of the note, the fewer the resonance peaks that contribute to its sound. High notes therefore contain fewer harmonics than low notes. Another consideration is that the reflection characteristics of the bell are independent of the pitch of the note. Compare the waveforms and harmonic analysis of a low note from a tenor horn (below) with that of a high note.



IMPEDANCE PATTERN OF A MODERN TRUMPET is annotated to show what happens when a player sounds the notes C_4 or G_4. When he blows into the horn, a "regime of oscillation" is set up in which several impedance maxima of the air column collaborate with oscillations of his lips to generate energy in a steady oscillation that contains several harmonically related frequency components. The regime of oscillation for the C_4 note involves the second, fourth, sixth and eighth peaks in the curve. When the trumpeter plays very softly, the second peak is dominant, but because this peak is not tall the beginner may produce a wobbly note. As he plays louder the other peaks become more influential and the oscillation becomes stabilized. The dominant oscillation for the G_4 note corresponds to the third impedance peak; since it is taller than the second peak, G_4 is easier than C_4 to play pianissimo. As the trumpeter plays louder the tall sixth peak comes in and greatly stabilizes the regime of oscillation, making the G_4 one of the easiest notes of all to play.



You can almost be convinced that the frequency spectra are both contained within the same envelope.

As the instrument doesn't change size or shape when it is being played, the resonance curve is fixed and therefore contributes differently to notes at different frequencies. A good illustration of this is the violin with its wood and cavity resonances. These are all at particular frequencies which are independent of the note being played, giving the violin a very marked formant.

<u>Demo:</u> Another illustration of the change of harmonic content with pitch is given by this clarinet recording:

1. 880 Hz normal speed
2. Same note at ½ speed
3. Same note at ¼ speed
4. 440 Hz normal speed
5. Same note at ½ speed
6. 220 Hz normal speed

Transposing notes to the same pitch gives a good comparison of their character.

Instrumental synthesis

A successful analysis of the sound of an instrument should yield a physical description from which a sound could be synthesised that, to a listener, is almost indistinguishable from the original. What must go into such a synthesis? There are probably four main things to consider:

1. **Harmonic content.** Getting the harmonic content of note correct is obviously important. However, if all notes have the same harmonic content, the sound will not match that of a real instrument very well at all. It will sound artificial and clearly produced electronically.
2. **Temporal evolution.** We have already seen how important the temporal evolution of each note is. It is characteristic of the particular instrument and the transient, steady state and decay must all be correct. For example, upon initiation of a note on a brass instrument, there will be a faster build-up of low harmonics than high. With harmonic content and temporal evolution, the synthesised instrument is rather more convincing, but it will still sound rather electronic.
3. **Formant.** The change of harmonic content with pitch and loudness is also very important and is characteristic of the instrument. You can expect more high frequencies with increasing loudness. Change of harmonic content with pitch has already been examined. A synthesised passage of music with correct harmonic content, temporal evolution and formant will sound very like the instrument it is trying to imitate. However, it is not completely convincing as it will still sound as if it is being produced by a box of electronics, which it is. The notes and the performance will seem very mechanical and clearly not performed by a human being.
4. **Quasi-random effects.** No human being can play a passage of music in an identical manner every time. There will be slight variations in amplitude, pitch, speed, use of vibrato, timing and many other things. Most instruments are also subject to non-linear effects in the production of the sound, e.g. when the superposition of waves produces partials that cannot be explained by a simple theory. (The next lecture will consider some non-linearities in the production and detection of sound.) All these effects are very difficult to produce either electronically or by computer processing, but if they are present even only approximately, the resulting sound can be very convincing indeed, assuming it has the correct harmonic content, temporal evolution and formant. Without knowing the source of the sound, a listener could mistake the synthesised sound for a genuine performance by a human being.

Beats

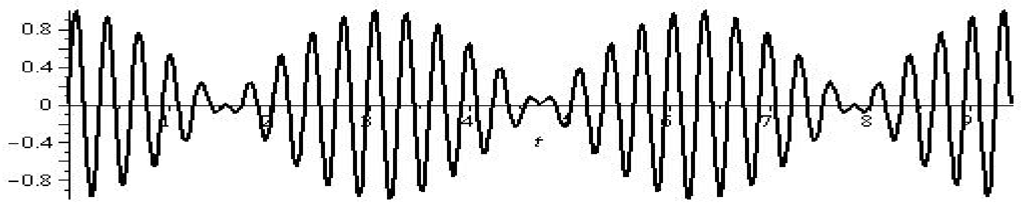
When two notes are sounded together with slightly different frequencies a phenomenon known as beats is heard – the intensity of the combined sound rises and falls in a regular pattern.

Let the two sound waves be $\sin(\omega + \Delta\omega)t$ and $\sin(\omega - \Delta\omega)t$ where only the time variation is considered. The difference in frequency is the small quantity $2\Delta\omega$.

Using a trig identity, the combined sound wave can be written as:

$$\sin(\omega + \Delta\omega)t + \sin(\omega - \Delta\omega)t = 2\cos(\Delta\omega t)\sin(\omega t)$$

The cosine wave varies slowly and can be regarded as the amplitude of the sine wave, which varies rapidly. The frequency of the sine wave is the mean of the frequencies of the two original waves.



Note that there are two beats for each period of the cosine wave. Since this has a frequency of $\Delta\omega$ Hz, the beat frequency is $2\Delta\omega$, which is the difference in frequency between the two original waves.

Beats are normally used to tune two musical instruments together. When they are exactly in tune and playing the same note, no beats can be heard.

Combination tones

So far we have only considered the linear superposition of sound. That is, if $\phi_1(x,t)$ and $\phi_2(x,t)$ are two sound waves, then their superposition is simply $\phi_1+\phi_2$ and the waves add linearly. This occurs when we add harmonics to synthesise a sound or when two instruments play together. For example, when a flute and a clarinet play together, both instruments are easily recognisable in that they still sound like a flute and a clarinet and no new sound is produced. However, if non-linearities occur either in the production, detection or recording of sound then something new is produced. This is what we will examine.

Consider only the time dependence of the waves, so two waves can be represented by $\phi_1 = \sin(\omega_1 t)$ and $\phi_2 = \sin(\omega_2 t)$.

A completely linear detector of sound will be sensitive only to $\phi_1 + \phi_2$. However, let us represent a non-linear detector by making it sensitive to $\phi + a\phi^2 + b\phi^3 + \cdots$ where $\phi = \phi_1 + \phi_2$ and the coefficients a and b etc. are expected to be small.

Now let us see what the ϕ^2 term gives:

$$\phi^2 = (\phi_1 + \phi_2)^2 = \phi_1^2 + 2\phi_1\phi_2 + \phi_2^2 = \sin^2(\omega_1 t) + 2\sin(\omega_1 t)\sin(\omega_2 t) + \sin^2(\omega_2 t)$$

To get rid of products of sine functions, we use trig identities to give

$$\phi^2 = \frac{1}{2}\left(1-\cos(2\omega_1 t)\right)+\cos(\omega_1-\omega_2)t-\cos(\omega_1+\omega_2)t+\frac{1}{2}\left(1-\cos(2\omega_2 t)\right)$$

We see that the combined wave contains waves of frequencies $2\omega_1$, ω_1-ω_2, ω_1+ω_2, and $2\omega_2$ which are not present in the original sound wave.

These are normally referred to as the **sum tones:** $2\omega_1$, $\omega_1+\omega_2$, $2\omega_2$
and the **difference tone:** $\omega_1-\omega_2$

The cubic combination tones resulting from the ϕ^3 term have frequencies:

Sum tones: $3\omega_1$, $2\omega_1+\omega_2$, $\omega_1+2\omega_2$, $3\omega_2$
Difference tones: $2\omega_1-\omega_2$, $\omega_1-2\omega_2$

The ear is not a perfectly linear detector of sound and, under certain circumstances, sum or difference tones can be heard. These are produced within the ear and are referred to as **aural harmonics**.

The first person to document such sounds was Tartini (1692 – 1770), known nowadays as a composer of Baroque music, but was a celebrated violinist when he was alive. He claimed to use difference tones to tune his violin. For example, to tune the perfect 5th G to D, where the frequencies are in the ratio 2 : 3, the first difference tone is of relative frequency 1 and is an octave below G. If the difference tone is exactly in tune with the G then the 5th must be perfectly in tune.

Demo: Sound recording using two signal generators. The recording is deliberately of poor quality to introduce non-linearlities. The frequencies vary with time as shown in the diagram. Listen for the falling tone which is the difference frequency between the two original sounds. There are lots of other tones as well. You can hear beats towards the end as the two frequencies approach each other.

Demo: Use two descant recorders. One plays a high G and the other glissandos down to the C below. A rising tone can be heard – the difference tone – which appears to come from the middle of your head and there is no sense of the direction of the sound.

Examples of difference tones:

- A Victorian policeman's whistle produces two tones about 200Hz apart. When the whistle is blown loudly, the difference tone is quite audible.
- A descant recorder will typically play two octaves above middle C (1046Hz). A major 3rd above this with another recorder (frequency ratio 4 : 5) produces a difference tone of middle C which is in the middle of the audible range and can be quite prominent especially to the players.
- Some pipe organs have what is known as an acoustic bass. This consists of a 16ft rank of pipes playing simultaneously with a 10⅔ ft rank (frequency ratio 2 : 3). The idea is that the difference tones will mimic a 32ft rank, i.e. an octave lower than the pipes actually in the organ. It is rarely convincing.
- In Bach's G major organ fantasia, there is a note (B_1) which is below the bottom note on the pedal board. On CDs of the work, this note is dubbed in, but what do you do in a live performance? My organ teacher suggests playing the B an octave up (B_2) and the F# a 5th above this simultaneously. The frequency ratio is 2:3, so the difference tone is at the pitch of the required B_1. Most organists seem to know this trick, but I haven't come across one that understands why it might work. The pipes of these two notes in standard organ layout are at different sides of the instrument so the chance of a non-linear interaction is close to zero. I've never heard it work on any organ.

- It is possible to play a three-note chord on a French horn. The first note is played in the normal way. The second note is produced by modulating the breath, i.e. by humming it. The third note is the difference tone. This is an example of a non-linear source of sound.

Scales and temperament

Consonant intervals

We have used the Pythagorean intervals a number of times, but now let us look at more modern intervals as well. The following presents the frequency ratios of intervals in complementary pairs. The combination of the intervals in each pair gives an octave.

1 : 1 unison	1 : 2 octave
2 : 3 perfect 5^{th}	3 : 4 perfect 4^{th}
4 : 5 major 3^{rd}	5 : 8 minor 6^{th}
5 : 6 minor 3^{rd}	3 : 5 major 6^{th}

Note that combining intervals is done by multiplying the frequency ratios. Chords tuned to these very simple ratios are said to use just intonation. It produces the most perfect tuning possible and is used especially by unaccompanied choirs singing early music.

The natural scale

Can these intervals be used as the basis of a complete musical scale? The following gives the frequency ratios of one possibility:

C	D	E	F	G	A	B	C
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

This is based on just intonation for the chords CEG, FAC, GBD. The first interval in each chord is a major 3^{rd} and the second is a minor 3^{rd}. The combination is a perfect 5^{th}. These chords are perfectly in tune, but notice that the interval D → A, which should be a perfect 5^{th} is not. Its frequency ratio is $\frac{5}{3}\Big/\frac{9}{8} = \frac{40}{27}$ instead of $\frac{3}{2}$. Many other intervals are badly out of tune as well, making this scale completely unusable for most harmony.

Demo: Set up the natural scale on the computer and play the in-tune 5ths C → G and F → C, then try the out-of-tune D → A.

An unaccompanied choir uses just intonation because each chord they sing can be adjusted to be perfectly in tune. This means that any particular note is not absolute in pitch, but is altered slightly according to the chord being sung. However, if the tuning of the instrument is fixed, as it is with a keyboard instrument, it is clear it cannot be exactly in tune – the numbers don't fit.

The Pythagorean scale

The earliest practical scale is known as the Pythagorean scale. It is based on the Pythagorean intervals of octave, perfect 5^{th} and perfect 4^{th} which is all that the ancient Greek music of Pythagoras required. No other harmonies were needed. The scale is obtained by going up in 5^{th}s using the ratio 3/2 each time and transposing the note down an octave when required. Starting at the top C, intervals of a 5^{th} can also be assigned going down, using the ratio 2/3. This gives:

C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B	C
1	$\frac{3^7}{2^{11}}$	$\frac{3^2}{2^3}$	$\frac{2^5}{3^3}$	$\frac{3^4}{2^6}$	$\frac{2^2}{3}$	$\frac{3^6}{2^9}$	$\frac{3}{2}$	$\frac{3^8}{2^{12}}$	$\frac{3^3}{2^4}$	$\frac{2^4}{3^2}$	$\frac{3^5}{2^7}$	2

- Sharps are assigned going up in pitch, i.e. F# is a perfect 5th above B, and flats are assigned going down – Bb is a 5th below F, etc..
- Note that Eb is not a perfect 5th above G# in this scale, but D# is:

  D# = $3^9 / 2^{14}$ and Eb = $2^5 / 3^3$ Ratio of D# / Eb = $3^{12} / 2^{19}$ = 1.014
- Note the badly out of tune major 3rd C → E = 81/64 instead of 5 / 4.

This scale was in use until about the 15th century. It became unsatisfactory when musical harmony made increasing use of the major 3rd from the 13th century onwards.

Demo: Set up the Pythagorean scale on the computer and demonstrate the just 5ths, the out-of-tune 3rds (play a just 3rd for comparison) and the badly out of tune 5th G# → Eb.

Mean-tone temperament

Just intonation on some intervals concentrates the mistuning onto others. If some of the intervals are 'tempered', the mistuning is more evenly spread. This gives rise to a 'temperament' rather than a scale and there are hundreds of ways of doing this. We will look at two of them.

The diagram shows two ways of producing the interval $C_4 \rightarrow E_6$ and they don't coincide. Using 4 5ths, the frequency ratio is $(3/2)^4 = 81/16$, while two octaves and a 3rd give $2^2 \times 5/4 = 5$. These can be reconciled by making the ratio for a 5th equal to $\sqrt[4]{5}$ = 1.496 instead of 3/2. The mistuning is slight and quite acceptable. Using this as the ratio for a 5th, the mean tone temperament can be set up in the same manner as the Pythagorean scale:

C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B	C
1	$\frac{5^{7/4}}{16}$	$\frac{5^{1/2}}{2}$	$\frac{4}{5^{3/4}}$	$\frac{5}{4}$	$\frac{2}{5^{1/4}}$	$\frac{5^{3/2}}{8}$	$5^{1/4}$	$\frac{25}{16}$	$\frac{5^{3/4}}{2}$	$\frac{4}{5^{1/2}}$	$\frac{5^{5/4}}{4}$	2

- Note that C# ≠ Db since Db = $8/5^{5/4}$ and G# ≠ Ab as Ab = $8/5$. Playing G# in place of Ab will therefore sound out of tune. This means that modulation into any key is not possible. The extreme keys are therefore A major (3 sharps) and Bb major (2 flats).
- Note that the thirds are pure, the 5ths are slightly flat (but acceptably so) and the 4ths are slightly sharp.
- The interval G# → Eb is far from being a perfect 5th with a frequency ratio of $\frac{8}{5^{3/4}} \times \frac{16}{25} = 1.53$.

In the natural scale, the interval C → D is $\frac{9}{8}$, while D → E is $\frac{5}{4} \times \frac{8}{9} = \frac{10}{9}$. Both of these are meant to be whole tones, yet they are different. Taking the geometric mean of these gives $\sqrt{\frac{9}{8} \times \frac{10}{9}} = \frac{\sqrt{5}}{2}$, which is the ratio for a whole tone in the mean tone temperament, hence its name. It came into common use during the 15th century to avoid the unacceptably out of tune major 3rd of the Pythagorean scale.

Demos: Set up the mean tone temperament on the computer and demonstrate the slightly flattened 5ths (slow beats) and the interval G# → Eb.

Play “O come all ye faithful” (in A) in 3-part harmony. It should sound in tune, but “Good King Wenceslas” (in Ab) is badly out of tune. “While shepherds” (in F) is fine.

Now retune G# to Ab and C# to Db. “GKW” is now beautifully in tune while “O come” is hopelessly out of tune. “While shepherds” is still fine.

Equal temperament

Mean-tone dominated the tuning of keyboard instruments from its inception till the middle of the 19th century. It solved the problem of the use of 3rds in harmony and was perfectly satisfactory for Renaissance music.

From 1600 onwards (early Baroque) the use of accidentals and modulation into different keys became an additional means of musical expression. The limitations of the mean-tone temperament gradually became a severe constraint.

From the time of J. S. Bach (1685 – 1750), many modifications of mean-tone temperament were tried. More than 200 are known about and some are still used for particular types of music.

The modern solution to the problem of tuning keyboard instruments is to spread the mistuning evenly over all intervals. The octave is divided into 12 equal semitones, each of ratio $2^{1/12}$ (= 1.059). Since the semitones are all equal, there is no longer any distinction between G# and Ab etc. and all keys are equally out of tune. This allows free modulation into any desired key without change of mistuning.

<u>Demo</u>: Set up equal temperament on the computer and demonstrate fifths (slow beats) and thirds (much faster beats). Compare with just intonation.

Play “O come” and “GKW” – they are both quite playable and give beats (acceptably slow) all the way through.

Comparison of intervals

Since we combine intervals by multiplying and dividing frequency ratios, it is sensible to put frequency ratios on a logarithmic scale so intervals can be combined by adding.

The standard unit is 1/100 of a semitone, called a cent, so we can calculate an interval in cents as:

$$\text{interval in cents} = \frac{1200}{\ln(2)} \ln\left(\frac{f_1}{f_2}\right)$$

This gives 100 cents to the equal-tempered semitone and 1200 cents to the octave. Intervals are combined by adding cents instead of multiplying ratios. A comparison of intervals for different scales and temperaments is now easily made:

Interval	Unison	Third	Fourth	Fifth
Note	C	E	F	G
Just intonation	0	386	498	702
Pythagorean scale	0	408	498	702
Mean-tone temperament	0	386	503	697
Equal temperament	0	400	500	700

With the intervals in cents, it is seen that the Pythagorean third is 22 cents sharp compared with just intonation, i.e. 0.22 of a semitone. Equal tempered thirds are 14 cents sharp. To the purists, this is unacceptable, and mean-tone is still sometimes used to play early keyboard music. Sustaining instruments like an organ are particularly sensitive to the tuning, whereas it is not so obvious on pianos and harpsichords.

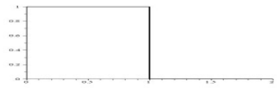
Concert hall acoustics

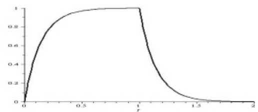
Concert hall acoustics is concerned with providing the best environment for listening to an orchestral concert. It is a large and specialist subject, so only a flavour of the topic can be given in a single lecture.

Reverberation

Imagine a room that is heavily carpeted and curtained with well-upholstered furniture. It will be very quiet and conversations would only be possible with people very close to you. It would not be very satisfactory for listening to music except through earphones. Most people would describe it as “dead”. Imagine another room, something like York Minster, where each sound seems to last a long time as it is reflected from the stone surfaces that are all around. Sounds become jumbled up and indistinct and it is also not good for most kinds of music.

The difference between the two rooms is the **reverberation**. This is the property of the sound remaining audible after all sources have ceased. The reason why the sound may linger is that it reflects off surfaces in the room and is only gradually absorbed. Absorption takes place almost entirely upon reflection; absorption by the air is usually negligible except at high frequencies. In our first room, there was too little reverberation and in the second there was too much.





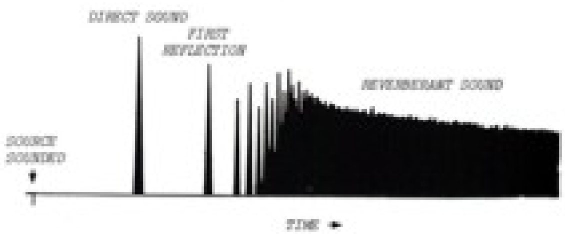
The upper diagram shows the intensity of a sound source, which operates for a limited time. The lower diagram shows the build-up of sound intensity in the room when the source is first activated and the decay of sound after it is switched off. The steady state is reached when the rate of sound production is balanced by the rate of absorption. The rate of decay is measured by the **reverberation time** which is defined as:

Reverberation time is the time for the sound energy density to decay to 10^{-6} of its original value after all sources have ceased.

This definition is due to Sabine, who was a physicist at Harvard University and pioneered the first research into room acoustics in about 1900.

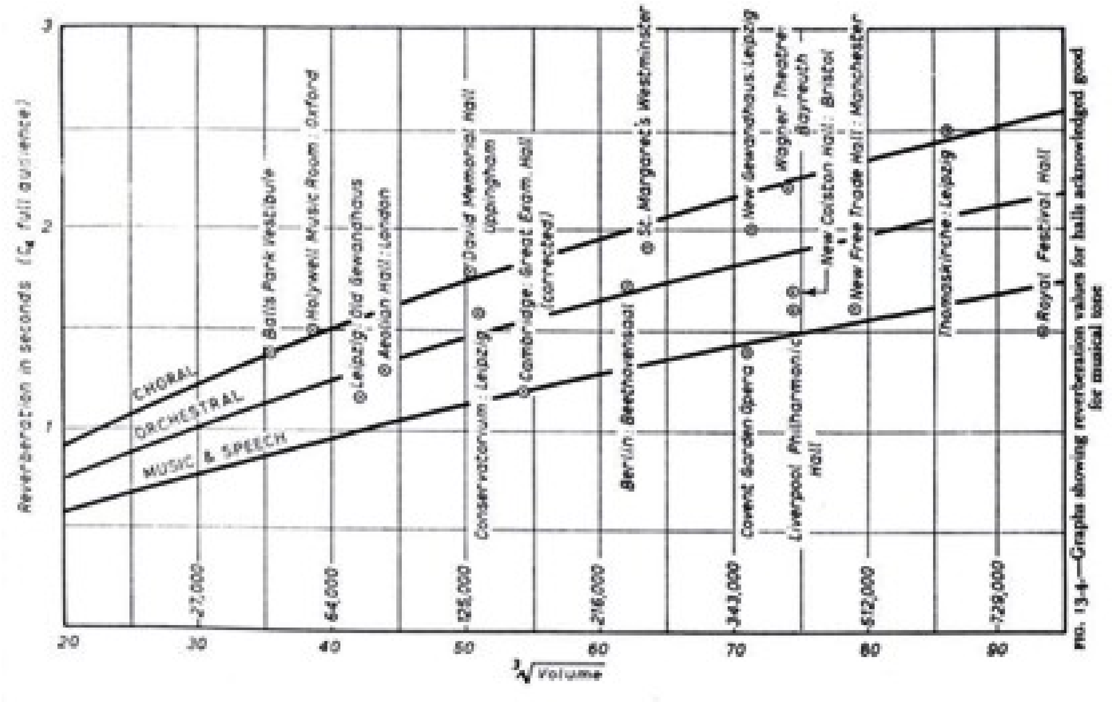
What should the reverberation time be? Clearly, some reverberation is good, because people like to sing in the bath or shower where the walls are tiled and reflect a lot of the sound. The reflections seem to support the voice and enhance its quality. The diagram on the next page shows the reverberation times for a number of concert halls and it is clear that the time is a function of the size of hall – larger halls require more reverberation. Less reverberation is required for speech than music, since clear speech requires that consonants are well-defined and do not overlap. The diagram also shows that some halls have much too short a reverberation time, e.g. the Royal Festival Hall in London before its refurbishment.

Reverberation time can be measured by recording the sound of a starting pistol in the hall. This will record all the reflections from surfaces and give a measure of sound energy density as a function of time.



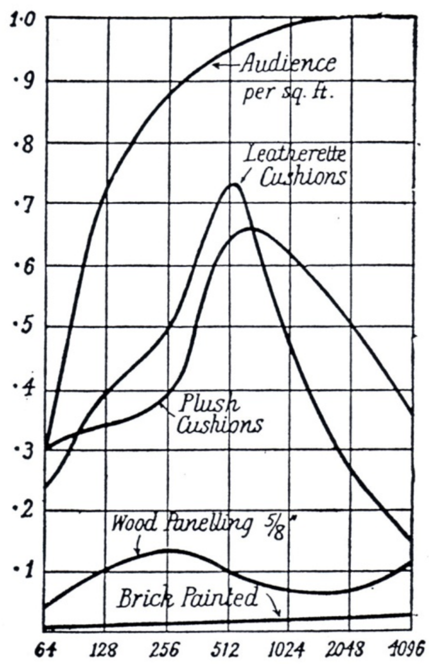
Demo: Sound of pistol shot replayed at slow speed allows individual reflections to be heard.

It is a common misconception that reverberation time is the most important feature of room acoustics. This may be because it is easy to understand and easy to measure – physicists love measuring things. It is also an easy way to introduce the topic. Reverberation time is important, but there are more important things to come – keep reading.



Sound absorption

Reverberation time depends largely upon the nature of the surfaces in the hall and their disposition. Hard, smooth surfaces like stone or polished wood will reflect most of the sound incident upon them while curtains, thick carpets and upholstery will absorb most of the sound, as shown in the diagram. Audiences also absorb sound. Note that absorption is a function of frequency and, therefore, so must the reverberation time. The frequency at which reverberation time is measured should normally be quoted. Absorption is quantified by the **absorption coefficient**, α, defined by:



$$1-\alpha = \frac{\text{energy density of sound after reflection}}{\text{incident energy density}}$$

Reverberation time formula

Sabine derived an empirical formula for reverberation time from his experimental results, but his formula is now regarded as being too simple. A better formula, derived in 1930, is:

$$T = \frac{4V\ln(10^{-6})}{v\,S\ln(1-\alpha)}$$

where V = volume of hall α = mean absorption coefficient for all surfaces
S = total surface area v = speed of sound

Note that $T \propto V$: big V gives a longer sound path between reflections and therefore a lower rate of absorption at surfaces.

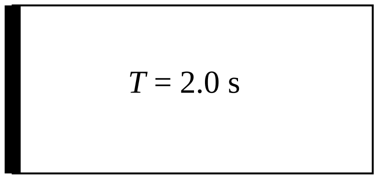
Also, $T \propto \frac{1}{S}$: big S gives a higher rate of absorption at surfaces.

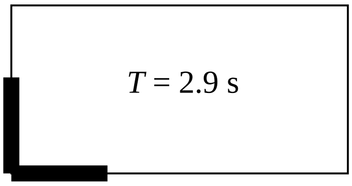
This is a typical physicist's formula in that it makes huge assumptions in order to arrive at any kind of result. Its main fault is that it does not take into account the shape of the hall or the disposition of the surfaces within it. It assumes that all directions of the sound are equivalent (as if all concert halls were spherical) and this gives an overestimate of the actual reverberation time. It was not realised that this formula was systematically wrong till the early 1960s.

The reverberation time for the Royal Festival Hall, designed before 1950, was estimated from a formula similar to the one above, resulting in an actual value of T far smaller than it should have been. Fortunately, acoustic engineers can now carry out much more accurate calculations and this mistake is unlikely to be repeated. The reverberation time is now a much more acceptable 2.4 seconds after the recent refurbishment.

Distribution of surfaces

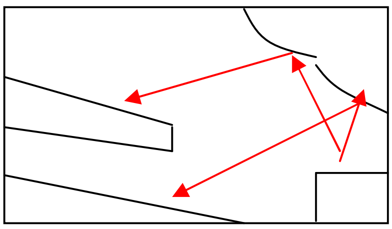
A simple example of how the distribution of surfaces in an enclosure affects the reverberation time is given by this 2D example. The calculation of T was carried out by computer ray tracing.





In the first diagram, three walls are perfectly reflecting and the fourth is perfectly absorbing, giving T = 2.0 secs. In the second diagram, there is the same amount of absorbing surface, but it is distributed over two walls as shown. The reverberation time is now 2.9 secs.

Sound diffusion



Reflections that arrive at the ears within 35ms of a previous reflection are perceived as part of the same sound, i.e. they add to the reverberation. Reflections with a time gap of 60ms or more are perceived as an echo and this is a serious acoustical fault. Sound gaps between these times give a perception of reverberation or echo according to each individual listener. Sound will travel about 12m in 35ms, which is not a very big distance in a large hall. It is therefore good design to avoid long sound paths between source and listener. A common device is to use sound baffles above the orchestra which reflect the sound more directly towards the audience before it reaches the far corners of the hall. These are convex surfaces, as in the diagram, so the sound is distributed in different directions.

Concave surfaces focus the sound as shown in the cross section of a fictitious opera house. This is also a serious acoustical fault as the focussing of sound on part of the audience means the rest of the audience gets less sound. Also the sound that is focussed will dominate, leaving the rest of the sound comparatively weaker.

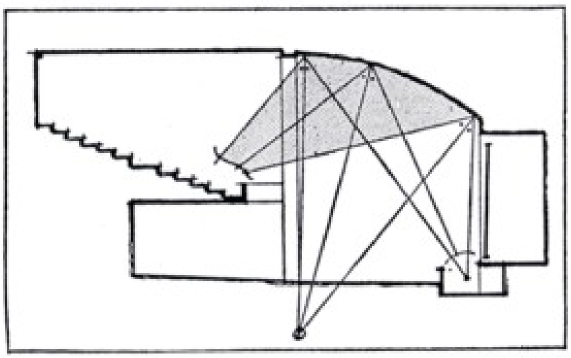
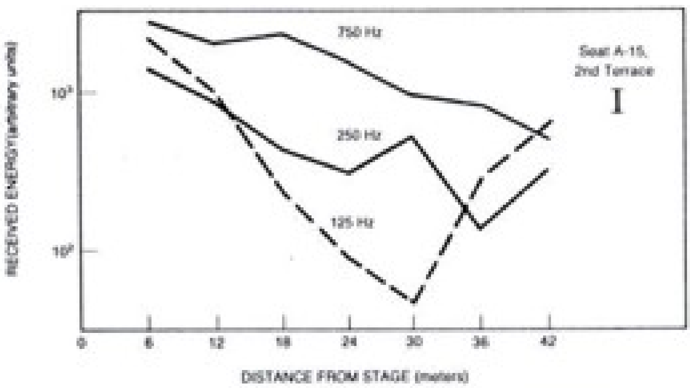


FIG. 13.8.—Undesirable concentration of sound by a concave proscenium

Spectrum balance

This is the story of the acoustical disaster in Philharmonc Hall, at the Lincoln Center in New York City, which is the home of the New York Philharmonic Orchestra. Despite hiring top quality acoustical engineers to help with the design of the hall, it had a number of obvious acoustical faults when it was built. One of them was that the sound of the orchestra varied enormously throughout the hall. This is illustrated in the diagram which shows the sound energy density at three different frequencies as a function of distance from the stage. At a distance of about 30m, there is hardly any sound at all at 125 Hz. This made the cellos inaudible yet, upon sitting further back, they were perfectly clear.

The fault was traced to the baffles above the orchestra, which absorbed heavily at this frequency. This part of the hall relied upon reflections from the baffles to obtain a good proportion of the sound received. It is not a good idea to use reflecting surfaces whose absorption coefficients are very dependent upon frequency.



Sound energy transmission from stage to main floor for New York's Philharmonic Hall in its early configuration. Note the unusually large attenuation of the lowest octave band (around 125 Hz) in the center of the main floor compared to the higher octave around 750 Hz. At seat A-15, in the second terrace, the variation of energy with frequency was much smaller; in fact, this was judged "the best seat in the house." Figure 4

Correcting acoustic faults is very expensive and the only real cure is to demolish the hall and start again. Because of other faults in Philharmonic Hall, including slight echoes in various places, it was decided to do just that – knock the hall down and redesign it from scratch. New acoustical engineers were hired and the hall rebuilt. It is now called the Avery Fisher Hall after the local millionaire who donated much of the cash and, fortunately, it is a significant improvement over the original hall.

Interaural coherence

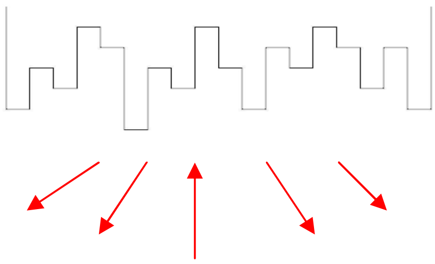
A number of concert halls built in the 19th century are extremely good acoustically, despite the fact that this was before any research had been done on room acoustics. In contrast, a number of concert halls built in the mid 20th century are very poor, even though by that time some knowledge of room acoustics was available. Why the difference?

A 19th century hall is typically a "shoe box" shape. It is fairly narrow with the side walls close together because of the limited span of the roof – supporting pillars in the middle of a concert hall are definitely not on. The ceiling is fairly high because a large volume of air is required, otherwise it would become stale with a large audience.

A mid 20th century hall, on the other hand, tends to be wide. This is made possible by better building materials which can span a longer distance and it allows a good proportion of the audience to be near the stage. In addition, the ceiling is low because there is a smaller volume to heat and air conditioning can now replace the air before it becomes stuffy.

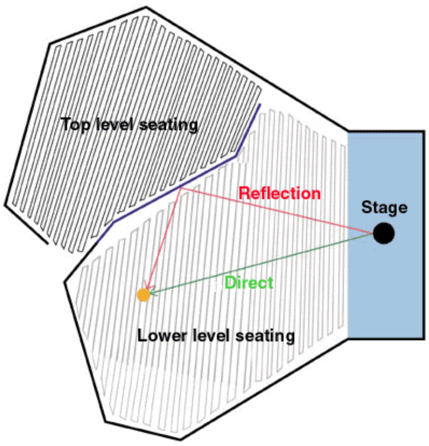
It turns out that the "shoe box" is the best basic shape that a hall can have from an acoustical point of view. People want to feel immersed in sound and this is achieved by giving each ear a different sound signal. Even simple sound systems using CDs or mp3 players are now at least in stereo, giving a different signal to each ear. This is achieved in the concert hall by making sure that the early sound reflections are received from the walls rather than the ceiling. Sound coming at you from the side gives a very different signal in each ear, whereas sound coming from the ceiling will give the same signal in each ear. A narrow hall with a high ceiling ensures that early reflections come from the walls. Symphony Hall in Birmingham is designed this way.

The technical name to describe this is **interaural coherence** and for good listening, it should be low. This is probably the most important aspect of concert hall acoustics – get this right and you've automatically got a good hall. It takes more than this to produce an excellent hall, but low interaural coherence gives you a good start.



Modern concert halls are all designed with this in mind and many will even use sound phase gratings in the ceiling to cut down direct reflections from the ceiling even more. The diagram shows a sound phase grating in profile where the sizes of the protrusions are comparable with sound wavelengths. The effect is to alter the phases of reflected sound waves, so wave fronts are built up that propagate the reflected sound towards the walls. Such structures in the ceiling are often designed and coloured to be decorative, but they have a serious acoustical purpose.

The latest design of concert halls is known as the "vineyard". The audience is divided into sections with low vertical walls between them. Everybody is therefore near a wall and receives early lateral reflections of sound. This makes interaural coherence low for everyone. The new halls in Berlin (Berlin Philharmonie), Cardiff (St. David's Hall – picture below) and Manchester (Bridgewater Hall) are all vineyard designs.



Handout, which serves as a brief summary of the lecture and fills some gaps:

Criteria for good concert hall acoustics

The items in this list are headed with words a musician might use to describe the required effect. Each one is translated into more physical terms. They are roughly in order of importance, though this is a matter of personal taste.

1. **intimacy** – music sounds as if it is being played in a small hall – time delay between direct sound and first reflection should be < 20 ms – lateral reflections more important than overhead.
2. **liveness –** optimum reverberation time.
3. **warmth –** fullness of bass tone – reverberation time below 250 Hz should be longer than that at middle and high frequencies.
4. **clarity** – level of early + direct sound should be greater than reverberant sound everywhere.
5. **uniformity** – good spacial distribution of sound – use diffuse or irregular reflecting surfaces, avoid focused sound or sound shadows or echoes.
6. **balance** – no change in quality of sound with position – reflecting surfaces should have absorption coefficients independent of frequency, especially near the stage.
7. **freedom from noise** – well insulated building – no noise from traffic or from the ventilating system.

End of lecture notes

Supplementary notes

1st year elective given since 1982 following the course given by Deryk Goodwin.

Mathematics is music for the mind;
Music is mathematics for the soul.

Science and music

E Eugene Helm (Dec, 1967) Sci. Am., *The vibrating string of the Pythagoreans.*
J Bronowski (1973) Ascent of Man pp.156-157 *Music of the spheres.*
Franchino Gafurio (1492) *Theorica Musice*

Preliminaries

The general mathematical form of a wave, $\phi(x-vt)$, satisfies the wave equation: $\frac{\partial^2\phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}$ where ϕ is anything which characterises the wave – displacement, pressure, density, strain, velocity of medium, electric field etc. This is a second order partial differential equation which has many applications in mathematical physics. It is a linear equation which means the principle of superposition applies. The effect of this is that two sources of sound can be distinguished by the ear when sounded simultaneously and do not produce something entirely new. Individual instruments can be heard in the midst of a full orchestra.

Waves on a string

Consider an element of string: length = δx; displacement = ϕ; tension = T; linear density = ρ

Upwards force on element at $x \;= -T\sin\theta \approx -T\tan\theta = -T\frac{\partial\phi}{\partial x}$ assuming the slope of the string is small.

Upwards force at $x+\delta x \;= T\frac{\partial}{\partial x}\left(\phi + \frac{\partial\phi}{\partial x}\delta x\right) = T\left(\frac{\partial\phi}{\partial x} + \frac{\partial^2\phi}{\partial x^2}\delta x\right)$

Therefore, total upwards force on element $= T\frac{\partial^2\phi}{\partial x^2}\delta x$

This force produces an acceleration of $\frac{\partial^2\phi}{\partial t^2}$

Since the mass of the element is constant at $\rho\delta x$, Newton's 2nd law of motion gives:

$$T\frac{\partial^2\phi}{\partial x^2}\delta x = \rho\,\delta x\frac{\partial^2\phi}{\partial t^2} \quad\Rightarrow\quad \frac{\partial^2\phi}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2\phi}{\partial t^2}$$

i.e. the displacement ϕ obeys the wave equation. Hence, a transverse disturbance of the string will be propagated as a wave with speed: $v = \sqrt{\frac{T}{\rho}}$

This analysis of the motion of a vibrating string is due to Lagrange. Mersenne was the first to determine directly the frequency of a musical sound and the relationship between string length and frequency. He used a long cord and timed the vibrations by eye using his pulse.

Energy associated with each mode

Let the *n*th mode be $\phi = a_n \sin\left(\pi n \frac{x}{l}\right)\sin\left(\pi n \frac{vt}{l}\right)$

then the transverse velocity $= \frac{\partial \phi}{\partial t} = a_n \pi n \frac{v}{l} \sin\left(\pi n \frac{x}{l}\right)\cos\left(\pi n \frac{vt}{l}\right)$

At $t = 0$, all energy is kinetic, so total energy

$$= \frac{1}{2} m \left(\frac{\partial \phi}{\partial t}\right)^2 = \frac{1}{2} m \left(a_n \pi n \frac{v}{l}\right) \int_0^l \sin^2\left(\pi n \frac{x}{l}\right) dx = \frac{1}{4} m \left(a_n \pi n \frac{v}{l}\right)^2 = m\left(a_n \pi n f\right)^2$$

where m = total mass of string and f = frequency = $\frac{v}{2l}$.

Resonance

The tuning fork was invented in 1711 by John Shore, a trumpeter in Handel's orchestra. Remarkably, there's a YouTube video of a person breaking a wine glass using only his voice.

The violin

The motion of the bowed string was determined by Helmholtz in 1860 and explained by Raman in 1921. The simplest description was given by Raman as a pair of velocity waves travelling in opposite directions, i.e. a standing velocity wave.

Bowing the string exactly at its centre doesn't produce a musical note – explanation in my musical acoustics notes.

Percussion

Chladni lectured in many courts in Europe, including that of Napoleon, who was delighted by his demonstrations. "He makes sounds visible!" Napoleon contributed money to pay for the translation of Chladni's book into French.

Flue organ pipes

N H Fletcher & S Thwaites (Jan, 1983) Sci. Am. *The physics of organ pipes*

Reed organ pipes

Bernoulli's equation: $\frac{1}{2}\rho v^2 + \rho g z + p = \mathit{constant}$ along any stream line

i.e. kinetic energy + potential energy density + pressure = constant

Therefore, increased v results in lower p.

Brass instruments

A H Benade (July 1973) Sci. Am., *The physics of brasses.*

Cardwell (1966) JASA, **40,** 1252

R A Smith and G J Daniell (1976) Nature, **262,** 761-765. *Systematic approach to the correction of intonation in wind instruments.*