Encoding Quantified CSPs as Quantified Boolean Formulae

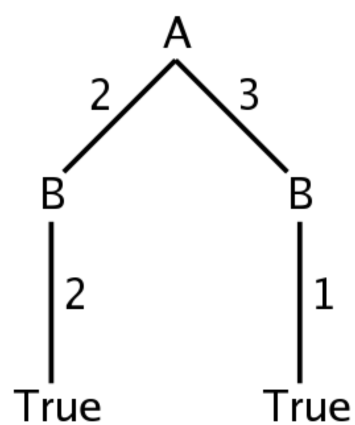
Ian P. Gent, Peter Nightingale and Andrew Rowley

Finite domain constraint satisfaction problem (CSP)

- Variables with a finite domain
  - e.g. $A \in \{2, 3\}, B \in \{1, 2, 4\}$
- Constraints placed on variables
  - $A \neq B$, $A + B = 4$
- A solution is a valid assignment to all variables
  - $A = 3$, $B = 1$
- NP-complete decision problem

Introducing quantifiers (QCSP)

- Existential ($\exists$) and universal ($\forall$) quantifiers
- $A \in \{2, 3\}, B \in \{1, 2, 4\}$, $\exists A \exists B$, $A \neq B$, $A + B = 4$
- $\forall A \exists B$, $A + B = 4$
  - Solution tree (strategy)



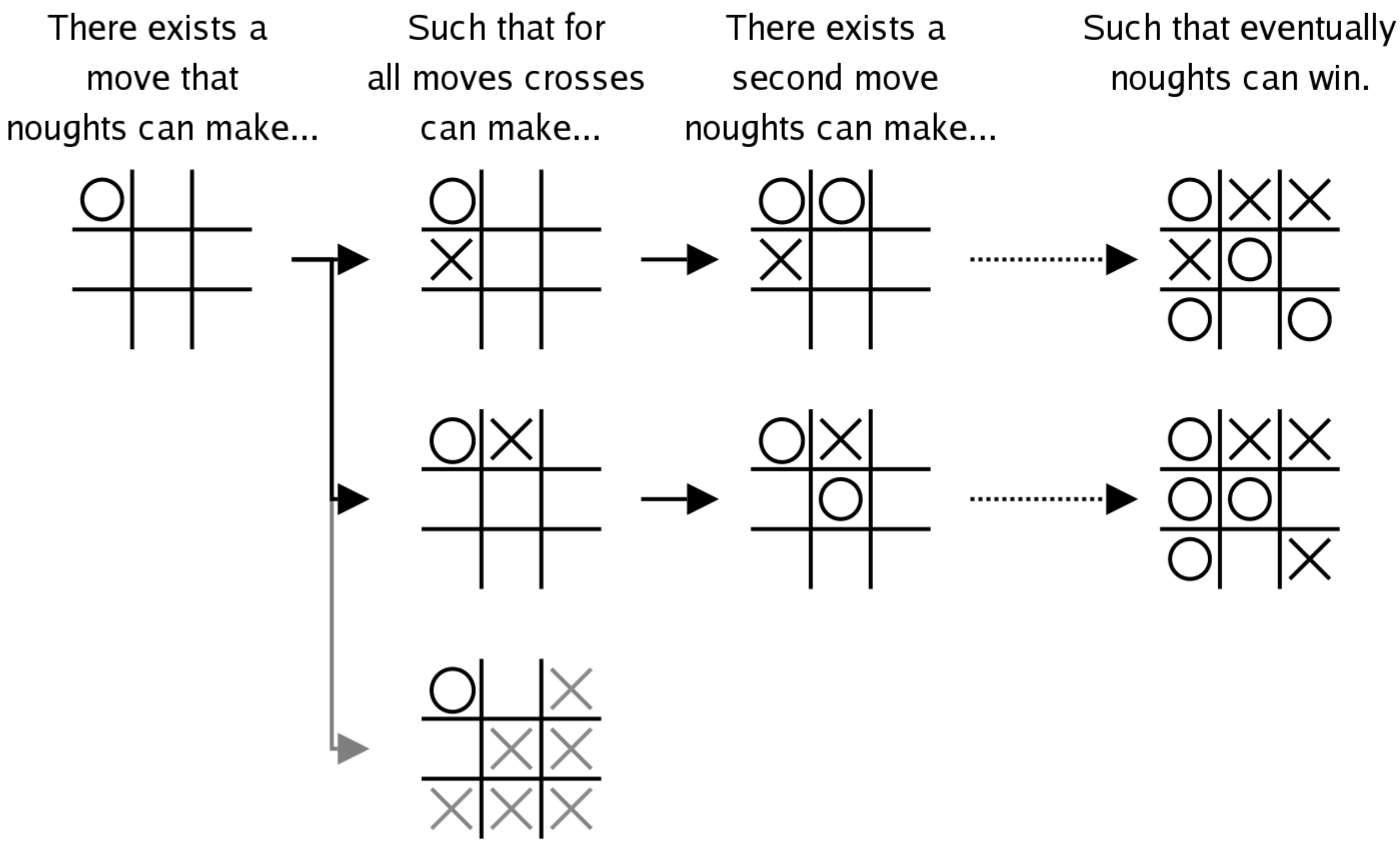
Introducing quantifiers (QCSP)

- Quantification order is significant
  - $\forall A \exists B,\ A + B = 4$
  - $\exists B \forall A,\ A + B = 4$
- PSPACE-complete decision problem
  - PSPACE algorithm traverses solution tree
- Exponential space to provide a solution

The game of QCSP

- QCSP can be thought of as a game
- Players are existential and universal
- Some games map into QCSP
  - Connect-4 (Gent and Rowley)
  - A variant of Go (Lichtenstein and Sipser)
  - Othello (Iwata and Kasai)

Noughts and crosses



Why consider QCSP?

- Natural generalization of CSP
- Problem solving with uncertainty
  - Uncertain data at solution time e.g. delivery time 10 am$\pm$1 hour
    - (Minimal) Covering set of solutions (Yorke-Smith and Gervet)
  - Uncertainty resolved during execution of plan
    - Game against the environment

Quantified Boolean Formulae (QBF)

- Subset of QCSP (also PSPACE-complete)

- We consider conjunctive normal form QBF in prenex form

$$\forall a, b \exists c, (a \vee \neg c) \wedge (\neg a \vee \neg b \vee \neg c)$$

- Unit propagation rules similar to SAT — slightly stronger

Why encode?

- QBF is the subject of recent research
  - Basic complete algorithm based on Davis Putnam Logemann Loveland algorithm
  - Conflict and solution directed backjumping (Guinchiglia, Narizzano and Tacchella)
  - Efficient watched data structures (Gent, Guinchiglia, Narizzano, Rowley and Tacchella)

- Take advantage of fast QBF solvers for QCSP

Direct encoding

- We consider binary QCSP for this work
- Encode CSP variable v with SAT variables x_i^v for each value i
- At-least-one clause $(\bigvee_{i=1}^{d} x_i^v)$ (v takes at least one value)
- At-most-one clauses $\bigwedge_{i=1}^{d} \bigwedge_{j=i+1}^{d} (\neg x_i^v \vee \neg x_j^v)$
- Conflict clauses $(\neg x_i^v \vee \neg x_j^w)$

Global Acceptability Encoding for QCSP

- Considerably more involved than direct encoding
- *Acceptable* assignment to the encoded QBF corresponds to QCSP assignment
- The formula is required to be *true* for some unacceptable assignments — where universal variables take $\neq 1$ values
- Additional literal z in most clauses
- Conflict clauses $(\neg x_i^v \vee \neg x_j^w \vee z)$
- Prevents unit propagation until innermost universal variable is set

Local Acceptability Encoding (refinement of above)

- Local z_u variables are set earlier than z and allow unit propagation
- $\ldots \forall x_i^v \ldots \forall x_j^w \ldots (\neg x_i^v \vee \neg x_j^w \vee z_w)$
- $\ldots \forall x_h^u \ldots \exists x_i^v \ldots \exists x_j^w \ldots (\neg x_i^v \vee \neg x_j^w \vee z_u)$
- Simulates forward checking (Mamoulis and Stergiou)
- Large number of unacceptable assignments

Adapted Log Encoding (further refinement)

- Unary encoding of universal variables has $O(2^d)$ unacceptable assignments — Log encoding has $O(d)$ unacceptable assignments

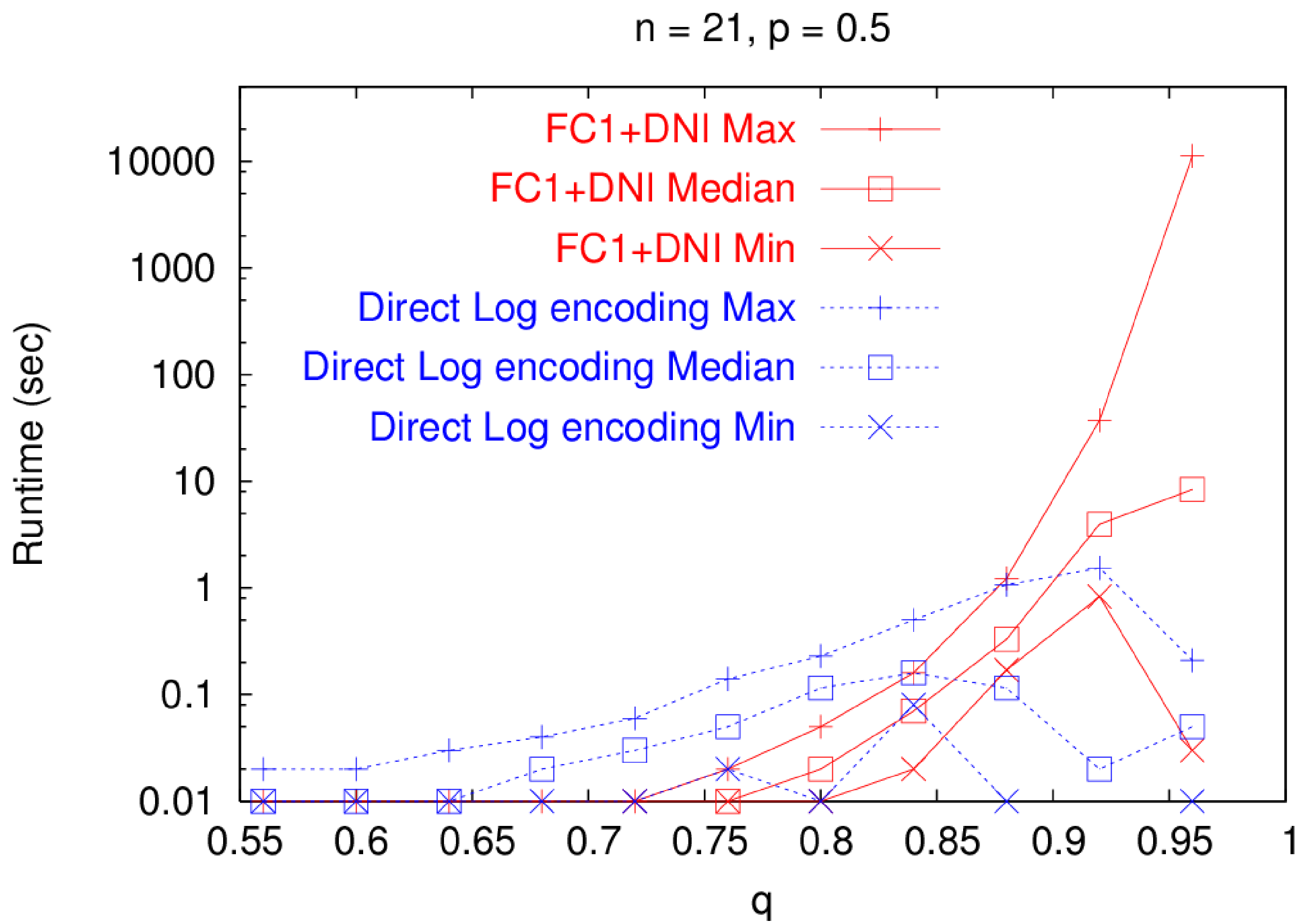
- Proven correct

- Channel log encoding to unary encoding

$$
\begin{array}{c}
(z_u \vee x_1^v \vee b_2^v \vee b_1^v \vee b_0^v) \\
(z_u \vee x_2^v \vee b_2^v \vee b_1^v \vee \neg b_0^v) \\
(z_u \vee x_3^v \vee b_2^v \vee \neg b_1^v \vee b_0^v) \\
(z_u \vee x_4^v \vee b_2^v \vee \neg b_1^v \vee \neg b_0^v) \\
(z_u \vee x_5^v \vee \neg b_2^v \vee b_1^v \vee b_0^v)
\end{array}
$$

- One-way channelling preserves pure literal propagation

Direct solution vs. encoding



Flaws in QCSPs

- Some instances trivially false
- Universals $u_1 \ldots u_7$ followed by existential e
- Each value of e conflicts with some value of some u_i
- Artificially shifts phase transition
- Recent work on controlling parameters to avoid this

Conclusions

- Encoding outperforms direct solution on some problems
  - Sometimes by orders of magnitude
- Low implementation effort
- Support encoding remains open
- Good benchmark problems required

Thank you