Supplementary Information: Inner approximations to the marginal cones of small causal structures

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In this document we present inner approximations to the causal structures mentioned in Section IV of the main text, which are listed as structures 4, 5 and 6 in [1] (see Figure 1). We provide these in terms of one vector on each extremal ray of the corresponding marginal entropy cone and we give strategies for recovering these vectors in each case, proving that our extremal rays define an inner approximation. (Note that the inner approximations to causal structures 1, 2 and 3 have been analysed in the main text).

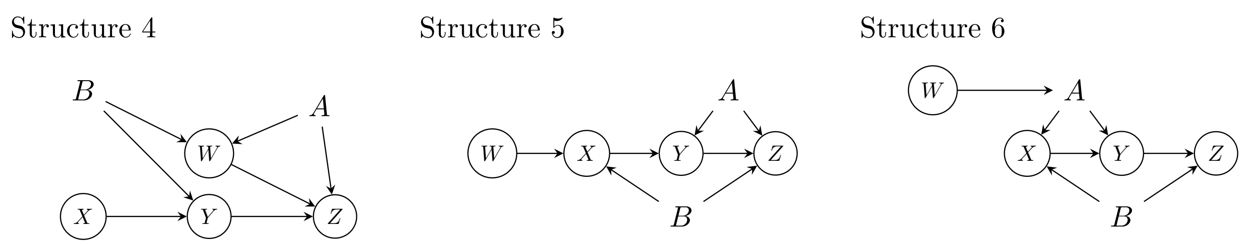

Figure 1: Structures 4, 5 and 6 with observed variables W, X, Y and Z and unobserved A and B.

1 Causal Structure 4

We provide the vertex description of an inner approximation to the marginal entropy cone of Causal Structure 4 in terms of one vector on each extremal ray, with components ordered as

$$(H(W),\ H(X),\ H(Y),\ H(Z),\ H(WX),\ H(WY),\ H(WZ),\ H(XY),\ H(XZ),\ H(YZ),\ H(WXY),\ H(WXZ),\ H(WYZ),\ H(XYZ),\ H(WXYZ))\ .$$

(1)	2 2 3 2 4 4 4 4 4 5 5 4 5 5 5	(21)	1 1 1 1 2 2 2 2 2 2 3 2 3 3 3
(2)	2 2 2 1 4 4 3 4 3 3 5 4 5 4 5	(22)	1 1 1 1 2 2 2 2 2 2 3 3 3 2 3
(3)	1 2 2 2 3 3 3 3 3 4 4 3 4 4 4	(23)	1 1 1 1 2 2 2 2 2 2 2 2 2 2 2
(4)	2 1 2 2 3 3 3 3 3 4 4 3 4 4 4	(24)	1 1 1 1 2 2 2 2 1 2 3 2 3 2 3
(5)	2 1 2 2 3 3 4 3 3 3 4 4 4 3 4	(25)	1 1 1 1 2 2 1 2 2 2 2 2 2 2 2
(6)	1 1 2 2 2 3 3 3 3 4 4 4 4 4 4	(26)	1 1 1 1 2 2 2 1 2 2 2 2 2 2 2
(7)	2 1 1 2 3 3 4 2 3 3 4 4 4 4 4	(27)	1 1 1 1 2 2 2 2 1 2 2 2 2 2 2
(8)	2 1 2 1 3 3 3 3 2 3 4 3 4 3 4	(28)	1 1 1 1 2 2 2 2 2 1 2 2 2 2 2
(9)	1 1 1 3 2 2 3 2 3 3 3 3 3 3 3	(29)	1 0 1 1 1 2 2 1 1 2 2 2 2 2 2
(10)	1 1 1 2 2 2 3 2 3 3 3 3 3 3 3	(30)	1 1 1 0 2 2 1 2 1 1 2 2 2 2 2
(11)	1 1 2 1 2 3 2 3 2 3 3 3 3 3 3	(31)	0 0 0 1 0 0 1 0 1 1 0 1 1 1 1
(12)	2 1 1 1 3 3 3 2 2 2 3 3 3 3 3	(32)	0 0 1 0 0 1 0 1 0 1 1 0 1 1 1
(13)	1 1 1 2 2 2 2 2 3 3 3 3 3 3 3	(33)	0 1 0 0 1 0 0 1 1 0 1 1 0 1 1
(14)	1 1 1 2 2 2 3 2 2 3 3 3 3 3 3	(34)	1 0 0 0 1 1 1 0 0 0 1 1 1 0 1
(15)	1 1 1 2 2 2 3 2 3 2 3 3 3 3 3	(35)	0 0 1 1 0 1 1 1 1 1 1 1 1 1 1
(16)	1 1 1 2 2 2 2 2 2 3 3 2 3 3 3	(36)	0 1 1 0 1 1 0 1 1 1 1 1 1 1 1
(17)	1 1 1 2 2 2 3 2 2 2 3 3 3 2 3	(37)	1 0 0 1 1 1 1 0 1 1 1 1 1 1 1
(18)	1 1 1 2 2 2 2 2 2 2 2 2 2 2 2	(38)	1 0 1 0 1 1 1 1 0 1 1 1 1 1 1
(19)	1 1 2 1 2 2 2 2 2 2 2 2 2 2 2	(39)	0 1 1 1 1 1 1 1 1 1 1 1 1 1 1
(20)	1 1 1 1 2 2 2 2 2 2 3 3 3 3 3	(40)	1 0 1 1 1 1 1 1 1 1 1 1 1 1 1

Below, we list strategies to recover an entropy vector on each of the extremal rays listed above. C_1, C_2, C_3, C_4 and C_5 are uniformly random bits, $\oplus$ denotes addition modulus 2.

(1) $A = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $W = (W_1,\ W_2) = (B_1,\ A \oplus B_2)$, $X = (X_1,\ X_2) = (C_4,\ C_5)$, $Y = (Y_1,\ Y_2,\ Y_3) = (B_1,\ X_1,\ B_2 \oplus X_2)$ and $Z = (W_2 \oplus Y_2, A \oplus W_2 \oplus Y_1 \oplus Y_3)$.

(2) $A = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $W = (W_1,\ W_2) = (B_1,\ A \oplus B_2)$, $X = (X_1,\ X_2) = (C_4,\ C_5)$, $Y = (Y_1,\ Y_2) = (B_1 \oplus X_2,\ B_2 \oplus X_1)$ and $Z = (A \oplus W_1 \oplus W_2 \oplus Y_2)$.

(3) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = (X_1,\ X_2) = (C_3,\ C_4)$, $Y = (Y_1,\ Y_2) = (X_1,\ B \oplus X_2)$ and $Z = (A \oplus W \oplus Y_2,\ W \oplus Y_1)$.

(4) $A = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $W = (W_1,\ W_2) = (B_1,\ A \oplus B_2)$, $X = C_4$, $Y = (Y_1,\ Y_2) = (B_1,\ B_2 \oplus X)$ and $Z = (W_2,\ A \oplus W_1 \oplus W_2 \oplus Y_2)$.

(5) $A = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $W = (W_1,\ W_2) = (B_1,\ A \oplus B_2)$, $X = C_4$, $Y = (Y_1,\ Y_2) = (B_1,\ B_2 \oplus X)$ and $Z = (Y_2,\ A \oplus W_1 \oplus W_2 \oplus Y_2)$.

(6) $A = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $W = A \oplus B_1$, $X = C_4$, $Y = (Y_1,\ Y_2) = (B_1 \oplus X,\ B_2)$ and $Z = (A \oplus Y_2,\ A \oplus W)$.

(7) $A = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $W = (W_1,\ W_2) = (A \oplus B_1, B_2)$, $X = C_4$, $Y = B_1 \oplus X$ and $Z = (W_2 \oplus Y,\ A \oplus Y)$.

(8) $A = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $W = (W_1,\ W_2) = (A \oplus B_1,\ A \oplus B_2)$, $X = C_4$, $Y = (Y_1,\ Y_2) = (B_1 \oplus X,\ B_2 \oplus X)$ and $Z = A \oplus W_2 \oplus Y_1$.

(9) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = C_3$, $Y = B \oplus X$ and $Z = (A \oplus W,\ Y,\ A)$.

(10) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = C_3$, $Y = B \oplus X$ and $Z = (A \oplus Y,\ W \oplus Y)$.

(11) $A = 0$, $B = (B_1,\ B_2) = (C_1,\ C_2)$, $W = B_1$, $X = C_3$, $Y = (Y_1,\ Y_2) = (B_1 \oplus X,\ B_2)$ and $Z = W \oplus Y_2$.

(12) $A = 0$, $B = (B_1,\ B_2) = (C_1,\ C_2)$, $W = (W_1,\ W_2) = (B_1,\ B_2)$, $X = C_3$, $Y = B_1 \oplus X$ and $Z = W_2 \oplus Y$.

(13) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = C_3$, $Y = B \oplus X$ and $Z = (A \oplus W,\ A)$.

(14) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = C_3$, $Y = B \oplus X$ and $Z = (W \oplus Y,\ A)$.

(15) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = C_3$, $Y = B \oplus X$ and $Z = (A \oplus Y,\ A)$.

(16) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = C_3$, $Y = B \oplus X$ and $Z = (W,\ A \oplus W \oplus Y)$.

(17) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = C_3$, $Y = B \oplus X$ and $Z = (Y,\ A \oplus W \oplus Y)$.

(18) $A = 0$, $B = C_1$, $W = B$, $X = C_2$, $Y = B \oplus X$ and $Z = (W,\ Y)$.

(19) $A = 0$, $B = C_1$, $W = B$, $X = C_2$, $Y = (Y_1,\ Y_2) = (B, X)$ and $Z = (W \oplus Y_2)$.

(20) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = C_3$, $Y = B \oplus X$ and $Z = A$.

(21) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = C_3$, $Y = B \oplus X$ and $Z = A \oplus Y$.

(22) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = C_3$, $Y = B \oplus X$ and $Z = A \oplus W$.

(23) $A = 0$, $B = (B_1,\ B_2) = (C_1,\ C_2)$, $W = (W_1,\ W_2) = (B_1,\ B_2)$, $X = (X_1,\ X_2) = (C_3,\ C_4)$, $Y = (Y_1,\ Y_2) = (B_2 \oplus X_1 \oplus X_2,\ B_1 \oplus X_1)$ and $Z = (W_1 \oplus W_2 \oplus Y_1,\ W_2 \oplus Y_2)$. [1]

(24) $A = C_1$, $B = C_2$, $W = A \oplus B$, $X = C_3$, $Y = B \oplus X$ and $Z = A \oplus W \oplus Y$.

(25) $A = 0$, $B = C_1$, $W = B$, $X = C_2$, $Y = B \oplus X$ and $Z = W$.

(26) $A = 0$, $B = 0$, $W = C_1$, $X = C_2$, $Y = X$ and $Z = W \oplus Y$.

(27) $A = 0$, $B = C_1$, $W = B$, $X = C_2$, $Y = B \oplus X$ and $Z = W \oplus Y$.

(28) $A = 0$, $B = C_1$, $W = B$, $X = C_2$, $Y = B \oplus X$ and $Z = Y$.

(29) $A = 0$, $B = 0$, $W = C_1$, $X = 0$, $Y = C_2$ and $Z = W \oplus Y$.

(30) $A = 0$, $B = C_1$, $W = B$, $X = C_2$, $Y = B \oplus X$ and $Z = 0$.

(31) $A = 0$, $B = 0$, $W = 0$, $X = 0$, $Y = 0$ and $Z = C_1$.

(32) $A = 0$, $B = 0$, $W = 0$, $X = 0$, $Y = C_1$ and $Z = 0$.

(33) $A = 0$, $B = 0$, $W = 0$, $X = C_1$, $Y = 0$ and $Z = 0$.

(34) $A = 0$, $B = 0$, $W = C_1$, $X = 0$, $Y = 0$ and $Z = 0$.

(35) $A = 0$, $B = 0$, $W = 0$, $X = 0$, $Y = C_1$ and $Z = Y$.

(36) $A = 0$, $B = 0$, $W = 0$, $X = C_1$, $Y = X$ and $Z = 0$.

(37) $A = 0$, $B = 0$, $W = C_1$, $X = 0$, $Y = 0$ and $Z = W$.

(38) $A = 0$, $B = C_1$, $W = B$, $X = 0$, $Y = B$ and $Z = 0$.

(39) $A = 0$, $B = 0$, $W = 0$, $X = C_1$, $Y = X$ and $Z = Y$.

(40) $A = 0$, $B = C_1$, $W = B$, $X = 0$, $Y = B$ and $Z = Y$.

[1] Note that this strategy recovers double the entropy vector listed above.

2 Causal Structure 5

We provide the vertex description of an inner approximation to the marginal entropy cone of Causal Structure 5 in terms of one vector on each extremal ray, with components ordered as

$$(H(W),\ H(X),\ H(Y),\ H(Z),\ H(WX),\ H(WY),\ H(WZ),\ H(XY),\ H(XZ),\\ H(YZ),\ H(WXY),\ H(WXZ),\ H(WYZ),\ H(XYZ),\ H(WXYZ))\ .$$

(1)	2 2 2 3 3 4 4 4 4 5 5 5 5 5 5	(22)	1 1 1 2 1 2 2 2 2 2 2 2 2 2 2
(2)	2 2 2 3 3 4 4 3 3 4 4 4 4 4 4	(23)	1 1 1 2 2 2 2 1 2 2 2 2 2 2 2
(3)	2 2 2 2 3 4 4 3 3 4 4 4 4 4 4	(24)	1 2 1 1 2 2 1 2 2 2 2 2 2 2 2
(4)	2 2 1 3 4 3 4 3 4 4 5 5 5 5 5	(25)	1 1 1 1 2 2 1 2 2 2 3 2 2 3 3
(5)	2 2 1 2 4 3 3 3 4 3 5 5 4 5 5	(26)	1 1 1 1 1 2 2 2 2 2 2 2 2 2 2
(6)	2 2 1 2 3 3 3 3 3 3 4 4 4 4 4	(27)	1 1 1 1 2 2 2 1 2 2 2 2 2 2 2
(7)	1 1 2 2 2 3 3 3 3 4 4 4 4 4 4	(28)	0 1 1 1 1 1 1 2 2 2 2 2 2 2 2
(8)	1 1 2 2 2 3 3 2 2 3 3 3 3 3 3	(29)	1 1 0 1 2 1 2 1 2 1 2 2 2 2 2
(9)	1 2 1 2 2 2 2 3 3 3 3 3 3 3 3	(30)	0 0 0 1 0 0 1 0 1 1 0 1 1 1 1
(10)	1 2 1 2 2 2 2 2 2 2 2 2 2 2 2	(31)	0 0 1 0 0 1 0 1 0 1 1 0 1 1 1
(11)	1 1 1 3 2 2 3 2 3 3 3 3 3 3 3	(32)	0 1 0 0 1 0 0 1 1 0 1 1 0 1 1
(12)	1 1 1 2 2 2 3 2 3 3 3 3 3 3 3	(33)	1 0 0 0 1 1 1 0 0 0 1 1 1 0 1
(13)	1 1 1 2 2 2 2 2 3 3 3 3 3 3 3	(34)	1 1 1 1 1 2 1 2 1 2 2 1 2 2 2
(14)	1 1 1 2 2 2 3 2 2 3 3 3 3 3 3	(35)	1 1 1 1 2 2 1 1 2 2 2 2 2 2 2
(15)	1 1 2 1 2 3 2 2 2 3 3 3 3 3 3	(36)	0 0 1 1 0 1 1 1 1 1 1 1 1 1 1
(16)	1 1 1 2 2 2 2 2 2 3 3 2 3 3 3	(37)	0 1 0 1 1 0 1 1 1 1 1 1 1 1 1
(17)	1 1 1 2 2 2 2 2 3 2 3 3 2 3 3	(38)	0 1 1 0 1 1 0 1 1 1 1 1 1 1 1
(18)	1 2 1 1 2 2 2 2 2 2 2 2 2 2 2	(39)	1 1 0 0 1 1 1 1 1 0 1 1 1 1 1
(19)	1 1 1 1 2 2 2 2 2 2 3 3 3 3 3	(40)	0 1 1 1 1 1 1 1 1 1 1 1 1 1 1
(20)	2 2 2 2 3 4 3 3 3 4 4 4 4 4 4	(41)	1 1 1 0 1 1 1 1 1 1 1 1 1 1 1
(21)	1 1 1 1 2 2 2 2 2 2 3 3 2 3 3	(42)	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

In the following we list strategies to recover entropy vectors on all of the above rays. C_1, C_2, C_3, C_4 and C_5 are uniform bits, $\oplus$ denotes addition modulus 2.

(1) $A = (A_1,\ A_2) = (C_1,\ C_2)$, $B = C_3$, $W = (W_1,\ W_2) = (C_4,\ C_5)$, $X = (X_1,\ X_2) = (W_1,\ B \oplus W_2)$, $Y = (Y_1,\ Y_2) = (A_1 \oplus X_1,\ A_2 \oplus X_2)$ and $Z = (A_1 \oplus Y_1,\ B \oplus Y_2,\ A_1 \oplus A_2 \oplus B)$.

(2) $A = C_1$, $B = C_2$, $W = (W_1,\ W_2) = (C_3,\ C_4)$, $X = (X_1,\ X_2) = (B \oplus W_1,\ B \oplus W_2)$, $Y = (Y_1,\ Y_2) = (X_1,\ A \oplus X_2)$ and $Z = (A \oplus B \oplus Y_1,\ A \oplus Y_2,\ A \oplus B)$.

(3) $A = C_1$, $B = C_2$, $W = (W_1,\ W_2) = (C_3,\ C_4)$, $X = (X_1,\ X_2) = (B \oplus W_1,\ B \oplus W_2)$, $Y = (Y_1,\ Y_2) = (X_1,\ A \oplus X_2)$ and $Z = (A \oplus B \oplus Y_1,\ A \oplus Y_2)$.

(4) $A = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $W = (W_1,\ W_2) = (C_4,\ C_5)$, $X = (X_1,\ X_2) = (B_1 \oplus W_1,\ B_2 \oplus W_2)$, $Y = A \oplus X_1$ and $Z = (A \oplus B_1 \oplus Y,\ A \oplus Y,\ A \oplus B_2)$.

(5) $A = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $W = (W_1,\ W_2) = (C_4,\ C_5)$, $X = (X_1,\ X_2) = (B_1 \oplus W_1,\ B_2 \oplus W_2)$, $Y = A \oplus X_1$ and $Z = (A \oplus B_1 \oplus Y,\ B_1 \oplus B_2 \oplus Y)$.

(6) $A = C_1$, $B = C_2$, $W = (W_1,\ W_2) = (C_3,\ C_4)$, $X = (X_1,\ X_2) = (B \oplus W_1,\ W_2)$, $Y = A \oplus X_2$ and $Z = (A \oplus Y,\ A \oplus B)$.

(7) $A = (A_1,\ A_2) = (C_1,\ C_2)$, $B = C_3$, $W = C_4$, $X = B \oplus W$, $Y = (Y_1,\ Y_2) = (A_1 \oplus X,\ A_2)$ and $Z = (A_1 \oplus A_2,\ A_1 \oplus B)$.

(8) $A = C_1$, $B = C_2$, $W = C_3$, $X = B \oplus W$, $Y = (Y_1,\ Y_2) = (X,\ A)$ and $Z = (Y_1,\ A \oplus B \oplus Y_1)$.

(9) $A = C_1$, $B = C_2$, $W = C_3$, $X = (X_1,\ X_2) = (W,\ B)$, $Y = A \oplus X_1 \oplus X_2$ and $Z = (A \oplus B \oplus Y,\ B \oplus Y)$.

(10) $A = 0$, $B = C_1$, $W = C_2$, $X = (X_1,\ X_2) = (B,\ W)$, $Y = X_1 \oplus X_2$ and $Z = (B \oplus Y,\ B)$.

(11) $A = C_1$, $B = C_2$, $W = C_3$, $X = B \oplus W$, $Y = A \oplus X$ and $Z = (B,\ A,\ A \oplus Y)$.

(12) $A = C_1$, $B = C_2$, $W = C_3$, $X = B \oplus W$, $Y = A \oplus X$ and $Z = (B,\ A)$.

(13) $A = C_1$, $B = C_2$, $W = C_3$, $X = B \oplus W$, $Y = A \oplus X$ and $Z = (A \oplus B \oplus Y,\ A)$.

(14) $A = C_1$, $B = C_2$, $W = C_3$, $X = B \oplus W$, $Y = A \oplus X$ and $Z = (A \oplus Y,\ A \oplus B)$.

(15) $A = C_1$, $B = C_2$, $W = C_3$, $X = B \oplus W$, $Y = (Y_1,\ Y_2) = (A,\ X)$ and $Z = A \oplus B$.

(16) $A = C_1$, $B = C_2$, $W = C_3$, $X = B \oplus W$, $Y = A \oplus X$ and $Z = (A \oplus B \oplus Y,\ A \oplus Y)$.

(17) $A = C_1$, $B = C_2$, $W = C_3$, $X = B \oplus W$, $Y = A \oplus X$ and $Z = (Y,\ A \oplus B \oplus Y)$.

(18) $A = 0$, $B = C_1$, $W = C_2$, $X = (X_1,\ X_2) = (W,\ B)$, $Y = X_1 \oplus X_2$ and $Z = B$.

(19) $A = C_1$, $B = C_2$, $W = C_3$, $X = B \oplus W$, $Y = A \oplus X$ and $Z = B \oplus Y$.

(20) $A = C_1$, $B = C_2$, $W = (W_1,\ W_2) = (C_3,\ C_4)$, $X = (X_1,\ X_2) = (B \oplus W_1,\ W_2)$, $Y = (Y_1,\ Y_2) = (X_1,\ A \oplus X_2)$ and $Z = (A \oplus B \oplus Y_1,\ A \oplus Y_2)$.

(21) $A = C_1$, $B = C_2$, $W = C_3$, $X = B \oplus W$, $Y = A \oplus X$ and $Z = A \oplus B$.

(22) $A = C_1$, $B = 0$, $W = C_2$, $X = W$, $Y = A \oplus X$ and $Z = (A,\ Y)$.

(23) $A = 0$, $B = C_1$, $W = C_2$, $X = B \oplus W$, $Y = X$ and $Z = (B,\ Y)$.

(24) $A = 0$, $B = C_1$, $W = C_2$, $X = (X_1,\ X_2) = (W,\ B)$, $Y = X_1 \oplus X_2$ and $Z = B \oplus Y$.

(25) $A = C_1$, $B = C_2$, $W = C_3$, $X = B \oplus W$, $Y = A \oplus X$ and $Z = A \oplus B \oplus Y$.

(26) $A = C_1$, $B = 0$, $W = C_2$, $X = W$, $Y = A \oplus X$ and $Z = A$.

(27) $A = 0$, $B = C_1$, $W = C_3$, $X = B \oplus W$, $Y = X$ and $Z = B$.

(28) $A = C_1$, $B = C_2$, $W = 0$, $X = B$, $Y = A \oplus X$ and $Z = A$.

(29) $A = 0$, $B = C_1$, $W = C_2$, $X = B \oplus W$, $Y = 0$ and $Z = B$.

(30) $A = 0$, $B = 0$, $W = 0$, $X = 0$, $Y = 0$ and $Z = C_1$.

(31) $A = 0$, $B = 0$, $W = 0$, $X = 0$, $Y = C_1$ and $Z = 0$.

(32) $A = 0$, $B = 0$, $W = 0$, $X = C_1$, $Y = 0$ and $Z = 0$.

(33) $A = 0$, $B = 0$, $W = C_1$, $X = 0$, $Y = 0$ and $Z = 0$.

(34) $A = C_1$, $B = 0$, $W = C_2$, $X = W$, $Y = A \oplus X$ and $Z = A \oplus Y$.

(35) $A = 0$, $B = C_1$, $W = C_2$, $X = B \oplus W$, $Y = X$ and $Z = B \oplus Y$.

(36) $A = 0$, $B = 0$, $W = 0$, $X = 0$, $Y = C_1$ and $Z = Y$.

(37) $A = 0$, $B = C_1$, $W = 0$, $X = B$, $Y = 0$ and $Z = B$.

(38) $A = 0$, $B = 0$, $W = 0$, $X = C_1$, $Y = X$ and $Z = 0$.

(39) $A = 0$, $B = 0$, $W = C_1$, $X = W$, $Y = 0$ and $Z = 0$.

(40) $A = 0$, $B = 0$, $W = 0$, $X = C_1$, $Y = X$ and $Z = Y$.

(41) $A = 0$, $B = 0$, $W = C_1$, $X = W$, $Y = X$ and $Z = 0$.

(42) $A = 0$, $B = 0$, $W = C_1$, $X = W$, $Y = X$ and $Z = Y$.

3 Causal Structure 6

We provide the vertex description of an inner approximation to the marginal entropy cone of Causal Structure 6 in terms of one vector on each extremal ray, with components ordered as

$$(H(W),\ H(X),\ H(Y),\ H(Z),\ H(WX),\ H(WY),\ H(WZ),\ H(XY),\ H(XZ),\\ H(YZ),\ H(WXY),\ H(WXZ),\ H(WYZ),\ H(XYZ),\ H(WXYZ))\ .$$

(1)	2 3 2 2 4 4 3 5 4 4 5 5 4 5 5
(2)	2 3 2 2 4 3 3 5 5 4 5 5 4 5 5
(3)	2 3 1 2 4 3 3 4 4 3 5 5 4 5 5
(4)	2 2 1 2 4 3 3 3 4 3 5 5 4 5 5
(5)	2 2 2 1 3 3 2 4 3 3 4 3 3 4 4
(6)	1 3 1 2 3 2 2 4 4 3 4 4 3 4 4
(7)	2 1 2 1 3 4 3 3 2 3 4 4 4 4 4
(8)	1 2 1 2 3 2 3 3 4 3 4 4 3 4 4
(9)	1 2 1 2 3 2 2 3 4 3 4 4 3 4 4
(10)	1 2 2 1 2 3 2 3 3 3 3 3 3 3 3
(11)	2 1 1 2 3 3 3 2 3 2 3 3 3 3 3
(12)	1 1 2 2 2 3 3 2 2 3 3 3 3 3 3
(13)	2 2 1 1 3 3 2 3 2 2 3 3 3 3 3
(14)	1 2 1 2 2 2 2 3 3 2 3 3 2 3 3
(15)	1 2 1 2 2 2 2 2 2 2 2 2 2 2 2
(16)	1 1 2 1 2 3 2 3 2 3 3 3 3 3 3
(17)	1 2 1 1 3 2 2 3 3 2 3 3 3 3 3
(18)	2 1 1 1 3 3 3 2 2 2 3 3 3 3 3
(19)	1 1 2 1 2 3 2 2 2 3 3 3 3 3 3
(20)	2 1 1 1 3 3 2 2 2 2 3 3 3 3 3
(21)	1 2 1 1 2 2 2 3 3 2 3 3 2 3 3
(22)	1 1 1 2 2 2 2 2 3 2 3 3 2 3 3
(23)	1 1 1 2 2 2 2 2 2 2 2 2 2 2 2
(24)	1 1 2 1 2 2 2 2 2 2 2 2 2 2 2
(25)	1 2 1 1 2 2 2 2 2 2 2 2 2 2 2
(26)	1 1 1 1 2 2 2 2 2 2 3 3 3 3 3
(27)	1 1 1 1 2 2 2 2 2 2 3 3 2 3 3
(28)	1 2 1 1 2 2 1 3 2 2 3 2 2 3 3
(29)	1 1 1 1 2 2 2 2 2 2 2 2 2 2 2
(30)	1 1 1 2 2 2 2 1 2 2 2 2 2 2 2
(31)	1 2 1 1 2 2 1 2 2 2 2 2 2 2 2
(32)	1 1 1 1 2 2 1 2 2 2 3 2 2 3 3
(33)	1 1 1 1 2 1 2 2 2 2 2 2 2 2 2
(34)	1 1 1 1 2 2 1 2 2 2 2 2 2 2 2
(35)	1 1 1 1 2 2 2 1 2 2 2 2 2 2 2
(36)	1 1 1 1 2 2 2 2 1 2 2 2 2 2 2
(37)	1 1 1 1 2 2 2 2 2 1 2 2 2 2 2
(38)	0 1 1 1 1 1 1 2 2 2 2 2 2 2 2
(39)	1 1 0 1 2 1 2 1 2 1 2 2 2 2 2
(40)	1 1 1 0 2 2 1 2 1 1 2 2 2 2 2
(41)	1 1 1 1 2 2 1 1 2 2 2 2 2 2 2
(42)	0 0 0 1 0 0 1 0 1 1 0 1 1 1 1
(43)	0 0 1 0 0 1 0 1 0 1 1 0 1 1 1
(44)	0 1 0 0 1 0 0 1 1 0 1 1 0 1 1
(45)	1 0 0 0 1 1 1 0 0 0 1 1 1 0 1
(46)	0 0 1 1 0 1 1 1 1 1 1 1 1 1 1
(47)	0 1 0 1 1 0 1 1 1 1 1 1 1 1 1
(48)	0 1 1 0 1 1 0 1 1 1 1 1 1 1 1
(49)	1 0 1 0 1 1 1 1 0 1 1 1 1 1 1
(50)	1 1 0 0 1 1 1 1 1 0 1 1 1 1 1
(51)	0 1 1 1 1 1 1 1 1 1 1 1 1 1 1
(52)	1 0 1 1 1 1 1 1 1 1 1 1 1 1 1
(53)	1 1 1 0 1 1 1 1 1 1 1 1 1 1 1
(54)	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

In the following we list strategies to recover entropy vectors on all of the above rays. C_1, C_2, C_3, C_4 and C_5 are uniform bits, $\oplus$ denotes addition modulus 2.

(1) $W = (W_1,\ W_2) = (C_1,\ C_2)$, $B = (B_1,\ B_2) = (C_3,\ C_4)$, $A = (A_1,\ A_2,\ A_3) = (W_1,\ W_2,\ C_5)$, $X = (X_1,\ X_2,\ X_3) = (A_1,\ A_1 \oplus A_3 \oplus B_2,\ A_3 \oplus B_1)$, $Y = (Y_1,\ Y_2) = (A_3 \oplus X_1 \oplus X_3,\ A_1 \oplus A_2 \oplus A_3 \oplus X_2)$ and $Z = (B_1 \oplus Y_1,\ B_2)$.

(2) $W = (W_1,\ W_2) = (C_1,\ C_2)$, $B = (B_1,\ B_2) = (C_3,\ C_4)$, $A = (A_1,\ A_2,\ A_3) = (W_1,\ W_2,\ C_5)$, $X = (X_1,\ X_2,\ X_3) = (A_1,\ A_2 \oplus A_3 \oplus B_1,\ A_2 \oplus A_3 \oplus B_2)$, $Y = (Y_1,\ Y_2) = (A_1 \oplus A_2,\ A_1 \oplus A_2 \oplus A_3 \oplus X_2)$ and $Z = (B_1 \oplus Y_1 \oplus Y_2,\ B_2)$.

(3) $W = (W_1,\ W_2) = (C_1,\ C_2)$, $B = (B_1,\ B_2) = (C_3,\ C_4)$, $A = (A_1,\ A_2,\ A_3) = (W_1,\ W_2,\ C_5)$, $X = (X_1,\ X_2,\ X_3) = (A_1 \oplus A_2,\ A_2 \oplus A_3 \oplus B_1,\ A_3 \oplus B_2)$, $Y = A_3 \oplus X_1 \oplus X_3$ and $Z = (B_2 \oplus Y,\ B_1)$.

(4) $W = (W_1,\ W_2) = (C_1,\ C_2)$, $B = (B_1,\ B_2) = (C_3,\ C_4)$, $A = (A_1,\ A_2,\ A_3) = (W_1,\ W_2,\ C_5)$, $X = (X_1,\ X_2) = (A_1 \oplus A_3 \oplus B_1,\ A_2 \oplus B_2)$, $Y = A_3 \oplus X_1$ and $Z = (B_1 \oplus Y,\ B_1 \oplus B_2)$.

(5) $W = (W_1,\ W_2) = (C_1,\ C_2)$, $B = C_3$, $A = (A_1,\ A_2,\ A_3) = (W_1,\ W_2,\ C_4)$, $X = (X_1,\ X_2) = (A_1,\ A_2 \oplus A_3 \oplus B)$, $Y = (Y_1,\ Y_2) = (A_1 \oplus A_2,\ A_3 \oplus X_2)$ and $Z = Y_2 \oplus B$.

(6) $W = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $A = (A_1,\ A_2) = (W,\ C_4)$, $X = (X_1,\ X_2,\ X_3) = (A_1,\ B_1,\ A_2 \oplus B_2)$, $Y = A_1 \oplus A_2 \oplus X_2 \oplus X_3$ and $Z = (B_1 \oplus B_2 \oplus Y,\ B_2)$.

(7) $W = (W_1,\ W_2) = (C_1,\ C_2)$, $B = C_3$, $A = (A_1,\ A_2,\ A_3) = (W_1,\ W_2,\ C_4)$, $X = A_3 \oplus B$, $Y = (Y_1,\ Y_2) = (A_1 \oplus A_3 \oplus X,\ A_2 \oplus X)$ and $Z = B$.

(8) $W = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $A = (A_1,\ A_2) = (W,\ C_4)$, $X = (X_1,\ X_2) = (B_1,\ A_2 \oplus B_2)$, $Y = A_1 \oplus A_2 \oplus X_2$ and $Z = (B_1 \oplus Y,\ B_2)$.

(9) $W = C_1$, $B = (B_1,\ B_2) = (C_2,\ C_3)$, $A = (A_1,\ A_2) = (W,\ C_4)$, $X = (X_1,\ X_2) = (A_1 \oplus B_1,\ A_2 \oplus B_2)$, $Y = A_1 \oplus A_2 \oplus X_2$ and $Z = (B_2 \oplus Y,\ B_1 \oplus B_2)$.

(10) $W = C_1$, $B = C_2$, $A = (A_1,\ A_2) = (W,\ C_3)$, $X = (X_1,\ X_2) = (A_2 \oplus B,\ A_1)$, $Y = (Y_1,\ Y_2) = (X_1,\ A_1 \oplus A_2)$ and $Z = B$.

(11) $W = (W_1,\ W_2) = (C_1,\ C_2)$, $B = C_3$, $A = (A_1,\ A_2) = (W_1,\ W_2)$, $X = A_2 \oplus B$, $Y = A_1 \oplus X$ and $Z = (B \oplus Y, B)$.

(12) $W = C_1$, $B = C_2$, $A = (A_1,\ A_2) = (W,\ C_3)$, $X = A_1 \oplus B$, $Y = (Y_1,\ Y_2) = (X,\ A_2)$ and $Z = (Y_1,\ B \oplus Y_2)$.

(13) $W = (W_1,\ W_2) = (C_1,\ C_2)$, $B = C_3$, $A = (A_1,\ A_2) = (W_1,\ W_2)$, $X = (X_1,\ X_2) = (A_1, A_2 \oplus B)$, $Y = A_1 \oplus A_2 \oplus X_2$ and $Z = B \oplus Y$.

(14) $W = C_1$, $B = C_2$, $A = (A_1,\ A_2) = (W,\ C_3)$, $X = (X_1,\ X_2) = (A_1, A_2 \oplus B)$, $Y = A_2 \oplus X_1 \oplus X_2$ and $Z = (B \oplus Y,\ B)$.

(15) $W = C_1$, $B = C_2$, $A = W$, $X = (X_1,\ X_2) = (A, B)$, $Y = X_1 \oplus X_2$ and $Z = (B \oplus Y,\ B)$.

(16) $W = C_1$, $B = C_2$, $A = (W,\ C_3)$, $X = B$, $Y = (X \oplus A_2,\ A_1 \oplus A_2)$ and $Z = B \oplus Y$.

(17) $W = C_1$, $B = C_2$, $A = (W,\ C_3)$, $X = (X_1,\ X_2) = (A_1 \oplus B,\ A_2)$, $Y = A_1 \oplus A_2$ and $Z = B$.

(18) $W = (W_1,\ W_2) = (C_1,\ C_2)$, $B = C_3$, $A = (A_1,\ A_2) = (W_1,\ W_2)$, $X = A_1 \oplus B$, $Y = A_1 \oplus A_2 \oplus X$ and $Z = B$.

(19) $W = C_1$, $B = C_2$, $A = (A_1,\ A_2) = (W,\ C_3)$, $X = A_1 \oplus B$, $Y = (Y_1,\ Y_2) = (X,\ A_2)$ and $Z = B \oplus Y_1 \oplus Y_2$.

(20) $W = (W_1,\ W_2) = (C_1,\ C_2)$, $B = C_3$, $A = (A_1,\ A_2) = (W_1,\ W_2)$, $X = A_1 \oplus A_2 \oplus B$, $Y = A_2 \oplus X$ and $Z = B \oplus Y$.

(21) $W = C_1$, $B = C_2$, $A = (A_1,\ A_2) = (W,\ C_3)$, $X = (X_1,\ X_2) = (A_2 \oplus B,\ A_1)$, $Y = A_1 \oplus A_2 \oplus X_1$ and $Z = B$.

(22) $W = C_1$, $B = C_2$, $A = (A_1,\ A_2) = (W,\ C_3)$, $X = A_2 \oplus B$, $Y = A_1 \oplus A_2 \oplus X$ and $Z = (B \oplus Y,\ B)$.

(23) $W = C_1$, $B = C_2$, $A = W$, $X = B$, $Y = A \oplus X$ and $Z = (B \oplus Y,\ B)$.

(24) $W = C_1$, $B = C_2$, $A = W$, $X = A \oplus B$, $Y = (Y_1,\ Y_2) = (A,\ A \oplus X)$ and $Z = B$.

(25) $W = C_1$, $B = C_2$, $A = W$, $X = (X_1,\ X_2) = (A,\ B)$, $Y = X_1 \oplus X_2$ and $Z = B$.

(26) $W = C_1$, $B = C_2$, $A = W$, $X = A \oplus B$, $Y = X \oplus C_3$ and $Z = B \oplus Y$.

(27) $W = C_1$, $B = C_2$, $A = (A_1,\ A_2) = (W,\ C_3)$, $X = A_1 \oplus A_2 \oplus B$, $Y = X \oplus A_2$ and $Z = B$.

(28) $W = C_1$, $B = C_2$, $A = (A_1,\ A_2) = (W,\ C_3)$, $X = (X_1,\ X_2) = (A_1,\ A_2 \oplus B)$, $Y = A_1 \oplus A_2 \oplus X_2$ and $Z = B \oplus Y$.

(29) $W = (W_1,\ W_2) = (C_1,\ C_2)$, $B = (B_1,\ B_2) = (C_3,\ C_4)$, $A = (A_1,\ A_2) = (W_1,\ W_2)$, $X = (X_1,\ X_2) = (A_1 \oplus B_1,\ A_2 \oplus B_2)$, $Y = (Y_1,\ Y_2) = (A_1 \oplus A_2 \oplus X_1, A_1 \oplus X_2)$ and $Z = (B_1, B_2)$. [2]

(30) $W = C_1$, $B = C_2$, $A = W$, $X = A \oplus B$, $Y = X$ and $Z = (B,\ Y)$.

(31) $W = C_1$, $B = C_2$, $A = W$, $X = (X_1,\ X_2) = (B,\ A)$, $Y = A \oplus X_1$ and $Z = B \oplus Y$.

(32) $W = C_1$, $B = C_2$, $A = (A_1,\ A_2) = (W,\ C_3)$, $X = A_2 \oplus B$, $Y = A_1 \oplus A_2 \oplus X$ and $Z = B \oplus Y$.

(33) $W = C_1$, $B = C_2$, $A = W$, $X = A \oplus B$, $Y = A$ and $Z = B$.

(34) $W = C_1$, $B = C_2$, $A = W$, $X = B$, $Y = A \oplus X$ and $Z = B \oplus Y$.

(35) $W = C_1$, $B = C_2$, $A = W$, $X = A \oplus B$, $Y = X$ and $Z = B$.

(36) $W = C_1$, $B = C_2$, $A = W$, $X = B$, $Y = A \oplus X$ and $Z = B$.

(37) $W = C_1$, $B = C_2$, $A = W$, $X = A \oplus B$, $Y = A \oplus X$ and $Z = Y$.

(38) $W = 0$, $B = C_1$, $A = C_2$, $X = A \oplus B$, $Y = A$ and $Z = B$.

(39) $W = C_1$, $B = C_2$, $A = W$, $X = A \oplus B$, $Y = 0$ and $Z = B$.

(40) $W = C_1$, $B = C_2$, $A = W$, $X = B$, $Y = A \oplus X$ and $Z = 0$.

(41) $W = C_1$, $B = C_2$, $A = W$, $X = A \oplus B$, $Y = X$ and $Z = B \oplus Y$.

(42) $W = 0$, $B = 0$, $A = 0$, $X = 0$, $Y = 0$ and $Z = C_1$.

(43) $W = 0$, $B = 0$, $A = 0$, $X = 0$, $Y = C_1$ and $Z = 0$.

(44) $W = 0$, $B = 0$, $A = 0$, $X = C_1$, $Y = 0$ and $Z = 0$.

(45) $W = C_1$, $B = 0$, $A = 0$, $X = 0$, $Y = 0$ and $Z = 0$.

(46) $W = 0$, $B = 0$, $A = 0$, $X = 0$, $Y = C_1$ and $Z = Y$.

(47) $W = 0$, $B = C_1$, $A = 0$, $X = B$, $Y = 0$ and $Z = B$.

(48) $W = 0$, $B = 0$, $A = 0$, $X = C_1$, $Y = X$ and $Z = 0$.

(49) $W = C_1$, $B = 0$, $A = W$, $X = 0$, $Y = A$ and $Z = 0$.

(50) $W = C_1$, $B = 0$, $A = W$, $X = A$, $Y = 0$ and $Z = 0$.

(51) $W = 0$, $B = 0$, $A = 0$, $X = C_1$, $Y = X$ and $Z = Y$.

(52) $W = C_1$, $B = 0$, $A = W$, $X = 0$, $Y = A$ and $Z = Y$.

(53) $W = C_1$, $B = 0$, $A = W$, $X = A$, $Y = X$ and $Z = 0$.

(54) $W = C_1$, $B = 0$, $A = W$, $X = A$, $Y = X$ and $Z = Y$.

References

[1] Henson, J., Lal, R. & Pusey, M. F. Theory-independent limits on correlations from generalized Bayesian networks. *New Journal of Physics* **16**, 113043 (2014).

[2] Note that this strategy recovers double the entropy vector listed above.