A THEORY OF QUANTUM ENTANGLEMENT

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Entangled states are considered to be an important resource for quantum computation and quantum information processes. Various authors have developed theories of entanglement including entanglement measures and robustness but this work is somewhat different. Our work is inspired by a paper of Paul Busch:

The role of entanglement in
quantum measurement and information processing,

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The theme of Paul's paper is that one cannot transfer information without entanglement. Our present work involves measuring the amount of entanglement.

1 Classical Entanglement

It is frequently stated that entanglement is a strictly quantum phenomenon that is not present in classical theory. We do not believe this is actually true and begin with a classical theory of entangled measures. This theory is quite simple and does not have the depth and complexity of its quantum counterpart. However, we believe that it can be instructive and give insights into the quantum theory.

Let M be the set of probability measures on $\mathbb{N}$. We view $\underline{u} \in M$ as a probability vector $\underline{u} = \{u_i \colon i \in \mathbb{N}\}$, $u_i \geq 0$, $\sum u_i = 1$ and write $\|\underline{u}\|^2 = \sum u_i^2$. The *support* of $\underline{u}$ is $\operatorname{supp}(\underline{u}) = \{i \colon u_i \neq 0\}$. The *entanglement index* of $\underline{u}$ is the cardinality of $\operatorname{supp}(\underline{u})$ and is denoted by $n(\underline{u})$. The *entanglement number* of $\underline{u}$ is

$$e(\underline{u}) = \left[1 - \|\underline{u}\|^2\right]^{1/2} = \left[\sum_{i \neq j} u_i u_j\right]^{1/2} = \left[\sum_i u_i(1 - u_i)\right]^{1/2}$$

We see that $e(\underline{u})^2$ is the average deviation of $\underline{u}$ from 1. We say that $\underline{u}$ is a *point* (or *Dirac*) *measure* if $u_i = 1$ for some $i \in \mathbb{N}$. We say $\underline{u}$ is *uniform* if $u_i = u_j$ whenever, $u_i, u_j \neq 0$. If $\underline{u}$ is uniform, then $n(\underline{u}) < \infty$ and $u_i = 1/n(\underline{u})$ whenever $u_i \neq 0$.

Theorem 1.1. (a) $e(\underline{u}) = 0$ *iff* $\underline{u}$ *is a point measure.*
(b) *If* $n(\underline{u}) < \infty$*, then* $e(\underline{u}) \leq [(n(\underline{u}) - 1)/n(\underline{u})]$ *and equality is achieved iff* $\underline{u}$ *is uniform.*

If $\underline{u}$ is uniform and $n(\underline{u}) \neq 1$, we say that $\underline{u}$ is *maximally entangled* with index $n(\underline{u})$. There is precisely one maximally entangled $\underline{u}$ for every finite support in $\mathbb{N}$. Also, $0 \leq e(\underline{u}) < 1$.

Example 1. (a) If $u_1 = u_2 = 1/2$, then $e(\underline{u}) = 1/\sqrt{2}$ and $\underline{u}$ is maximally entangled with index 2. (b) If $u_1 = u_2 = u_3 = 1/3$ then $e(\underline{u}) = \sqrt{2/3}$ and $\underline{u}$ is maximally entangled with index 3. (c) If $u_1 = 1/2$, $u_2 = 1/3$, $u_3 = 1/6$, then $e(\underline{u}) = \sqrt{11/18}$. (d) If $u_1 = 1/9$, $u_2 = 1/9$, $u_3 = 7/9$, then $e(\underline{u}) = \sqrt{30}/9$. We have

$$\frac{\sqrt{30}}{9} < \frac{1}{\sqrt{2}} < \sqrt{\frac{11}{18}} < \sqrt{\frac{2}{3}}$$

so $(d) < (a) < (c) < (b)$. □

If $\underline{u}, \underline{v} \in M$ and $\lambda \in [0,1]$, then $\lambda\underline{u} + (1-\lambda)\underline{v} \in M$ is a *mixture* of $\underline{u}$ and $\underline{v}$. It is easy to check that n is *concave*, that is

$$n\left[\lambda\underline{u} + (1-\lambda)\underline{v}\right] \geq \lambda n(\underline{u}) + (1-\lambda)n(\underline{v})$$

Theorem 1.2. *If* $\underline{u}, \underline{v} \in M$, $\lambda \in [0,1]$ *we have*

$$e\left[\lambda \underline{u} + (1-\lambda)\underline{v}\right] \geq \lambda e(\underline{u}) + (1-\lambda)e(\underline{v})$$

and if $\lambda \in (0,1)$ *we have equality iff* $\underline{u} = \underline{v}$.

Let $M \times M$ be the set of probability measures on $\mathbb{N} \times \mathbb{N}$. Then $\underline{u} \in M \times M$ if $\underline{u} = \{u_{ij} \colon i, j \in \mathbb{N}\}$, $u_{ij} \geq 0$, $\sum u_{ij} = 1$. As before

$$e(\underline{u}) = \left[1 - ||\underline{u}||^2\right]^{1/2} = \left(1 - \sum u_{ij}^2\right)^{1/2}$$

Also, $e(\underline{u}) = 0$ iff $\underline{u}$ is a point measure. If $\underline{v}, \underline{w} \in M$ we define $\underline{u} = \underline{v} \times \underline{w} \in M \times M$ by $u_{ij} = v_i w_j$. We say $\underline{u} \in M \times M$ is *factorized*, if $\underline{u} = \underline{v} \times \underline{w}$ for some $\underline{v}, \underline{w} \in M$. If $\underline{u}$ is not factorized, then $\underline{u}$ is *entangled.* Notice that if $e(\underline{u}) = 0$ then $\underline{u}$ is factorized. The converse does not hold because there are factorized $\underline{u} \in M \times M$ that are not point measures. This does not hold in quantum mechanics and gives an important difference between quantum mechanics and the classical theory.

Example 2. (a) Let $\underline{u} \in M \times M$ be $u_{11} = u_{12} = 1/2$. Then $e(\underline{u}) = 1/\sqrt{2}$ but $\underline{u}$ is factorized. (b) Let $\underline{u} \in M \times M$ be $u_{11} = u_{12} = u_{22} = 1/3$. Then $e(\underline{u}) = \sqrt{2/3}$ and $\underline{u}$ is entangled. $\square$

2 Quantum Statistics

Let H be a complex, finite-dimensional Hilbert space. A *pure state* is a one-dimensional projection $P_\phi = |\phi\rangle\langle\phi|$ on H. A unit vector $\phi \in H$ is a *vector state.* A *context* is a set of mutually orthogonal pure states $P\phi_i$ such that $\sum P_{\phi_i} = I$. Equivalently, a context is an orthonormal basis $\{\phi_i\}$ of vector states. A context is a complete set of minimal nonzero sharp events. There are uncountably many contexts for a quantum system. For a classical system described by $\mathbb{N}$, the minimal nonzero sharp events are the points of $\mathbb{N}$ so the only context is $\mathbb{N}$ itself.

Let $\mathcal{L}(H)$ be the set of linear operators on H. For $A \in \mathcal{L}(H)$ define $|A| = (A^*A)^{1/2} \geq 0$. A *state* is a $\rho \in \mathcal{L}(H)$ such that $\rho \geq 0$, $\operatorname{tr}(\rho) = 1$. Denote the set of states by $\mathcal{S}(H)$. If $\rho \in \mathcal{S}(H)$ and $A \in \mathcal{L}(H)$ the *ρ-expectation* of A is $E_\rho(A) = \operatorname{tr}(\rho A)$ and the *ρ-variance* of A is

$$V_\rho(A) = E_\rho\left[|A - E_\rho(A)I|^2\right]$$

In particular, for a pure state P_ϕ

$$E_\phi(A) = E_{P_\phi}(A) = \langle\phi, A\phi\rangle$$
$$V_\phi(A) = V_{P_\phi}(A) = \left\langle\phi, |A - \langle\phi, A\phi\rangle I|^2\,\phi\right\rangle$$

The complex vector space $\mathcal{L}(H)$ becomes a Hilbert space under the H-S inner product $\langle A, B\rangle = \operatorname{tr}(A^*B)$. The H-S norm becomes

$$||A|| = [\operatorname{tr}(A^*A)]^{1/2} = \left[\operatorname{tr}(|A|^2)\right]^{1/2}$$

Theorem 2.1. (a) $V_\rho(A) = E_\rho\left(|A|^2\right) - |E_\rho(A)|^2$
(b) $|E_\rho(A)|^2 \leq E_\rho\left(|A|^2\right)$ *and* $V_\rho(A) = 0$ *iff* $A\rho^{1/2} = c\rho^{1/2}$ *for some* $c \in \mathbb{C}$.

Corollary 2.2. $V_\phi(A) = \left\langle \phi, |A|^2 \phi\right\rangle - |\langle \phi, A\phi\rangle|^2$ *and* $V_\phi(A) = 0$ *iff* $A\phi = c\phi$ *for some* $c \in \mathbb{C}$*; that is* ϕ *is an eigenvector of* A *with eigenvalue* c*.*

For a context $\mathcal{A} = \{\phi_i\}$, $A \in \mathcal{L}(H)$ is *measurable* with respect to $\mathcal{A}$ if $AP_{\phi_i} = P_{\phi_i}A$ for all i. Then $A\phi_i = E_{\phi_i}(A)\phi_i$ for all i. The only operators accurately described by $\mathcal{A}$ are operators measurable with respect to $\mathcal{A}$. We define the *context coefficient* of A with respect to $\mathcal{A}$ by

$$c_{\mathcal{A}}(A) = \left[\sum V_{\phi_i}(A)\right]^{1/2}$$

By Corollary 2.2, $c_{\mathcal{A}}(A) = 0$ if A is measurable with respect to $\mathcal{A}$. We consider $c_{\mathcal{A}}(A)$ as an indicator of how close A is to being measurable with respect to $\mathcal{A}$. Notice that A is normal ($AA^* = A^*A$) iff $c_{\mathcal{A}}(A) = 0$ for some context $\mathcal{A}$.

For $A\in\mathcal{L}(H)$ and context $\mathcal{A}=\{\phi_i\}$ we can write

$$A=\sum_i\langle\phi_i,A\phi_i\rangle|\phi_i\rangle\langle\phi_i|+\sum_{i\neq j}\langle\phi_i,A\phi_j\rangle|\phi_i\rangle\langle\phi_j|$$

We define the linear maps $L_{\mathcal{A}},R_{\mathcal{A}}\colon\mathcal{L}(H)\to\mathcal{L}(H)$ by

$$L_{\mathcal{A}}(A)=\sum_i\langle\phi_i,A\phi_i\rangle|\phi_i\rangle\langle\phi_i|$$

$$R_{\mathcal{A}}(A)=\sum_{i\neq j}\langle\phi_i,A\phi_j\rangle|\phi_i\rangle\langle\phi_j|$$

Then $A=L_{\mathcal{A}}(A)+R_{\mathcal{A}}(A)$ and we call $L_{\mathcal{A}}$ the *context map* and $R_{\mathcal{A}}$ the *residual map*. Notice that $L_{\mathcal{A}}$ preserves self-adjointness, positivity and states. In fact, $L_{\mathcal{A}}$ is a completely positive map and is a special case of a quantum channel. Also $L_{\mathcal{A}}(A)$ is measurable with respect to $\mathcal{A}$ iff $L_{\mathcal{A}}(A)=A$ or equivalently $R_A(\mathcal{A})=0$.

Theorem 2.3. *For any $A\in\mathcal{L}(H)$ and context $\mathcal{A}$, $||R_{\mathcal{A}}(A)||=c_{\mathcal{A}}(A)$.*

Thus, $c_{\mathcal{A}}(A)=||A-L_{\mathcal{A}}(A)||$ so $c_{\mathcal{A}}(A)$ is a measure of the closeness of A to $L_{\mathcal{A}}(A)$.

3 Quantum Entanglement

We incorporate Section 1 and Section 2 to develop a general theory of quantum entanglement. We restrict attention to bipartite systems. Let H_1, H_2 be finite-dimensional complex Hilbert spaces and let $H = H_1 \otimes H_2$. A state $\rho \in \mathcal{S}(H)$ is *factorized* if there exist $\rho_1 \in \mathcal{S}(H_1)$, $\rho_2 \in \mathcal{S}(H_2)$ with $\rho = \rho_1 \otimes \rho_2$. $\rho \in \mathcal{S}(H)$ is *separable* if $\rho = \sum \lambda_i \rho_i \otimes \sigma_i$, $\lambda_i \geq 0$, $\sum \lambda_i = 1$, $\rho_i \in \mathcal{S}(H_1)$, $\sigma_i \in \mathcal{S}(H_2)$. If $\rho \in \mathcal{S}(H)$ is not separable, ρ is *entangled*. A vector state $\psi \in H$ is *factorized* if there exist vectors $\phi_1 \in H_1$, $\phi_2 \in H_2$ with $\psi = \phi_1 \otimes \phi_2$. If ψ is not factorized, ψ is *entangled*.

Lemma 3.1. *If $|\psi\rangle\langle\psi| \in \mathcal{S}(H)$ is a pure state, the following statements are equivalent* (a) $|\psi\rangle\langle\psi|$ *is factorized.* (b) $|\psi\rangle\langle\psi|$ *is separable.* (c) ψ *is factorized.*

Let $\mathcal{A} = \{\phi_i\}$, $\mathcal{B} = \{\psi_i\}$ be contexts for H_1, H_2 with $\dim H_1 = \dim H_2 = n$ and let

$$M_n = \{\underline{\lambda} \in M \colon \operatorname{supp}(\underline{\lambda}) \subseteq \{1, 2, \ldots, n\}\}$$

If $\underline{\lambda} \in M_n$ we call $(\underline{\lambda}, \mathcal{A}, \mathcal{B})$ an *entanglement* and $(M_n, \mathcal{A}, \mathcal{B})$ an *entanglement system.*

Corresponding to $E = (\underline{\lambda}, \mathcal{A}, \mathcal{B})$ we have a vector state

$$\psi_E = \sum \sqrt{\lambda_i}\, \phi_i \otimes \psi_i \in H_1 \otimes H_2$$

a pure state $P_E = P_{\psi_e}$, a separable state

$$\rho_E = \sum \lambda_i P_{\phi_i \otimes \psi_i} = \sum \lambda_i P_{\phi_i} \otimes P_{\psi_i}$$

and an *entanglement operator*

$$B_E = \sum_{i \neq j} \sqrt{\lambda_i \lambda_j}\, |\phi_i\rangle\langle\phi_j| \otimes |\psi_i\rangle\langle\psi_j|$$

From Section 1, since $\underline{\lambda} \in M$ we have defined $e(\underline{\lambda})$. We use this to define the *entanglement numbers*

$$e(\psi_E) = e(P_E) = e(\underline{\lambda})$$

Conversely, if $\psi \in H_1 \otimes H_2$ is a vector state, then there exists a *Schmidt decomposition* $(\underline{\lambda}, \mathcal{A}, \mathcal{B})$ where $\underline{\lambda} \in M_n$ is unique and $\psi = \sum \sqrt{\lambda_i}\, \phi_i \otimes \psi_i$. Thus, ψ determines an entanglement $E = (\underline{\lambda}, \mathcal{A}, \mathcal{B})$ such that $\psi = \psi_E$ although $\mathcal{A}$, $\mathcal{B}$ are not unique. We then have

$$P_E = |\psi_E\rangle\langle\psi_E| = \rho_E + B_E$$

We consider ρ_E as the separable part and B_E as describing the entangled part of P_E. Letting

$$\mathcal{D} = \mathcal{A} \otimes \mathcal{B} = \{\phi_i \otimes \psi_j\}$$

be the corresponding context for $H = H_1 \otimes H_2$ we have that

$$P_E = L_{\mathcal{D}}(P_E) + R_{\mathcal{D}}(P_E)$$

Where $L_{\mathcal{D}}$ and $R_{\mathcal{D}}$ are the context and residual maps of Section 2. Then $||B_E|| = ||P_E - \rho_E||$ gives a measure of the entanglement of P_E. The next theorem shows that our three entanglement measures coincide.

Theorem 3.2. $C_{\mathcal{D}}(B_E) = ||B_E|| = e(\psi_E)$

Let $E=(\underline{\alpha},\mathcal{A},\mathcal{B})$, $F=(\underline{\beta},\mathcal{A},\mathcal{B})$ be entanglements belonging to the entanglement system $(M_n,\mathcal{A},\mathcal{B})$. We have the corresponding vector states $\psi_E=\sum\sqrt{\alpha_i}\,\phi_i\otimes\psi_i$, $\psi_F=\sum\sqrt{\beta_i}\,\phi_i\otimes\psi_i$. For $\lambda\in(0,1)$ we have the entanglement

$$G=\big(\lambda\underline{\alpha}+(1-\lambda)\underline{\beta},\mathcal{A},\mathcal{B}\big)$$

and the vector state

$$\psi_G=\sum\sqrt{\lambda\alpha_i+(1-\lambda)\eta_i}\,\phi_i\otimes\psi_i$$

By Theorem 1.2 we have

$$\begin{aligned} e(\psi_G)&=e\left[\lambda\underline{\alpha}+(1-\lambda)\underline{\beta}\right]\geq\lambda e(\underline{\alpha})+(1-\lambda)e(\underline{\beta})\\ &=\lambda e(\psi_E)+(1-\lambda)e(\psi_F)\end{aligned}$$

Example 3. Let $(M_3,\mathcal{A},\mathcal{B})$ be an entanglement system with $\mathcal{A}=\{\phi_i\}$, $\mathcal{B}=\{\psi_i\}$. Define the vector states

$$\begin{aligned}\alpha&=\tfrac{1}{\sqrt{2}}\,\psi_1\otimes\psi_1+\tfrac{1}{\sqrt{2}}\,\phi_2\otimes\psi_2\\ \beta&=\tfrac{1}{\sqrt{3}}\,\phi_1\otimes\psi_1+\tfrac{1}{\sqrt{3}}\,\phi_2\otimes\psi_2+\tfrac{1}{\sqrt{3}}\,\phi_3\otimes\psi_3\\ \gamma&=\tfrac{1}{\sqrt{2}}\,\phi_1\otimes\psi_1+\tfrac{1}{\sqrt{3}}\,\phi_2\otimes\psi_2+\tfrac{1}{\sqrt{6}}\,\phi_3\otimes\psi_3\\ \delta&=\tfrac{1}{\sqrt{3}}\,\phi_1\otimes\psi_1+\tfrac{1}{3}\,\phi_2\otimes\psi_2+\sqrt{\tfrac{7}{9}}\,\phi_3\otimes\psi_3\end{aligned}$$

As in Example 1, $e(\alpha)=1/\sqrt{2}$, $e(\beta)=\sqrt{2/3}$, $e(\gamma)=\sqrt{11/81}$, $e(\delta)=\sqrt{30}/9$ and we have

$$e(\delta)<e(\alpha)<e(\gamma)<e(\beta)\qquad\square$$

Until now we considered the entanglement number for a pure state P_ϕ. We next discussed mixed states. If ρ is not pure, then ρ has an uncountable number of decompositions $\rho = \sum \lambda_i P_i$, $\lambda_i > 0$, $\sum \lambda_i = 1$ where P_i are pure states. Also ρ has a *spectral decomposition* in which the P_i are mutually orthogonal pure states.

Example 4. Let $H = \mathbb{C}^2 \otimes \mathbb{C}^2$, $\{\phi_1, \phi_2\}$ a context for $\mathbb{C}^2$ and define $\phi = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2)$. Consider the separable state

$$\rho = \tfrac{1}{2}\left(|\phi \otimes \phi\rangle\langle\phi \otimes \phi| + |\phi_1 \otimes \phi_1\rangle\langle\phi_1 \otimes \phi_1|\right)$$

The nonzero eigenvalues of ρ are $1/4$ and $3/4$ with eigenvectors

$$\psi_1 = \frac{1}{2\sqrt{3}}\left[(3\phi_1 + \phi_2) \otimes \phi_1 + (\phi_1 + \phi_2) \otimes \phi_2\right]$$
$$\psi_2 = \tfrac{1}{2}\left[(\phi_2 - \phi_1) \otimes \phi_1 + (\phi_1 + \phi_2) \otimes \phi_2\right]$$

The unique spectral decomposition of ρ is

$$\rho = \tfrac{1}{4} P_{\psi_1} + \tfrac{3}{4} P_{\psi_2}$$

It is easy to check that ψ_1 and ψ_2 are entangled. □

Example 4 shows that a spectral decomposition cannot be used to determine an entanglement number for a mixed state. Indeed, since ρ is separable, its entanglement number should be zero, yet $e(P_{\psi_1}), e(P_{\psi_2}) > 0$.

We now define the entanglement number for a mixed state ρ. Suppose $\rho = \sum \lambda_i P_i$, $\lambda_i > 0$, $\sum \lambda_i = 1$, $P_i \neq P_j$, $i \neq j$ is a decomposition of ρ into pure states P_i. Let $\mathcal{A} = \{P_i\}$ and define $e_{\mathcal{A}}(\rho) = \sum \lambda_i e(P_i)$. We define the *entanglement number* $e(\rho)$ by

$$e(\rho) = \inf_{\mathcal{A}} \left[e_{\mathcal{A}}(\rho) \right] \tag{1}$$

Since a pure state has the unique decomposition $P = P$, (1) reduces to the usual definition of entanglement number for pure states. We say that the infimum in (1) is *attained* if there exists an $\mathcal{A}$ such that $e(\rho) = e_{\mathcal{A}}(\rho)$.

Theorem 3.3. *The infimum $e(\rho)$ is attained for some $\mathcal{A}$.*

Theorem 3.4. *A state ρ is separable iff $e(\rho) = 0$.*

Theorem 3.5. *$e(\psi)$ is continuous in the norm topology of H.*

It is an open problem whether $e(\rho)$ is continuous in the operator topology of H. We conjecture that this is true.