Subspace constraints for joint measurability

Jukka Kiukas

Aberystwyth University

Mathematical Foundations of Quantum Mechanics
in memoriam Paul Busch
York, June 2019

Starting point: connection to Paul's work

- General topic - joint measurements of unsharp observables (as explained to some extent in Pekka's talk and [1])
- Noise bounds for joint measurability [2]
- Geometric structure of quantum effects [3] [4]

- Comment: most of Paul's work on operational quantum mechanics is directly relevant for the study of quantum information and correlations — regardless of whether one agrees with (or cares about) the philosophy of unsharp reality and individual state interpretation.

[1] J. Kiukas, P. Lahti, J.-P. Pellonpaa, K. Ylinen, FOOP special issue 2019
[2] P. Busch, T. Heinosaari, J. Schultz, N. Stevens, 2013
[3] P. Busch, S.P. Gudder, LMP 1999
[4] P. Busch, H.-J. Schmidt, Quant. Inf. Proc. 2010

Outline

- Basic idea and some motivation
  - Context: joint measurability (compatibility) / steering under quantum noise[5] [6]
  - Problem: incompatibility vs subspace compatibility $\simeq$ quantum coherence?
  - Coherence understood on the spatial level (as opposed to noncommutativity)
  - Added motivation: steering in strongly correlated spin networks
- Method: subspace constraints for positivity
  - Strength of an effect along a ray[7] - Schur complements and complementarity[8]
  - Positivity constraints in term of the strength function
  - Coherent extension of subspace observables
- Coherent extension of subspace models for joint measurability
  - General idea
  - One systematic method (works in some cases)
  - Application: loss of incompatibility due to decoherence + subspace noise

[5] J. Kiukas, C. Budroni, R. Uola, J.-P. Pellonpaa, PRA 2017

[6] T. Heinosaari, J. Kiukas, D. Reitzner, JPA 2017

[7] P. Busch, S.P. Gudder, LMP 1999

[8] J. Kiukas, P. Lahti, J.-P. Pellonpaa, K. Ylinen, FOOP special issue 2019

Motivation: joint measurability and EPR-steering

- Fix a quantum state σ and outcome space Ω. A *(σ-consistent) ensemble* is a family of states $(\sigma_\omega)_{\omega\in\Omega}$ plus a probability measure μ such that $\int_\Omega \sigma_\omega d\mu(\omega) = \sigma$.
- An *assemblage* is a set $\{(\sigma_{\omega|x}, \mu_x)\}_x$ of ensembles. It is *non-steerable*[9] if there is an ensemble $((\rho_g)_{g\in\mathcal{G}}, \mu)$ and probability densities $\omega \mapsto D_x(\omega, g)$ w.r.t. μ_x, such that

$$\sigma_{\omega|x} = \int D_x(\omega, g)\, \rho_g\, \mu(dg), \text{ for each } \omega, x \qquad \text{(trace class Bochner integral)}$$

- Ensemble-measurement duality[10] $\int_U \sigma_\omega d\mu(\omega) = \sigma^{\frac{1}{2}} F(U) \sigma^{\frac{1}{2}}$
  - 1-1 between ensembles (σ_ω, μ) and observables (normalised POVMs) F
  - Operational meaning through Bayes theorem (inversion of conditional probabilities)
  - Works in separable Hilbert spaces due to Radon-Nikodym property of the trace class

$\rightsquigarrow$ A set of observables $\{E_x\}$ is *jointly measurable (compatible)* iff the assemblage $\{(\sigma_{\omega|x}, \mu_x)\}_x$ given by $\int_U \sigma_{\omega|x} d\mu_x(\omega) = \sigma^{\frac{1}{2}} F(U) \sigma^{\frac{1}{2}}$ and $\mu_x(\cdot) = \mathrm{tr}[E_x(\cdot)\sigma]$ is non-steerable[11] [12].

- The ensemble ρ_g of *hidden states* corresponds to the *joint observable* G via $\int_Z \rho_g\, d\mu(g) = \sigma^{\frac{1}{2}} G(Z) \sigma^{\frac{1}{2}}$, and is called the *(classical) model* for $\{E_x\}$.

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[9] H.M. Wiseman, S.J. Jones, A.C. Doherty, PRL 2007

[10] used independently of steering e.g. in Dall'Arno, D'Ariano, Sacchi, PRA 2011

[11] Uola, Moroder, Guhne, PRL 2014; Quintino, Vertesi, Brunner, PRL 113

[12] Uola, Budroni, Guhne, Pellonpaa, PRL 2015; Kiukas, Budroni, Uola, Pellonpaa, PRA 2017

Motivation: subspace models for joint measurability / steering

Let $\mathcal{H}_0$ be a (closed) subspace of a Hilbert space $\mathcal{H}$ with inclusion $V_0 : \mathcal{H}_0 \to \mathcal{H}$.

Definition

(a) A set $\mathcal{M}$ of observables on $\mathcal{H}$ are said to be *subspace compatible* (w.r.t $\mathcal{H}_0$) if the set of subspace observables $\{X \mapsto V_0^* F(X) V_0 \mid F \in \mathcal{M}\}$ is compatible.

(b) Their joint observable G is then called a *subspace model* for $\mathcal{M}$.

- Obviously, $\mathcal{M}$ is subspace compatible if it is compatible.

Problem (Extension of subspace models)

Find constraints under which a given subspace model extends to a joint observable in the full space.

Motivation:

- Subspace models are easier to find (smaller dimension)
- Could be implemented iteratively to solve the full problem
  - Probably not very efficient in general.
  - Could be useful in cases where the subspace split is natural / where models are expected to have specific structure.
- Two obvious applications: decoherence and subspace noise

Motivation: subspace models and (de)coherence

- I only look at the simplest case: $\mathcal{H}_0$ has codimension one (so $\dim \mathcal{H}_0 = 1$).
- The "block" form is useful: any $H \in \mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}_0^\perp \oplus \mathcal{H}_0)$ can be written as

$$H = \begin{pmatrix} p & \langle\psi| \\ |\psi\rangle & F \end{pmatrix}$$

- $\psi \in \mathcal{H}_0$ describes coherence between the subspaces.

Definition

A set $\mathcal{M}$ of observables is called *incompatible due to coherence* (w.r.t. $\mathcal{H}_0$) if it is incompatible but subspace compatible.

Problem

If $\mathcal{M}$ is incompatible due to coherence, how much decoherence is needed to break the incompatibility[a] *of $\mathcal{M}$?*

[a] *Incompatibility breaking channels (IBC)* introduced in [Heinosaari, Kiukas, Reitzner, JPA 2015]

Motivation: subspace models and (de)coherence

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If $\mathcal{M}$ is incompatible due to coherence, how much decoherence is needed to break the incompatibility[a] of $\mathcal{M}$?

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Formally:

- Def: A *decoherence semigroup* (w.r.t. $\mathcal{H}_0$) is a (continuous) semigroup $[0,1] \mapsto \Lambda_r$ of quantum channels such that $\Lambda_1 = \mathrm{id}$ and $(\mathbb{I} - P_0)\Lambda_0(H)P_0 = 0$ for all $H \in \mathcal{B}(\mathcal{H})$.
- Def: The subspace $\mathcal{H}_0$ is *decoherence-free* w.r.t a channel Λ if $\mathcal{B}(\mathcal{H}_0)$ is included in the fixed point space of Λ.
- If Λ_r is a decoherence semigroup w.r.t. a decoherence-free subspace $\mathcal{H}_0$, and $\mathcal{M}$ is incompatible due to coherence w.r.t $\mathcal{H}_0$, then

  $$r_0 := \sup\{r \in [0,1] \mid \Lambda_r(\mathcal{M}) \text{ compatible}\}$$

  is the minimal amount of decoherence needed to break incompatibility.
- Basic examples are Hadamard (Schur) multiplication channels - simplest case:

$$\Lambda_r\left(\begin{pmatrix} p & \langle\psi| \\ |\psi\rangle & F \end{pmatrix}\right) := \begin{pmatrix} p & r\langle\psi| \\ r|\psi\rangle & F \end{pmatrix}$$

Motivation: subspace models and (de)coherence

- A decoherence semigroup which also effects *subspace noise* is given by the amplitude damping channels

$$\Lambda_r\left(\begin{pmatrix} p & \langle\psi| \\ |\psi\rangle & F \end{pmatrix}\right) = \begin{pmatrix} p & r\langle\psi| \\ r|\psi\rangle & r^2F + (1-r^2)p\mathbb{I} \end{pmatrix}.$$

- Subspace incompatibility is lost first (due to mixing with a trivial observable), while overall incompatibility is lost (possibly later) due to decoherence.
- Two critical parameter values:

$$r_c := \sup\{r \in [0,1] \mid \Lambda_r(\mathcal{M}) \text{ is compatible}\}$$
$$r_{sc} := \sup\{r \in [0,1] \mid \Lambda_r(\mathcal{M}) \text{ is subspace compatible}\}$$

- $r_c \leq r_{sc}$.

Problem

Given a set $\mathcal{M}$ of incompatible observables, how large is the gap between r_c and r_{sc}?

- Note: qubit amplitude damping is entanglement-breaking iff $r = 0$, with (diamond norm) distance from nearest EBC at least $r^2/2$ [13] - the IBC problem is nontrivial.

[13] F. Leditzky, E. Kaur, N. Datta, M. M. Wilde PRA 97, 012332 (2018)

Extra motivation for qubit amplitude damping: steering in spin networks

- Strongly interacting network with N qubits and Hamiltonian H.
  - Assume that the total spin commutes with H so each K-excitation sector is invariant.
  - Restrict to the subspace of (at most) one excitation.
  - Alice has access to spin A of the chain, Bob has another (distant) spin B.
- State and entanglement transfer in such systems studied a lot [14] [15] [16]

Problem

Is this entanglement good enough for quantum steering across the chain?

- Excitation transfer is described by qubit amplitude damping [17]

$\rightsquigarrow$ The steering problem reduces to the IBC problem for the amplitude damping channel.

The IBC problem for amplitude damping is motivated by "practical" applications.

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[14] M. Christandl, N. Datta, A. Ekert, A. J. Landahl, PRL 92, 187902 (2004)

[15] T. J. Osborne, N. Linden, PRA 69, 052315 (2004)

[16] M. B. Plenio, F. L. Semiao, New J. Phys. 7 73 2005

[17] S. Bose, PRL 91, 207901 (2003)

Outline

Basic idea and motivation

- Context: joint measurability (compatibility) / steering under quantum noise[18] [19]
- Problem: incompatibility vs subspace compatibility $\simeq$ quantum coherence?
- Coherence understood on the spatial level (as opposed to noncommutativity)
- Added motivation: steering in strongly correlated spin networks

Method: subspace constraints for positivity

- Strength of an effect along a ray[20] - Schur complements and complementarity[21]
- Positivity constraints in term of the strength function
- Coherent extension of subspace observables

Coherent extension of subspace models for joint measurability

- General idea
- One systematic method (works in some cases)
- Application: loss of incompatibility due to decoherence + subspace noise

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[18] J. Kiukas, C. Budroni, R. Uola, J.-P. Pellonpaa, PRA 2017

[19] T. Heinosaari, J. Kiukas, D. Reitzner, JPA 2017

[20] P. Busch, S.P. Gudder, LMP 1999

[21] J. Kiukas, P. Lahti, J.-P. Pellonpaa, K. Ylinen, FOOP special issue 2019

Strength of an effect along a ray[22]

- For any effect E and $\psi \in \mathcal{H}$ define the strength of E along ψ by

$$\lambda(E,\psi) = \sup\{\lambda \geq 0 \mid \lambda|\psi\rangle\langle\psi| \leq E\}$$

Theorem

(a) $\lambda(E,\psi) > 0$ *iff* $\psi \in \mathrm{ran} E^{\frac{1}{2}}$, *in which case* $\lambda(E,\psi) = \|E^{-\frac{1}{2}}\psi\|^{-2}$.

(b) *For each* $p \in (0,1]$ *and* $\psi \in \mathcal{H}_0$, *the set*

$$\mathcal{F}_{p,\psi} := \{E \geq 0 \mid \lambda(E,\psi) \geq 1/p\}$$

is a convex cone.

- Note: if $\psi \in \mathrm{ran} E \subset \mathrm{ran} E^{\frac{1}{2}}$ then $\lambda(E,\psi) = \langle\psi|E^{-1}\psi\rangle^{-1}$.

Proposition (Two effects along the same ray)

For two effects E, F, *the following are equivalent:*

(i) $\min\{\lambda(E,\psi), \lambda(F,\psi)\} = 0$ *for all* $\psi \in \mathcal{H}$.

(ii) $\mathrm{ran}\, E^{\frac{1}{2}} \cap \mathrm{ran} F^{\frac{1}{2}} = \{0\}$.

(iii) *there is no* $\psi \in \mathcal{H}$ *such that* $|\psi\rangle\langle\psi| \leq E$ *and* $|\psi\rangle\langle\psi| \leq F$

(iv) E *and* F *are complementary*[a]

[a] As in Pekka's talk - see the FOOP special issue paper by Kiukas, Lahti, Pellonpaa, Ylinen

[22] P. Busch, S. P. Gudder, Lett. Math. Phys. 47 329 (1999)

Another perspective - subspace constraints for positivity

Let $\mathcal{H}_0 \subset \mathcal{H}$ be a subspace with codimension 1. Any selfadjoint $H \in \mathcal{B}(\mathcal{H})$ can be decomposed along the direct sum $\mathcal{H}_0^\perp \oplus \mathcal{H}_0$:

$$H = \begin{pmatrix} p & \langle\psi| \\ |\psi\rangle & F \end{pmatrix}, \quad p \in \mathbb{R},\ \psi \in \mathcal{H}_0,\ F \in \mathcal{B}(\mathcal{H}_0) \text{ selfadjoint.}$$

If $d := \dim \mathcal{H}_0 < \infty$ we define the Schur complements:

$$H/p := F - p^{-1}|\psi\rangle\langle\psi| \ \text{(for } p > 0), \qquad H/F := p - \langle\psi|F^{-1}\psi\rangle \ \text{(for } F > 0).$$

Observation (The strength function and Schur complements)

$H/F = p - \lambda(F,\psi)^{-1}$ *(and this works also for $d = \infty$).*

I also use the "determinant" $\mathfrak{M}(H) = (H/p)\,p = pF - |\psi\rangle\langle\psi|$.

Proposition (Subspace constraints for positivity)

The following are equivalent[a]*:*

(i) $H \geq 0$

(ii) $p \geq 0$ *and* $\mathfrak{M}(H) \geq 0$

(iii) $F \geq 0$ *and* $\lambda(F,\psi) \geq p^{-1}$

[a] Standard for matrices, general case: [Paulsen, Completely bounded maps and operator algebras]

Subspace constraints on states and effects

- Any state on $\mathcal{H}$ is of the form

$$\rho = \begin{pmatrix} q & \langle\varphi| \\ |\varphi\rangle & (1-q)\rho_0 \end{pmatrix}, \quad q \in [0,1],\ \rho_0 \text{ a state on } \mathcal{H}_0,\ \lambda(\rho_0,\varphi) \geq q^{-1} - 1.$$

- The vector φ describes the *coherence* of the state (w.r.t $\mathcal{H}_0$).
- Any effect on $\mathcal{H}$ is of the form

$$H = \begin{pmatrix} p & \langle\psi| \\ |\psi\rangle & F \end{pmatrix}, \quad p \in [0,1], \psi \in \mathcal{H}_0,\ F \in \mathcal{F}_{p,\psi},\ \mathbb{I} - F \in \mathcal{F}_{1-p,\psi}.$$

- Recall: $\mathcal{F}_{p,\psi} := \{E \geq 0 \mid \lambda(E,\psi) \geq 1/p\}$
- We call H *coherent* if $\psi \neq 0$, and $\|\psi\|^2$ the *coherence* of H. Note that ψ extracts the coherences in states operationally via

$$2\mathrm{Re}\langle\psi|\varphi\rangle = \mathrm{tr}[\rho\, H] - qp - (1-q)\mathrm{tr}[\rho_0 F]$$

Extension of subspace effects

For each $p \in \mathbb{R}$, $\psi \in \mathcal{H}_0$ define an extension map $I_{p,\psi} : \mathcal{B}(\mathcal{H}_0) \to \mathcal{B}(\mathcal{H})$ via

$$I_{p,\psi}(F) = \begin{pmatrix} p & \langle\psi| \\ |\psi\rangle & F \end{pmatrix}.$$

- If $p > 0$, $I_{p,\psi}$ is positivity preserving precisely on the cone $\mathcal{F}_{p,\psi} := \{E \geq 0 \mid \lambda(E,\psi) \geq 1/p\}$.
- $\mathbb{I} - I_{p,\psi}(F) = I_{1-p,-\psi}(\mathbb{I} - F)$.

Proposition (Extension of a single effect)

Let F be an effect on the subspace $\mathcal{H}_0$. Given $\psi \in \mathcal{H}_0$, there exists a $0 < p < 1$ such that $I_{p,\psi}(F)$ is an effect, iff

$$\lambda(F(\mathbb{I} - F), \psi) \geq 1 \quad \text{or, equivalently} \quad \|[F(\mathbb{I} - F)]^{-\frac{1}{2}}\psi\|^2 \leq 1.$$

Joint extensions and complementarity

- Consider a pair of effects E, F on the subspace $\mathcal{H}_0$.
- Definition (as in Pekka's talk): effects E, F are *complementary* if there is no effect $A \neq 0$ such that $A \leq E$ and $A \leq F$.
- E, F are complementary iff $\mathrm{ran} E^{\frac{1}{2}} \cap \mathrm{ran} F^{\frac{1}{2}} = \{0\}$

Proposition (Joint coherent positive extensions)

The following are equivalent:

(i) *There is no coherent extension $I_{p,\psi}$ such that $I_{p,\psi}(E) \geq 0$ and $I_{p,\psi}(F) \geq 0$.*

(ii) *E and F are complementary.*

- $\sup_{\psi \in \mathcal{H}_0} \min\{\lambda(E,\psi), \lambda(F,\psi)\} = \inf\{p \mid \psi \in \mathcal{H}_0,\, I_{p,\psi}(E) \geq 0,\, I_{p,\psi}(F) \geq 0\}$ "quantifies" deviation from complementarity.

Proposition (Joint coherent effect extensions)

The following are equivalent:

(i) *There is no coherent extension $I_{p,\psi}$ such that $I_{p,\psi}(E)$ and $I_{p,\psi}(F)$ are effects.*

(ii) *The effects $E(\mathbb{I} - E)$ and $F(\mathbb{I} - F)$ are complementary.*

Explicit coordinate form of effect extensions for $d < \infty$

- For each unit vector $\phi \in \mathcal{H}_0$, define

$$\sigma_\phi^1 := \begin{pmatrix} 0 & \langle\phi| \\ |\phi\rangle & 0 \end{pmatrix} \qquad \sigma_\phi^2 := \begin{pmatrix} 0 & -i\langle\phi| \\ i|\phi\rangle & 0 \end{pmatrix}, \qquad \sigma_\phi^3 := \begin{pmatrix} 1 & 0 \\ 0 & -|\phi\rangle\langle\phi| \end{pmatrix}$$

- Let F be any rank r subspace effect with eigendecomposition $F = \sum_{i=1}^r \lambda_i |\phi_i\rangle\langle\phi_i|$.
- Given any $\psi \in \mathrm{ran}F$ and $p \in [0,1]$, define $x_0 = (p + \mathrm{tr}[F])/(1+r)$, $x_3^i = (x_0 - \lambda_i)$, and write $\psi = \sum_i (x_1^i + i x_2^i)\phi_i$. The coherent effect extension is

$$I_{p,\psi}(F) = \begin{pmatrix} p & \langle\psi| \\ |\psi\rangle & F \end{pmatrix} = x_0 \mathbb{I} + \sum_{i=1}^r \sum_{k=1}^3 x_k^i \sigma_{\phi_i}^k.$$

- The strength function / Schur complement has an explicit form:

$$I_{p,\psi}(F)/F = p - \lambda(F,\psi)^{-1} = x_0 + \sum_{i=1}^r \frac{x_0 x_3^i - [x_1^i]^2 - [x_2^i]^2 - [x_3^i]^2}{x_0 - x_3^i}.$$

- Note: in the qubit case ($d = r = 1$) this reduces to $(x_0 - x_3)^{-1}(x_0^2 - x_1^2 - x_2^2 - x_3^2)$ (Minkowski distance divided by $x_0 - x_3$)

Problem

Can we generalise the qubit effect compatibility characterisation given in [a]?

[a] P. Busch, H.-J. Schmidt, Quant. Inf. Proc. 9 143 (2010)

Maximally coherent effect extensions

Consider the extension map

$$I_{p,\psi}(F) = \begin{pmatrix} p & \langle\psi| \\ |\psi\rangle & F \end{pmatrix}.$$

Proposition (Maximally coherent effect extensions)

(a) *The coherence of an effect extension $I_{p,\psi}(F)$ is at most $\|\psi\|^2 = f_0(1-f_0)$, where f_0 is the point in the spectrum of F closest to $1/2$.*

(b) *The maximum is attained (approximately) when ψ is a corresponding (approximate) eigenvector, and $p = 1 - f_0$.*

(c) *If $I_{p,\psi}(F)$ is maximally coherent effect extension of F then $\lambda(F,\psi) = p^{-1}$, $\lambda(\mathbb{I} - F, \psi) = (1-p)^{-1}$, and $\mathrm{rank} I_{p,\psi}(F) = \mathrm{rank}(F)$.*

(d) *A maximally coherent effect extension of F is a projection iff either F or $\mathbb{I} - F$ has rank one.*

(e) *F has a coherent effect extension iff F is not itself a projection.*

Extension of subspace observables

- Fix an outcome set Ω with σ-algebra $\mathcal{A}$. The Hilbert space $\mathcal{H}$ is separable.
- Task: find constraints for extending an observable $F : \mathcal{A} \to \mathcal{B}(\mathcal{H}_0)$ into an observable on the full space.
- Take any probability measure $\mu : \mathcal{A} \to \mathcal{B}(\mathcal{H}_0)$ and a measurable function $\Psi : \Omega \to \mathcal{H}_0$ such that $\int_\Omega \Psi(\omega) d\mu(\omega) = 0$.
- Define the μ-continuous vector measure $\Psi(X) = \int_X \Psi(\omega) d\mu(\omega)$ and the set of extensible subspace observables

$$\mathcal{M}_{\mu,\Psi} = \{F : \mathcal{A} \to \mathcal{B}(\mathcal{H}_0) \mid F(X) \in \mathcal{F}_{\mu(X),\Psi(X)} \text{ for all } X \in \mathcal{A}\}.$$

- For each $F \in \mathcal{M}_{\mu,\Psi}$ define the extension $I_{\mu,\Psi}$ through

$$[I_{\mu,\Psi}(F)](X) = I_{\mu(X),\psi(X)}(F(X)) = \begin{pmatrix} \mu(X) & \langle\Psi(X)| \\ |\Psi(X)\rangle & F(X) \end{pmatrix}, \quad X \in \mathcal{A}.$$

- Note: every observable $H : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ arises in this way from some subspace observable [23].

[23] μ-continuity of Ψ is due to the Radon-Nikodym property of $\mathcal{H}$ and the positivity constraint.

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[24] J. Kiukas, C. Budroni, R. Uola, J.-P. Pellonpaa, PRA 2017

[25] T. Heinosaari, J. Kiukas, D. Reitzner, JPA 2017

[26] P. Busch, S.P. Gudder, LMP 1999

[27] J. Kiukas, P. Lahti, J.-P. Pellonpaa, K. Ylinen, FOOP special issue 2019

Extension of subspace models - idea and a simple method

- Fix two extension maps $I_{\mu,\Psi}$ and $I_{\mu',\Psi'}$.
- Assuming F, F' are compatible subspace observables with outcome sets Ω, Ω', and a joint observable G (on product space), we look for a joint observable for the extensions $I_{\mu,\Psi}(F)$ and $I_{\mu',\Psi'}(F')$.

Definition

The observable $\tilde{G} := I_{\mu\times\mu',\Psi\times\mu'+\mu\times\Psi'}(G)$ is called the *coherence-additive model* for $I_{\mu,\Psi}(E)$ and $I_{\mu',\Psi'}(F)$.

- Note: in the discrete case, e.g. $\mu(\{i\}) = p_i$, $\Psi(\{i\}) = \psi_i p_i$, and

$$\tilde{G}_{ij} = \begin{pmatrix} p_i p'_j & p_i p'_j \langle \psi_i + \psi'_j | \\ p_i p'_j | \psi_i + \psi'_j \rangle & G_{ij} \end{pmatrix}$$

- It has the correct marginals: e.g. using $\Psi'(\Omega') = 0$ we get

$$(\Psi \times \mu' + \mu \times \Psi')(X \times \Omega') = \Psi(X)\mu'(\Omega') + \mu(X)\Psi'(\Omega') = \Psi(X).$$

$\rightsquigarrow$ The model is valid exactly when all positivity constraints are fulfilled:

$$\lambda\left(G(Z), (\Psi \times \mu' + \mu \times \Psi')(Z)\right) \geq (\mu\times\mu')(Z)^{-1} \qquad \text{(whenever } Z \text{ is not } \mu \times \mu'\text{-null).}$$

Extension of subspace models - idea and a simple method

Definition

$\tilde{G} := I_{\mu\times\mu',\Psi\times\mu'+\mu\times\Psi'}(G)$ is the *coherence-additive model* for $I_{\mu,\Psi}(E)$ and $I_{\mu',\Psi'}(F)$.

- The model is valid exactly when all positivity constraints are fulfilled:

$$\lambda\left(G(Z), (\Psi\times\mu'+\mu\times\Psi')(Z)\right) \geq (\mu\times\mu')(Z)^{-1} \quad \text{for all } Z$$

or, equivalently, $\mathfrak{M}\left(I_{\mu\times\mu',\Psi\times\mu'+\mu\times\Psi'}(G)(Z)\right) \geq 0$ for all Z .

Proposition (Example of the coherence-additive model)

Define the operator measure G on $\mathcal{H}_0$ by

$$G(X\times X') = \mu'(X')F(X) + \mu(X)F'(X') - \mu(X)\mu(X')\mathbb{I} + |\Psi(X)\rangle\langle\Psi'(X')| + |\Psi'(X')\rangle\langle\Psi(X)|.$$

For each X, X' of nonzero μ, μ'-measure, define the positive operator

$$\mathfrak{L}^{\mu,\mu'}_{\Psi,\Psi'}[F,F'](X,X') := \frac{\mathfrak{M}(I_{\mu,\Psi}(F)(X))}{\mu(X)^2} + \frac{\mathfrak{M}(I_{\mu',\Psi'}(F')(X'))}{\mu'(X')^2}.$$

If $\mathfrak{L}^{\mu,\mu'}_{\Psi,\Psi'}[F,F'](X,X') \geq \mathbb{I}$ for all X, X', then G is a joint subspace observable for F, F', and the coherence-additive model is a joint observable for $I_{\mu,\Psi}(F)$ and $I_{\mu',\Psi'}(F')$.

Application: decoherence + subspace smearing

- The *amplitude damping channel* is given for each $r \in [0,1]$ by

$$\Lambda_r\left(\begin{pmatrix} p & |\psi\rangle \\ |\psi\rangle & F \end{pmatrix}\right) = \begin{pmatrix} p & \langle r\psi| \\ |r\psi\rangle & r^2 F + (1-r^2)p\mathbb{I} \end{pmatrix}.$$

- It has $d+1$ Kraus operators

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & r\mathbb{I} \end{pmatrix}, \qquad K_i = \begin{pmatrix} 0 & \sqrt{1-r^2}\langle\phi_i| \\ 0 & 0 \end{pmatrix},$$

  where $\{\phi_i\}$ is any basis of $\mathcal{H}_0$.
- It forms a semigroup [28] $\Lambda_r \Lambda_{r'} = \Lambda_{rr'}$ for all $r, r' \in [0,1]$.
- It forms a decoherence semigroup since $\Lambda_1 = \mathrm{Id}$ and $\Lambda_0(H) = \mathrm{tr}[(\mathbb{I} - P_0)H]\mathbb{I}$.
- Hence[29], for any set $\mathcal{M}$ of observables there is a unique critical point r_c such that Λ_r breaks the incompatibility of $\mathcal{M}$ iff $r \leq r_c$.

[28] qubit case mentioned e.g. in V. Giovannetti, R. Fazio PRA 71, 032314 (2005)

[29] T. Heinosaari, J. Kiukas, D. Reitzner, JPA 2015

Application: decoherence + subspace smearing

- The amplitude damping channel is given for each $r \in [0,1]$ by

$$\Lambda_r\left(\begin{pmatrix} p & \langle\psi| \\ |\psi\rangle & F \end{pmatrix}\right) = \begin{pmatrix} p & r\langle\psi| \\ r|\psi\rangle & r^2F + (1-r^2)p\mathbb{I} \end{pmatrix}.$$

- Let F be a subspace observable, μ a probability measure (with same outcomes), and

$$F_\lambda^\mu(X) := \lambda F(X) + (1-\lambda)\mu(X)\mathbb{I}_0, \text{ for each } \lambda \in [0,1],$$

their usual mixture of F with a trivial observable.

Observation

Suppose that $I_{\mu,\Psi}(F)$ is a coherent extension of F (with some choice of Ψ). Then

$$\Lambda_r(I_{\mu,\Psi}(F)) = I_{\mu,r\Psi}(F^\mu_{r^2}).$$

Explicitly:

$$\Lambda_r\left(\begin{pmatrix} \mu(X) & \langle\Psi(X)| \\ |\Psi(X)\rangle & F(X) \end{pmatrix}\right) = \begin{pmatrix} \mu(X) & r\langle\Psi(X)| \\ r|\Psi(X)\rangle & F^\mu_{r^2}(X) \end{pmatrix}.$$

Hence, amplitude damping is a coherent extension of trivial subspace noise.

Application: decoherence + subspace smearing

- The amplitude damping channel is a coherent extension of subspace noise:

$$\Lambda_r(I^{\mu}_{\Psi}[F])(X) = \Lambda_r\left(\begin{pmatrix} \mu(X) & \langle\Psi(X)| \\ |\Psi(X)\rangle & F(X) \end{pmatrix}\right) = \begin{pmatrix} \mu(X) & r\langle\Psi(X)| \\ r|\Psi(X)\rangle & F^{\mu}_{r^2}(X) \end{pmatrix}.$$

- For any F, F', the mixtures $F^{\mu}_{r^2}$ and $(F')^{\mu'}_{r^2}$ are compatible for $r^2 \leq \frac{1}{2}$ [30].
- Can the extensions $\Lambda_r(I^{\mu}_{\Psi}[F])$ and $\Lambda_r(I^{\mu}_{\Psi}[F]')$ be incompatible in this case?

Proposition (Coherence-additive model for amplitude damping[a])

[a] This generalises the qubit version [J. Kiukas, C. Budroni, R. Uola, J.-P. Pellonpaa, PRA 2017]

$\Lambda_r(I^{\mu}_{\Psi}[F])$ *and* $\Lambda_r(I^{\mu}_{\Psi}[F'])$ *have a coherence-additive model for*

$$r^2 \geq \left(2 - \inf_{X,X} \inf \operatorname{spectrum}\left(\mathfrak{L}^{\mu,\mu'}_{\Psi,\Psi'}[F,F'](X,X')\right)\right)^{-1}.$$

Proof.

We have $\mathfrak{M}(\Lambda_r(I^{\mu}_{\Psi}[F](X))) = r^2\mathfrak{M}(I^{\mu}_{\Psi}[F](X)) + (1-r^2)\mu(X)^2\mathbb{I}$ and hence

$$\mathfrak{L}^{\mu,\mu'}_{r\Psi,r\Psi'}[F^{\mu}_{r^2},(F')^{\mu'}_{r^2}](X,X') = r^2\mathfrak{L}^{\mu,\mu'}_{\Psi,\Psi'}[F,F'](X,X') + 2(1-r^2)\mathbb{I}.$$

By the previous Prop. the coherence-additive model works when this is $\geq \mathbb{I}$. □

[30] P. Busch, T. Heinosaari, J. Schultz, N. Stevens, 2013

Application: 2-IBC problem for amplitude damping

Definition

A channel is Λ is *2-IBC*[a] if $\{\Lambda(F), \Lambda(F')\}$ is compatible for any pair of observables F, F'.

[a] T. Heinosaari, J. Kiukas, D. Reitzner, JPA 2015

Proposition (Prev. slide)

$\Lambda_r(I^\mu_\Psi[F])$ *and* $\Lambda_r(I^\mu_\Psi[F'])$ *have a coherence-additive model for*

$$r^2 \geq \left(2 - \inf_{X,X} \inf \operatorname{spectrum}\left(\mathfrak{L}^{\mu,\mu'}_{\Psi,\Psi'}[F,F'](X,X')\right)\right)^{-1}.$$

Corollary

Λ_r *is 2-IBC if and only if* $0 \leq r^2 \leq \frac{1}{2}$.

Proof.

By the previous Proposition, Λ_r is 2-IBC for all $r^2 \leq \frac{1}{2}$. For the converse, take any unit vector ϕ. Then $\Lambda_r(\frac{1}{2}(\mathbb{I} + \sigma^k_\phi)) = \frac{1}{2}(\mathbb{I} + r\sigma^k_\phi) + R_r$ for $k = 1, 2$, where R_r is supported in $\mathbb{I} - |\phi\rangle\langle\phi|$, so they are incompatible for $r^2 \geq \frac{1}{2}$ by Paul's unbiased qubit criterion[a]. □

[a] P. Busch, Phys. Rev. D 33, 2253 (1986)

Summary

- I studied the difference between *compatibility* and *subspace compatibility* for subspaces of co-dimension one.
- Effects decomposed in the block form

$$\begin{pmatrix} p & \langle\psi| \\ |\psi\rangle & F \end{pmatrix}$$

- This is an effect iff F is an effect and $\lambda(F,\psi) \geq p^{-1}$, $\lambda(\mathbb{I}-F,\psi) \geq (1-p)^{-1}$, where

$$\lambda(E,\psi) = \sup\{\lambda \geq 0 \mid \lambda|\psi\rangle\langle\psi| \leq E\}$$

  is the *strength of* E *along* ψ

- I discussed one simple extensible subspace model - works for amplitude damping (decoherence + trivial subspace noise)
- Qubit case also motivated by steering in spin networks
- Some open questions:
  - Generalisations of compatibility criteria for pairs of qubit effects
  - Precise connections between quantum coherence and incompatibility
  - Iterative search for joint measurements?