Uncertainty and complementarity

Two key notions in Paul's studies at the Heart of Quantum Mechanics

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The problem of approximate joint measurement of complementary observables and the relevance of the uncertainty relations to that question were at the heart of Paul's investigations into the foundations of quantum mechanics.

I have been lucky to follow and participate much of that work since the early 1981. I try to survey some steps taken in that research starting with Paul's first three papers on the subject matter and reaching its height in recent years.

Much of our common work, with many collaborators, is summarised in our last book *Quantum Measurement*, Springer 2016, coauthored by Juha-Pekka Pellonpää and Kari Ylinen.

Part of my talk is based on the paper "Complementary Observables in Quantum Mechanics", FOOP, *Paul Busch: At the Heart of Quantum Mechanics*, referred as [KLPY].

Intuitive ideas followed and a shared problem

- Bohr (1928,1935) and Pauli (1933): Position and momentum are complementary observables in the sense that all the experimental arrangements allowing their unambiguous operational definitions and measurements are mutually exclusive. Such observables cannot be defined and measured together.
- Heisenberg (1927): Complementary observables, like position and momentum, can be defined and measured jointly if sufficient ambiguities are allowed in their definitions. For the necessary defining ambiguities or measurement inaccuracies $\delta q, \delta p$ for position and momentum Heisenberg gave his (in?)famous relation $\delta q \cdot \delta p \sim h$. [Cp. Werner and Farrelly, FOOP 2019]
- **The problem**: How to express and possibly confirm/reject these intuitive ideas in quantum mechanics?

Paul's doctoral thesis: PUR *vs.* MUR

[**P2**] "Unbestimmtheitsrelation und simultane Messungen in der Quantentheorie", Cologne 1982.

- A first systematic attempt to *distinguish between* preparation (*statistical*) and measurement (*individualistic*) uncertainty relations, the latter being discussed in terms of fuzzy position and fuzzy momentum observables (as introduced by Ali, Emch, Prugovecki 1974-1977) and exemplified through an elaboration and extension of the Arthurs-Kelly model (1965) for an approximate joint measurement of position and momentum.

Complementarity

[**P4**] P^2: "On various joint measurements of position and momentum observables in quantum theory", PRD 29 (1984); many further elaborations, the latest one in [KLPY].

- Complementarity can be expressed in several alternative ways in terms of measurement outcome probabilities, observables, instruments, or measurement schemes.
- We follow(ed) a formulation of *complementarity* of observables as a *lack of joint tests* (expressed here directly on the level of effects constituting the observables).

. . . as lack of joint tests

- For any two effects $E, F \in \mathcal{E}(\mathcal{H})$, the yes-outcome of a yes-no measurement of a (nontrivial) dichotomic observable $\{0, A, I - A, I\}$, with $A \leq E, A \leq F$, gives probabilistic information on both E and F. Such a measurement is a *joint test* of E and F.

- For given observables $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and $\mathsf{F} : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ the lack of joint tests may vary between the two extremes:

  – $\mathsf{E}(X) \wedge \mathsf{F}(Y) = 0$ for all X, Y, $\mathsf{E}(X) \neq I \neq \mathsf{F}(Y)$

  – $\mathrm{l.b.}\{\mathsf{E}(X), \mathsf{F}(Y)\} \neq \{0\}$ for all X, Y, $\mathsf{E}(X) \neq 0 \neq \mathsf{F}(Y)$.

Where to put complementarity?

- For an operational definition of an observable $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ it suffices to know the effects $\mathsf{E}(X)$ for a semiring $\mathcal{R}$ which *generates* $\mathcal{A}$ and *covers* Ω in the sense of countable (disjoint) union.

- Observables E and F are *complementary* if $\mathsf{E}(X) \wedge \mathsf{F}(Y) = 0$ at least for some generating and covering semirings $\mathcal{R} \subset \mathcal{A}$ and $\mathcal{S} \subset \mathcal{B}$ (such that $\mathsf{E}(X) \neq I \neq \mathsf{F}(Y)$).

  [Corollary]: Complementary observables have no joint measurements.

Elementary facts

- For any two $E, F \in \mathcal{E}(\mathcal{H})$,

$$
\begin{aligned}
\text{l.b.}\{E,F\} &= \{A \in \mathcal{E}(\mathcal{H}) \,|\, A \leq E, A \leq F\} \text{ (typically no g.l.b.)} \\
&= \{A \in \mathcal{E}(\mathcal{H}) \,|\, A \leq E \wedge_{\mathcal{E}(\mathcal{H})} F\} \text{ (one projection)} \\
&= \{A \in \mathcal{E}(\mathcal{H}) \,|\, A \leq E \wedge_{\mathcal{E}(\mathcal{H})} F = E \wedge_{\mathcal{P}(\mathcal{H})} F\} \text{ (projections).}
\end{aligned}
$$

- For any $P \in \mathcal{P}(\mathcal{H})$, $P = P^2$, and $P(\mathcal{H}) = \mathrm{ran}(P) = \overline{\mathrm{ran}(P)}$.

- For any $E \in \mathcal{E}(\mathcal{H})$, $E^2 \leq E = \sqrt{E}\sqrt{E}$, and $\mathrm{ran}(E) \subset \overline{\mathrm{ran}(E)}$. Recall that $\overline{\mathrm{ran}(E)} = (\ker E)^{\perp} \equiv \mathcal{H}_E$ is the support space of E with the projection P_E so that $E \leq P_E$.

The prototypical example: Q and P

- Complementarity of Q and P follows from their Fourier equivalence which implies

$$\mathsf{Q}(X) \wedge \mathsf{P}(Y) = \mathsf{Q}(X) \wedge \mathsf{P}(\mathbb{R} \setminus Y) = \mathsf{Q}(\mathbb{R} \setminus X) \wedge \mathsf{P}(Y) = 0,$$

  for all bounded (finite measure) $X, Y \in \mathcal{B}(\mathbb{R})$, independently on the size $\ell(X), \ell(Y)$ of the sets.

- Many more examples of complementary observables are discussed in [KLPY], to mention number and canonical phase, any two (rotated) quadratures as well as the triples $(\mathsf{Q}, \mathsf{P}, \mathsf{H})$, where H is an energy observable, with the operator $H = \frac{1}{2m}P^2 + V(Q)$, where V is any function such that H has a purely discrete spectrum.

Uncertainty as a relaxation of complementarity?

Jauch theorem: $\forall\, X, Y$, if $\ell(X)\ell(Y) < \infty$,

then $\mathrm{l.b.}\{\mathsf{Q}(X), \mathsf{P}(Y)\} = \{0\}$, i.e. $\mathrm{ran}(\mathsf{Q}(X)) \cap \mathrm{ran}(\mathsf{P}(Y)) = \{0\}$.

- To break the complementarity of Q and P, fuzzy position and momentum $\mathsf{Q}_\mu \equiv \mu * \mathsf{Q}, \mathsf{P}_\nu \equiv \nu * \mathsf{P}$ were introduced:

$$\mathsf{Q}_\mu(X) = (\mu * \chi_X)(Q),\ Q = \int q\, d\mathsf{Q}(q),$$

$$\mathsf{P}_\nu(Y) = (\nu * \chi_Y)(P),\ P = \int p\, d\mathsf{P}(p),$$

$$d\mu(q) = |\varphi(q)|^2\, dq,\ d\nu(p) = |\psi(p)|^2\, dp,$$

  with an (intuitive) idea that $\Delta(\mu), \Delta(\nu)$ describe measurement inaccuracies.

Problem: Under which conditions $\mathrm{l.b.}\{\mathsf{Q}_\mu(X), \mathsf{P}_\nu(Y)\} \neq \{0\}$ and what is the role of $\Delta(\mu), \Delta(\nu)$ in that?

- By the time it was also known (Ali, Progovecki, Davies) that if $\psi = \hat{\varphi}$ then Q_μ and P_ν can be obtained as the margins of a phase space observable G^T, $T = |\varphi\rangle\langle\varphi|$,

$$\mathsf{G}^T(X \times Y) = \frac{1}{2\pi} \int_{X \times Y} W_{q,p} T W^*_{q,p} \, dqdp,$$

$$0 \neq G^T(X \times Y) \in \mathrm{l.b.}\{\mathsf{Q}_\mu(X), \mathsf{P}_\nu(Y)\}.$$

Clearly, $\Delta(\mu)\Delta(\nu) \geq \frac{1}{2}\hbar$.

Generalized Jauch theorem

[P3] "On joint lower bounds of position and momentum observables in quantum mechanics", JMP 25 (1984).

[P3] Theorem $(\widetilde{J})$. For any X, Y,

$$1.b. \{\mathsf{Q}_\mu(X), \mathsf{P}_\nu(Y)\} \neq \{0\} \Leftrightarrow \operatorname{ran}(\sqrt{\mathsf{Q}_\mu(X)}) \cap \operatorname{ran}(\sqrt{\mathsf{P}_\nu(Y)}) \neq \{0\}.$$

Since

$$\mathsf{Q}_\mu(X) \leq \mathsf{Q}(\operatorname{supp}(\mu * \chi_X)), \quad \mathsf{P}_\nu(Y) \leq \mathsf{P}(\operatorname{supp}(\nu * \chi_Y)),$$

$$\mathsf{Q}(\operatorname{supp}(\mu * \chi_X)) \wedge \mathsf{P}(\operatorname{supp}(\nu * \chi_Y)) = 0 \Longrightarrow \mathsf{Q}_\mu(X) \wedge \mathsf{P}_\nu(Y) = 0.$$

Question raised: Does the converse implication hold?

Further elaborations

[**P49**]: Paul with Stan, "Effects as Functions on Projective Hilbert Space", LMP 47 (1999) and [KLPY].

[**P49**, Theorem 3]: $\varphi \in \mathrm{ran}\sqrt{E} \Longleftrightarrow \exists \lambda > 0 \curvearrowright \lambda |\varphi\rangle\langle\varphi| \leq E$.

[Corollary]: For any two effects $E, F \in \mathcal{E}(\mathcal{H})$,

$$\mathrm{l.b.}\{E, F\} \neq \{0\} \Longleftrightarrow \mathrm{ran}(\sqrt{E}) \cap \mathrm{ran}(\sqrt{F}) \neq \{0\}.$$

[KLPY, Lemma 4]: Assume that the effect E is of the form $E = \int h\, d\mathsf{A}$ for some spectral measure $\mathsf{A} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ and a Borel function $h : \mathbb{R} \to [0, 1]$. Then $P_E \leq \mathsf{A}(\mathrm{supp}(h))$, but equality does not hold in general.

[Corollary]: For any two effects $E, F \in \mathcal{E}(\mathcal{H})$ of the above form $E = \int h\, d\mathsf{A}, F = \int k\, d\mathsf{B}$,

$$\begin{aligned}\mathrm{l.b.}\{E, F\} &\subseteq \{D \in \mathcal{E}(\mathcal{H}) \mid D \leq P_E \wedge P_F\} \\ &\subseteq \{D \in \mathcal{E}(\mathcal{H}) \mid D \leq \mathsf{A}(\mathrm{supp}(h)) \wedge \mathsf{B}(\mathrm{supp}(k))\}.\end{aligned}$$

All inclusions can be proper.

Things learned in between

"a reader is not necessarily interested in the accidental learning process of the authors" Araki to BCL in 1993.

- Any (phase space) observable M which is covariant under the phase space translations is of the form G^T, for some $T \geq 0, \mathrm{tr}[T] = 1$ [Holevo 1979, Werner 1984; revised proofs Cassinelli *et al* 2003, Kiukas *et al* 2006].
- Any observable that shares the symmetry properties of Q is of the form Q_μ for some probability measure μ, and similarly for P, P_ν [Carmeli, Heinonen (Heinosaari), Toigo 2004].
- Any Q_μ and P_ν are jointly measurable if and only if they can be obtained as the margins of a (covariant) G^T [CHT2005].

GJT revisited

For any pair $(\mathsf{Q}_\mu, \mathsf{P}_\nu)$:

$$\mathrm{l.b.}\{\mathsf{Q}_\mu(X), \mathsf{P}_\nu(Y)\} \neq \{0\} \ \Leftrightarrow \mathrm{ran}\left(\sqrt{\mathsf{Q}_\mu(X)}\right) \cap \mathrm{ran}\left(\sqrt{\mathsf{P}\nu(Y)}\right) \neq \{0\}.$$

[**P3**]: Since, for instance, $\mathrm{supp}(\mu * \chi_X) \subset \overline{\mathrm{supp}(\mu) + \overline{X}}$, Q_μ and P_ν remain complementary if the inaccuracy measures μ and ν have bounded supports, *independently of the size of* $\Delta(\mu)$ *and* $\Delta(\nu)$.

[**P3**]: For any bounded X, Y there are μ, ν *with arbitrary small* $\Delta(\mu)\Delta(\nu)$ such that $\mathrm{l.b.}\{\mathsf{Q}_\mu(X), \mathsf{P}_\nu(Y)\} \neq \{0\}$.

[KLPY, Proposition 13, answering Paul's question]: For any bounded intervals $X,\ Y \subset \mathbb{R}$ with lengths d_X, d_Y satisfying $d_X d_Y \leq \pi/2$, there exist probability measures μ, ν with finite variances, such that $\mathsf{Q}_\mu(X) \wedge \mathsf{P}_\nu(Y) = 0$, but $\mathsf{Q}(\mathrm{supp}(\mu * \chi_X)) \wedge \mathsf{P}(\mathrm{supp}(\nu * \chi_Y)) \neq 0$.

MUR reneissance

- The study of the measurement uncertainty relations got a new boost in the turn of the millenia. In addition to the revised operational tools of quantum measurement theory, part of this new wave of interest was triggered by the work of Mazano Ozawa in 2002 and 2003 on the error-disturbance relations. The proposed notions of Ozawa were independently critizised in the following two papers:

[**P61**] P^2&TH: "Noise and disturbance in quantum measurement", PLA **320** (2004).
[RW2004]: "The uncertainty relation for joint measurement of position and momentum", Quantum Inf. Comput. **4** (2004).

- Common to these investigations was a search for operationally meaningful measures of measurement error, noise, and disturbance, with the idea that such notions should be built on comparing the measurement outcome distributions of the target (ideal) and the approximating (actually measured, disturbed) observables – an idea advanced already by Ludwig (1984) but missed in Ozawa's work.

- Guided by his deep physical insight and careful conceptual analysis, Paul [**P64**] was led, with David Pearson, to study various practically motivated notions of measurement error such as the *calibration error* and the *error bar width*.

  With the methods developed in [RW2004] the relevant uncertainty relations for approximate position-momentum joint measurements were also obtained.

- In later work ([**P79,82,83**], P[2]&RW) the results of [RW2004] and [**P64**] were generalized to cover a whole range of measures to compare the target observables Q and P with their compatible approximators $\mathsf{M}_1, \mathsf{M}_2$ with a result that there is now an increasing flow of papers analysing in one or another form something like a *measurement uncertainty region* for two (or more) observables E_i, with the value spaces $(\Omega_i, \mathcal{A}_i)$:

$$\mathrm{MU}(\Omega_1, \Omega_2) = \{(\Delta_1(\mathsf{M}_1, \mathsf{E}_1), \Delta_2(\mathsf{M}_2, \mathsf{E}_2)) \mid \mathsf{M} : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})\}.$$

Unstability of complementarity

[KLPY, Proposition 11]: For any effect E and for any $\lambda, p \in (0,1)$ define

$$E_{\lambda,p} = \lambda E + (1-\lambda)p\, I. \tag{1}$$

Then $\mathrm{ran}\,\sqrt{E_{\lambda,p}} = \mathcal{H}$.

Complementarity of any two observables $\mathsf{E}_1, \mathsf{E}_2$ can thus easily be broken by mixing trivial noise in one of the observables.

Any two observables E_1 and E_2 can even be made compatible by mixing them with trivial noise:

$$\begin{aligned}\widetilde{\mathsf{E}}_1 &= \lambda \mathsf{E}_1 + (1-\lambda)\mu_1\, I \\ \widetilde{\mathsf{E}}_2 &= \gamma \mathsf{E}_2 + (1-\gamma)\mu_2\, I\end{aligned}$$

choosing the weights $0 < \lambda, \gamma < 1$ appropriately, for instance, $\gamma = 1 - \lambda$. See, for instance, [P**75**] P& TH, Scultz, Stevens.

[Lemma]: For any $\widetilde{\mathsf{Q}} = \lambda \mathsf{Q} + (1-\lambda)\mu\, I$, $0 < \lambda < 1$, the error, in the sense of the Wasserstein distance $\Delta_\alpha(\widetilde{\mathsf{Q}}, \mathsf{Q})$ of order $1 \leq \alpha < \infty$, is infinite.

Summary: a poetic formulation of the problem and its solution

- *One may view the world with the p-eye and one may view it with the q-eye but if one opens both eyes simultaneously then one gets crazy.*
  Wolfgang Pauli in a letter to Werner Heisenberg, 19 October 1926.
- *We hope to have demonstrated that one can safely open a pair of complementary 'eyes' simultaneously. He who does so may even 'see more' than with one eye only. The means of observation being part of the physical world, Nature Herself protects him from seeing too much and at the same time protects Herself from being questioned too closely: quantum reality, as it emerges under physical observation, is intrinsically unsharp. It can be forced to assume sharp contours – real properties – by performing repeatable measurements. But sometimes unsharp measurements will be both, less invasive and more informative.*
  Paul *et co* in the Epilogue of OQP, 1995.

We all miss Paul