Preparation uncertainty for multi-slit interferometry

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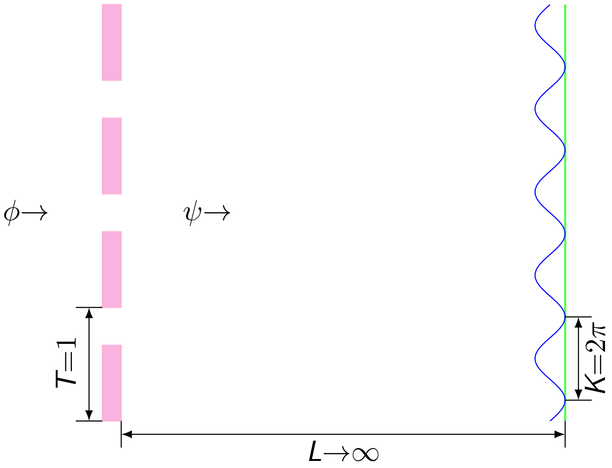
Joint work with Paul Busch and Jukka Kiukas

Setup



Background

- Modular momentum observable - Aharonov, Pendleton, Petersen (1969)
- $\left(\mathrm{P}_{MOD}\hat{\varphi}\right)(p) = (p - 2k\pi)\hat{\varphi}(p)$, where $(p - 2k\pi) \in [-\pi, \pi]$
- We seek a natural "which way" observable

Characterisation by covariance

- Idea: Complementarity for standard P / Q comes from the Fourier transform
- This generalises to locally compact Abelian groups
- Dual of a LCA group is the continuous homomorphisms $G \to \mathbb{T} \subset \mathbb{C}$
- Denoted $\widetilde{G}$
- Bidual: $\widetilde{\widetilde{G}} \cong G$ (Pontryagin)

Examples

$$\widetilde{\mathbb{R}} \cong \mathbb{R}$$

$$f_y(x) = e^{ixy}, \quad x, y \in \mathbb{R}$$

$$\widetilde{\mathbb{T}} \cong \mathbb{Z}$$

$$f_n\left(e^{i\theta}\right) = e^{ni\theta}, \quad n \in \mathbb{Z}, \theta \in [-\pi, \pi]$$

Problem

Spectrum: $\sigma\left(\mathrm{P}_{MOD}\right) = [-\pi, \pi]$ is not an LCA group

Two solutions

1. Identify points π and $-\pi$ and give the topology and group operation of $\mathbb{T} \implies$ dual is $\mathbb{Z}$
2. Consider $[-\pi, \pi]$ embedded in $\mathbb{R} \implies$ dual is $\mathbb{R}$

Covariance conditions

Circle case:

$$e^{in\mathrm{P}_{MOD}} E(k) e^{-in\mathrm{P}_{MOD}} = E(k+n), \quad k, n \in \mathbb{Z}$$

Interval case:

$$e^{ix\mathrm{P}_{MOD}} E(X) e^{-ix\mathrm{P}_{MOD}} = E(x+X), \quad x \in \mathbb{R}, X \in \mathcal{B}(\mathbb{R})$$

Uncertainty

- Can't define a mean for distributions on $\mathbb{T}$
- No mean $\implies$ no variance
- Use "metric error"[1] for a Borel measure μ on a metric space (X, d)

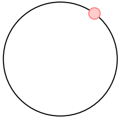
$$\delta_\alpha(\mu) = \left(\inf_{x_0} \int_X d(x, x_0)^\alpha d\mu(x) \right)^{\frac{1}{\alpha}}$$

- For $\alpha = 2$, $X = \mathbb{R}$, $d(x, y) = |x - y|$, recover standard deviation*
- For $\alpha = 2$, $X = \mathbb{T}$, $d(x, y) =$ arc length, recover Judge's circular variance[2]

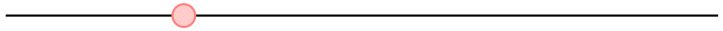
[1] P. Busch, P. Lahti, R.F. Werner, *Measurement uncertainty relations*, J. Math. Phys. 55 042111 (2014)

[2] D. Judge, *On the uncertainty relation for L_z and φ*, Phys. Lett. 5 129 (1963)

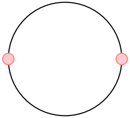
Classical examples



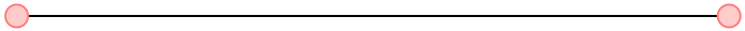
(a) $\delta = 0$



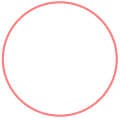
(b) $\delta = 0$



(c) $\delta = \frac{\pi}{2}$



(d) $\delta = \pi$



(e) $\delta = \frac{\pi}{\sqrt{3}}$

(f) $\delta = \frac{\pi}{\sqrt{3}}$

Quantum uncertainty

- Given an observable E and a state ρ define

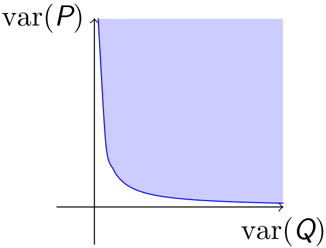
$$E^\rho : \mathcal{B}(X) \to [0,1]$$
$$E^\rho : A \mapsto \mathrm{tr}(E(A)\rho)$$

- For convenience we also define

$$\begin{aligned}\delta_\alpha(E,\rho) &= \delta_\alpha(E^\rho) \\ &= \left(\inf_{x_0} \int_X d(x,x_0)^\alpha \,\mathrm{tr}\left(E(dx)\rho\right)\right)^{\frac{1}{\alpha}}\end{aligned}$$

Goal

- Heisenberg uncertainty relation completely characterises variance preparation uncertainty between Q and P
- Want something similar for P_{MOD} and (some of) the observables obeying the covariance conditions



Diagonalising P_{MOD}

- Recall $(\mathrm{P}_{MOD}\hat{\varphi})(p) = (p - 2k\pi)\hat{\varphi}(p)$
- Set

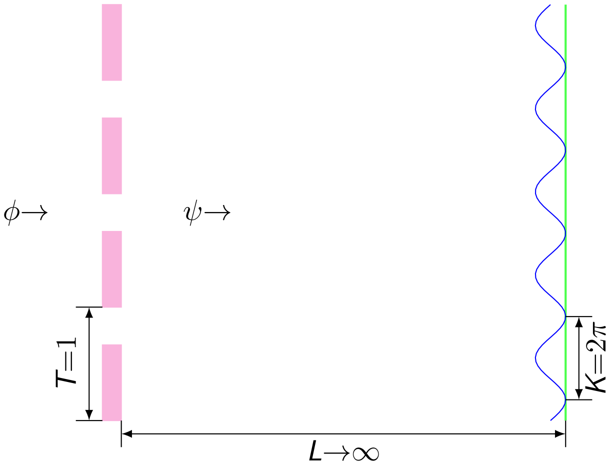
$$U : L^2(\mathbb{R}) \to L^2([-\pi, \pi], \mathcal{H}_0) \simeq L^2([-\pi, \pi]) \otimes \mathcal{H}_0$$
$$(U\varphi)(p) = \sum_{k \in \mathbb{Z}} \hat{\varphi}(p + 2k\pi)|k\rangle$$

- Then

$$U\mathrm{P}_{MOD}U^* = M_0 \otimes I,$$

where $(M_0\hat{\varphi})(p) = p\varphi(p)$, for $\varphi \in L^2([-\pi, \pi])$

Setup



Covariant path observables

- Observables obeying the covariance conditions may be characterised
- (Circle): $E(n) = V^* \left(E^{P_c}(n) \otimes I \right) V$, for $n \in \mathbb{Z}$, P_c is the usual differential operator on $\mathbb{T}$
- (Interval): $E(X) = V^* J^* \left(E^{P}(X) \otimes I \right) JV$, P is the differential operator on $\mathbb{R}$

$$(J\varphi)(p) = \begin{cases} \varphi(p), & p \in [-\pi, \pi] \\ 0, & \text{else} \end{cases}$$

- Where $V : L^2([-\pi,\pi], \mathcal{H}_0) \to L^2([-\pi,\pi], \mathcal{K}_0)$ is an isometry which commutes with $M_0 \otimes I$

Further restriction

- Lots of covariant observables
- We restrict to *variance free* observables[3]

$$N_\varphi(E) := \int_{\mathbb{R}} x^2 \langle \varphi | E(dx) \varphi \rangle - ||E[1]\varphi||^2 \geq 0$$

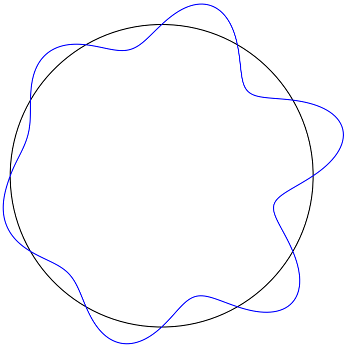
- Consider only observables such that $N_\varphi(E) \equiv 0$

[3] R. Werner, *Dilations of Symmetric Operators Shifted by a Unitary Group* J. Funct. Analysis. 92 166 (1990)

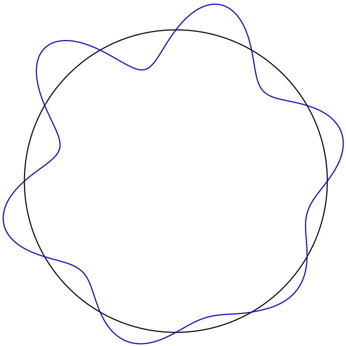
Variance free observables - circle

- $\delta(E,\varphi) = \inf_{n\in\mathbb{Z}} \left|\left|E[1]e^{in\mathrm{P}_{MOD}}\varphi\right|\right|$
- then E is a spectral measure
- we can choose $\mathcal{K}_0$, V such that V is unitary
- $E[1] = V^*(P_c \otimes I)V$ is self-adjoint
- the spectral measure of P_{MOD} is covariant under shifts generated by $E[1]$

Extra covariance - circle



(a) $\varphi(x)$



(b) $\left(e^{i\theta E[1]}\hat{\varphi}\right)(p) = \hat{\varphi}\,(p-\theta)$

Variance free observables - interval

- $\delta(E, \varphi) = \inf_{x \in \mathbb{R}} \left|\left| E[1] e^{ix\mathrm{P}_{MOD}} \varphi \right|\right|$
- There are no spectral measure solutions
- We can still choose V to be unitary to make E variance free
- There is no "extra covariance"

No extra covariance - interval

(a) $\varphi(x)$ (b) $\varphi(x-\theta)$

Metric error - circle[4]

- Domain $\mathcal{D} = \{\varphi \in L^2([-\pi,\pi]) \big|\ \text{abs. cont.}, \varphi(-\pi) = \varphi(\pi)\}$
- $P_c = -i\frac{d}{dx}$, is self-adjoint on $\mathcal{D}$ (as is P_c^2)

$$\delta(E,\varphi)^2 = \inf_{n\in\mathbb{Z}} \langle e^{inM_0\otimes I} VU\varphi | \left(P_c^2 \otimes I\right) e^{inM_0\otimes I} VU\varphi\rangle$$

$$\delta(\mathrm{P}_{MOD},\varphi)^2 = \inf_{p\in[-\pi,\pi]} \langle e^{ipP_c\otimes I} VU\varphi | \left(M_0^2 \otimes I\right) e^{ipP_c\otimes I} VU\varphi\rangle$$

- Uncertainty region convex - fully characterised by the Legendre transform

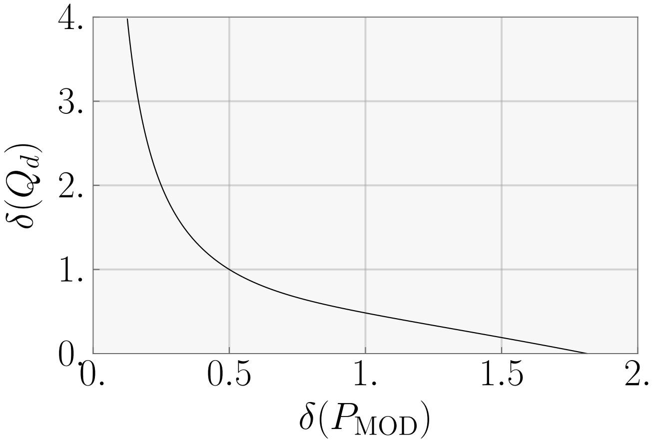
$$\delta(E,\varphi)^2 + \alpha\delta(\mathrm{P}_{MOD},\varphi) \geq \lambda(\alpha), \quad (1)$$

where $\lambda(\alpha)$ is the least eigenvalue of

$$H_\alpha = P_c^2 + \alpha M_0^2 \quad (2)$$

[4] P. Busch, J. Kiukas, R.F. Werner, *Sharp uncertainty relations for number and angle*, J. Math. Phys. (2018)

Circle diagram



Metric error - interval (1)

- Domain
  $\mathcal{D} = \left\{ \varphi \in L^2([-\pi,\pi]) \middle| \text{ abs. cont.}, \varphi(-\pi) = \varphi(\pi) = 0 \right\}$
- $P_0 = -i\frac{d}{dx}$, is closed and symmetric on $\mathcal{D}$ but not self-adjoint

$$\delta(E,\varphi)^2 = \inf_{n\in\mathbb{Z}} \langle e^{inM_0\otimes I} VU\varphi | \left(P_0^* P_0 \otimes I\right) e^{inM_0\otimes I} VU\varphi\rangle$$

$$\begin{aligned}\delta(\mathrm{P}_{MOD},\varphi)^2 &= \mathrm{var}\left(M_0\otimes I,\varphi\right)\\ &= \langle\varphi|(M_0^2\otimes I)\varphi\rangle - \langle\varphi|(M_0\otimes I)\varphi\rangle^2\end{aligned}$$

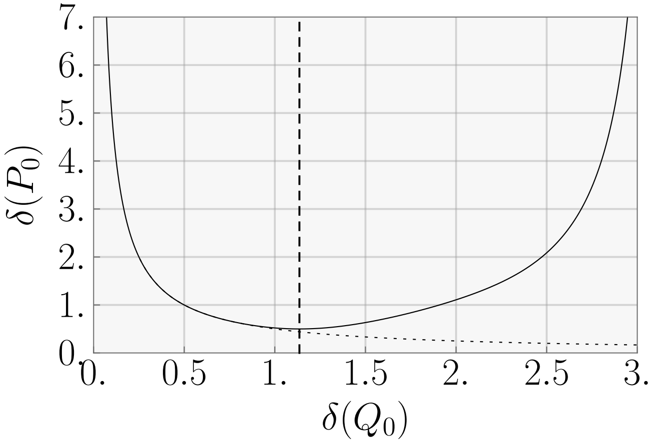
Metric error - interval (2)

- Extra assumption:
  - Fix a position uncertainty for particle on the interval
  - Consider the states with minimum "momentum" uncertainty among states with this position uncertainty
  - Assume one of these states has position expectation 0
- True if position uncertainty in $\left[\sqrt{\frac{\pi^2}{3} - 2}, \pi\right]$
- Even where this is not true the ground states ψ_α of H_α are states so

$$\{(\delta(M_0, \psi_\alpha), \delta(P_0, \psi_\alpha)|\alpha \in \mathbb{R}\} \tag{3}$$

upper bounds the boundary curve

Interval diagram



Outlook and open questions

- Left hand side of the interval diagram
- Explicit measurement scheme
- Measurement uncertainty
- Different covariance conditions (or a proof that there are none)

Thank you for your time