State-dependent Foster-Lyapunov criteria

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Outline

The general problem

Let $\Phi = \{\Phi_n, n \geq 0\}$ be a time-homogeneous Markov chain on a state space X. (Assume that Φ is phi-irreducible and aperiodic, for simplicity.)

We're interested in what can be said about the long-term behaviour of Φ. For example:

- does Φ converge to an equilibrium distribution?
- if so, in what norm does this convergence take place, and how fast?
- and how is this related to the average time spent between successive visits to certain sets?

Notation

- P^n: n-step transition kernel of Φ;
- for a non-negative function f, and measure μ:

$$P^n f(x) = \mathbb{E}_x[f(\Phi_n)]\,, \qquad \mu(f) = \int f(y)\mu(dy)\,;$$

- norm $\|\mu\|_g$:

$$\|\mu\|_g = \sup_{f:|f|\leq g} |\mu(f)|\,, \qquad \|\cdot\|_{\mathrm{TV}} \equiv \|\cdot\|_1\,;$$

- first return time to a set: $\tau_{\mathcal{A}} = \inf\{n \geq 1 \,:\, \Phi_n \in \mathcal{A}\}$;
- $\mathcal{C}$ is a *small set* if $\exists \varepsilon > 0$ and a measure ν s.t.

$$P(x,\cdot) \geq \varepsilon\nu(\cdot) \text{ for all } x \in \mathcal{C}\,.$$

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$$P(x, \cdot) \geq \varepsilon \nu(\cdot) \text{ for all } x \in \mathcal{C} .$$

Ergodicity

Φ is called *ergodic* if it has a finite invariant measure π ($\pi = \pi P$).

In this case, for any x,

$$\boxed{\|P^n(x,\cdot) - \pi\|_{\mathrm{TV}} \to 0 \text{ as } n \to \infty.} \tag{1}$$

Equivalently, we can find a small set $\mathcal{C}$ with

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But *how fast* does the convergence in (1) take place?

Definition

Φ is *geometrically ergodic* if there exists $r > 1$ with

$$\|P^n(x,\cdot) - \pi(\cdot)\|_{\mathrm{TV}} \leq M_x r^{-n}\,.$$

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Equivalently:

- there exists a *scale function* $V : \mathsf{X} \to [1, \infty)$, a small set $\mathcal{C}$, and constants $\beta \in (0, 1)$, $b < \infty$, with

$$\boxed{\mathbb{E}_x\left[V(\Phi_1)\right] = PV(x) \leq \beta V(x) + b\mathbf{1}_{\mathcal{C}}(x);}$$

- $\sup_{x\in\mathcal{C}} \mathbb{E}_x[\beta^{-\tau_{\mathcal{C}}}] < \infty$.

Drift conditions

The inequality

$$\boxed{PV(x) \leq \beta V(x) + b\mathbf{1}_{\mathcal{C}}(x);}$$

is called a *Foster-Lyapunov drift condition*.

- often easiest way of showing that Φ is geometrically ergodic;
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Subgeometric ergodicity is implied by a weaker drift condition:

$$\boxed{PV(x) \leq V(x) - \phi \circ V(x) + b\mathbf{1}_{\mathcal{C}}(x)}$$

for some concave non-negative function ϕ.

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1. When can we find a function $n : \mathsf{X} \to \mathbb{N}$ such that

$$\boxed{P^{n(x)} V(x) \leq \beta V(x) + b \mathbf{1}_{\mathcal{C}}(x)\,?} \qquad (2)$$

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2. Alternatively, if (2) holds for some n and V, what can be said about moments of the return time to $\mathcal{C}$?

3. And when is this useful?!

Establishing a subsampled drift condition

Theorem (1)

Assume that there exist a small set $\mathcal{D}$, a function $V : \mathsf{X} \to [1, \infty)$ and a continuously differentiable increasing concave function $\phi : [1, \infty) \to (0, \infty)$, such that $\sup_{\mathcal{D}} V < \infty$, $\inf_{[1,\infty)} \phi > 0$, and

$$\boxed{PV \leq V - \phi \circ V + b\mathbf{1}_{\mathcal{D}} \,.}$$

Fix $\beta \in (0, 1)$ and let $n : \mathsf{X} \to \mathbb{N}$ satisfy $n(x) \sim \frac{1}{\beta}\left(\frac{V}{\phi \circ V}\right)(x)$. Then for any $\beta < \beta' < 1$,

$$\boxed{P^{n(x)} W \leq \beta' W + b'\mathbf{1}_{\mathcal{C}} \,,}$$

where $W = \phi \circ V$.

Comments

- Theorem (1) says that we can deduce a (state-dependent) subsampled geometric drift condition from a one-step drift, but on a different scale;
- More general results (not requiring a drift condition for V) can be stated.

Examples

$$\boxed{PV \leq V - \phi \circ V + b\mathbf{1}_{\mathcal{D}}} \quad \Rightarrow \quad \boxed{P^{n(x)}W \leq \beta' W + b'\mathbf{1}_{\mathcal{C}}}$$

- **Polynomially ergodic:** if $\phi(t) \sim ct^{1-\alpha}$ for some $\alpha \in (0,1)$, then $n \sim V^{\alpha}$ and $W = V^{1-\alpha}$.

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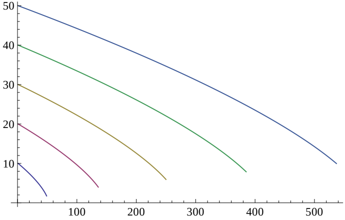
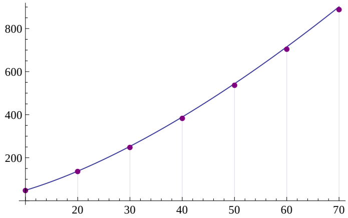
Motivation: Deterministic chain: $\Phi_{n+1} = \Phi_n - \Phi_n^{0.4}$ (so $V(x) = x$)



Decay of $W(\Phi_n) = \Phi_n^{0.4}$



$n : W(\Phi_n) \leq 0.2W(\Phi_0)$
(blue line $\propto \Phi_0^{0.6}$)

Examples

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- **Polynomially ergodic:** if $\phi(t) \sim ct^{1-\alpha}$ for some $\alpha \in (0,1)$, then $n \sim V^{\alpha}$ and $W = V^{1-\alpha}$.
- **Subgeometrically ergodic:** if $\phi(t) \sim t[\ln t]^{-\alpha}$ for some $\alpha > 0$, then $n \sim [\ln V(x)]^{\alpha}$ and $W = V[\ln V]^{-\alpha}$.

Examples

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- **Subgeometrically ergodic:** if $\phi(t) \sim t[\ln t]^{-\alpha}$ for some $\alpha > 0$, then $n \sim [\ln V(x)]^{\alpha}$ and $W = V[\ln V]^{-\alpha}$.

- **Logarithmically ergodic:** if $\phi(t) \sim [1 + \ln t]^{\alpha}$ for some $\alpha > 0$, then $n \sim \frac{V}{[1+\ln V]^{\alpha}}$ and $W = [1 + \ln V]^{\alpha}$.

Using a subsampled drift condition

Question 2

Suppose we know that

$$P^{n(x)}V(x) \leq \beta V(x) + b\mathbf{1}_{\mathcal{C}}(x)\,.$$

What can be said about moments of $\tau_{\mathcal{C}}$?

Using a subsampled drift condition

Question 2

Suppose we know that

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What can be said about moments of $\tau_{\mathcal{C}}$?

- If $n(x) = c$ then Φ is geometrically ergodic;

Using a subsampled drift condition

Question 2

Suppose we know that

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What can be said about moments of $\tau_{\mathcal{C}}$?

- If $n(x) = c$ then Φ is geometrically ergodic;
- Alternative drift condition

$$P^{n(x)}V(x) \leq \beta^{n(x)}V(x) + b\mathbf{1}_{\mathcal{C}}(x)$$

(with no relation assumed between n and V) shown by Meyn & Tweedie (1994) to also imply geometric ergodicity.

Theorem (2)

Assume that

$$P^{n(x)} V(x) \le \beta V(x) + b\mathbf{1}_{\mathcal{C}}(x)\,.$$

If there exists a strictly increasing function $R : (0,\infty) \to (0,\infty)$ *satisfying one of the following conditions*

(i) $t \mapsto R(t)/t$ *is non-increasing and* $R \circ n \le V$,

(ii) R *is a convex continuously differentiable function such that* R' *is log-concave and* $R^{-1}(V) - R^{-1}(\beta V) \ge n$,

then there exists a constant M *such that*

$$\boxed{\mathbb{E}_x[R(\tau_{\mathcal{C}})] \le M\,(V(x) + b\mathbf{1}_{\mathcal{C}}(x))\,.}$$

- If $n(x) \leq V(x)$ then taking $R(t) = t$ in (i) we obtain

$$\sup_{x \in \mathcal{C}} \mathbb{E}_x[\tau_{\mathcal{C}}] < \infty\,;$$

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- Part (i) can be applied when $V = \xi \circ n$ for some increasing concave function ξ (*i.e.* useful when $n \gg V$);

- Alternatively, if $n = \xi \circ V$ then

$$R^{-1}(t) \sim \int_1^t \frac{\xi(u)}{u}\,du$$

satisfies (ii) (*i.e.* this is useful when n/V decreasing).

Interplay of subsampling rates and moments

Geometric rates: if $n(x) = 1$, take $R(t) = \kappa^t$, with $1 \leq \kappa \leq \beta^{-1}$. Then

- R is convex and log-concave;
- $R^{-1}(V) - R^{-1}(\beta V) = (\ln \beta^{-1})/(\ln \kappa) \geq 1 = n$.

Thus (ii) shows that

$$\mathbb{E}_x\left[R(\tau_\mathcal{C})\right] = \mathbb{E}_x\left[\kappa^{\tau_\mathcal{C}}\right] \leq M\left(V(x) + b\mathbf{1}_\mathcal{C}(x)\right) ,$$

and so $\mathbb{E}_x\left[\beta^{-\tau_\mathcal{C}}\right] < \infty$.

Interplay of subsampling rates and moments

Polynomial rates: suppose $n(x) \sim V^{\frac{\alpha}{(1-\alpha)}}(x)$, for some $\alpha \in (0,1]$. Letting $R(t) \sim t^{1/\alpha - 1}$, we see that

- when $\alpha \leq 1/2$ then R satisfies (ii);
- when $\alpha \geq 1/2$ then $R(t)/t$ is non-increasing, and

$$R \circ n \sim (V^{\frac{\alpha}{(1-\alpha)}})^{1/\alpha - 1} = V\,,$$

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  and so R satisfies (i).

In either case, we obtain

$$\mathbb{E}_x\left[\tau_{\mathcal{C}}^{1/\alpha-1}\right] \leq cV(x)\,.$$

Logarithmic and **subgeometric** rates can be dealt with similarly.

Corollary (to Theorem (2))

If $\pi(V) < \infty$ *then there exists a small set* $\mathcal{D}$ *with*

$$\sup_{x \in \mathcal{D}} \mathbb{E}_x \left[\sum_{k=0}^{\tau_{\mathcal{D}}} R(k) \right] < \infty \,.$$

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Polynomial chains: if $PV \leq V - cV^{1-\alpha} + b\mathbf{1}_{\mathcal{C}}$ then (Thm (1))

$$P^{n(x)} W(x) \leq \beta' W(x) + b' \mathbf{1}_{\mathcal{D}} ,$$

where $W \sim V^{1-\alpha}$ and $n \sim V^{\alpha}$; furthermore, $\pi(W) < \infty$.

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where $W \sim V^{1-\alpha}$ and $n \sim V^{\alpha}$; furthermore, $\pi(W) < \infty$.

Using $R(t) \sim t^{1/\alpha - 1}$ in the Corollary we obtain

$$\boxed{\mathbb{E}_x \left[\tau_{\mathcal{C}}^{1/\alpha} \right] < \infty \,.}$$

Application: tame chains

Class of Markov chains introduced by SBC & Kendall (2007).

Definition

Φ is *tame* if the following two conditions hold:

(i) there exist $\delta \in (0, 1)$ and a deterministic function n satisfying $n(x) \leq W^\delta(x)$ such that

$$\mathbb{E}_x \left[W(\Phi_{n(x)}) \right] \leq \beta W(x) + b \mathbf{1}_C(x) \, ;$$

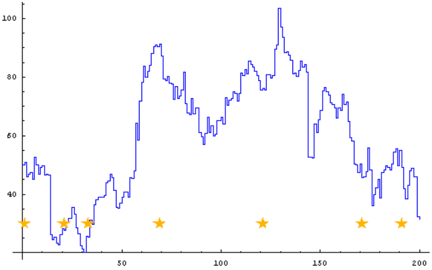
(ii) the constant δ satisfies $\ln \beta < \delta^{-1} \ln(1 - \delta)$.

i.e. Φ satisfies a subsampled geometric drift condition, where the subsampling time n is not too large.

Theorem (SBC & Kendall, 2007)

If Φ is tame then there exists a perfect simulation algorithm for Φ (using Dominated Coupling from the Past).

Idea of proof: there exists a simple dominating process for any geometrically ergodic chain (Kendall, 2004); *delay* this (using n) to produce dominating process for X.



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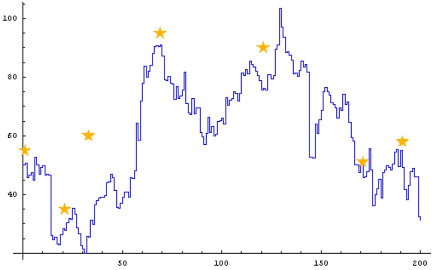

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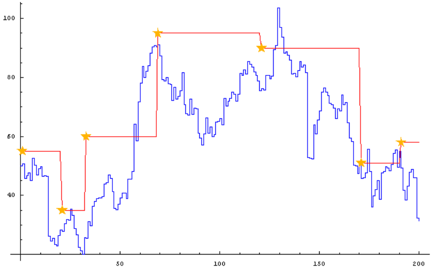


When is a chain tame?

- All geometrically ergodic chains are tame.

Proposition

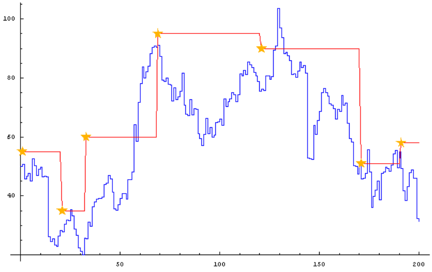
If $PV \leq V - cV^{1-\alpha} + b\mathbf{1}_{\mathcal{C}}$, with $\alpha \in (0, 1/2)$, then Φ is tame.

- Proof easy, using Theorem (1).
- Follows that chains with subgeometric drift ($\phi(t) \sim t[\ln t]^{-\alpha}$) are tame.
- Logarithmically ergodic chains ($\phi(t) \sim [1 + \ln t]^{\alpha}$) not covered by this result.

Dominating process

Can also use above theory to determine ergodic properties of the dominating process D for Φ in the perfect simulation algorithm.

- D does not satisfy a simple one-step drift condition;
- but establishing state-dependent drift is simple!
- Theorem (2) provides information about ergodic properties of D.



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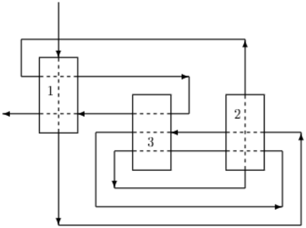
E.g. if $n(x) \sim W(x)^\gamma$ (with $\gamma \leq \delta$), then D is ergodic and converges to π_D polynomially fast (in total variation)

- but note that this isn't enough to guarantee that the mean run-time of the domCFTP algorithm is finite ...

Other applications/extensions

- Sufficient conditions for ergodicity of strong Markov processes
  - applications in queueing and network stability
  - continuum range of rates of convergence
  - explicit norm of convergence



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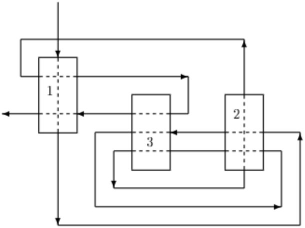

- Yüksel & Meyn (2012) use *random-time, state-dependent drift criteria* to prove stability results, but *not* convergence rates . . .

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