Shuffling Cards via One-sided Transpositions

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Joint work with Oliver Matheau-Raven and Michael Bate

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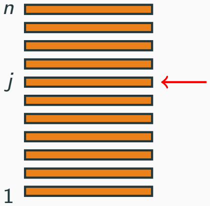

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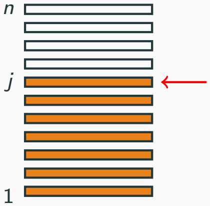

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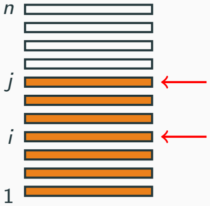

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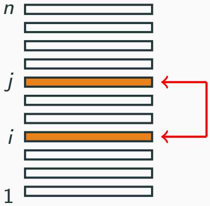
- The two chosen cards are **transposed**.



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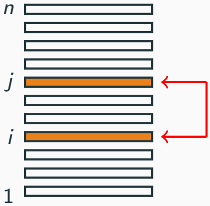
Natural question

How many shuffles does it take to "randomize" the deck?
(What is this shuffle's **mixing time**?)



Card shuffles as random walks

Most interesting card shuffles can be viewed as random walks on the symmetric group, S_n, with uniform stationary distribution π_n:

- top-to-random
- riffle shuffle
- random k-cycles
- adjacent transpositions
- semi-random transpositions (Right hand uniform, Left hand follows some other rule **independently of the Right hand**)

Methods of bounding the rate of convergence to equilibrium include:

- coupling
- strong uniform times
- eigenanalysis
- representation theory

Card shuffles as random walks

Measure distance from equilibrium using the **total variation metric**:

$$d_n(t) = \sup_{B \subset S_n} \left(P_n^t(B) - \pi_n(B) \right) = \frac{1}{2} \sum_{\sigma \in S_n} \left| P_n^t(\sigma) - \pi_n(\sigma) \right| .$$

- takes values in $[0, 1]$
- in general, will depend upon the starting state, but not if (as here) the Markov chain is transitive.

Define the ε-**mixing time** to be

$$t_n^{\text{mix}}(\varepsilon) = \min\{t \,:\, d_n(t) \leq \varepsilon\} .$$

The cutoff phenomenon

Many shuffles exhibit somewhat surprising convergence behaviour...

Definition

The sequence of shuffles generated by $(P_n)_{n\in\mathbb{N}}$ exhibits a **cutoff** at time t_n with *window* of size w_n if $w_n = o(t_n)$ and:

$$\begin{aligned} \lim_{c\to\infty} \liminf_{n\to\infty} d_n(t_n - cw_n) &= 1 \\ \lim_{c\to\infty} \limsup_{n\to\infty} d_n(t_n + cw_n) &= 0\,. \end{aligned}$$

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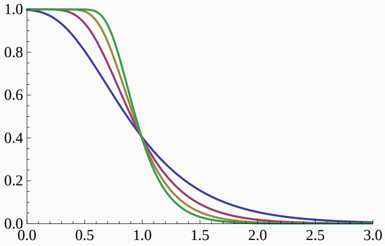
Cutoff implies that

$$t_n^{\text{mix}}(\varepsilon) \sim t_n \text{ for all } \varepsilon > 0.$$



The cutoff phenomenon

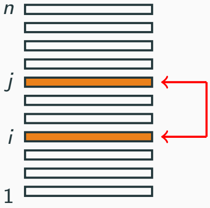
Previous results:

- top-to-random
  $\rightarrow$ cutoff at $n \log n$
- riffle shuffle
  $\rightarrow$ cutoff at $\frac{3}{2} \log_2 n$
- random k-cycles
  $\rightarrow$ cutoff at $\frac{n}{k} \log n$
- adjacent transpositions
  $\rightarrow$ cutoff at $\frac{n^2}{2\pi^2} \log n$
- semi-random transpositions (Right hand uniform, Left hand follows some other rule **independently of the Right hand**)
  $\rightarrow$ universal upper bound of $O(n \log n)$

The one-sided transposition shuffle

Our shuffle transposes cards in positions (i, j) with probability

$$P_n(i,j) = \frac{1}{jn}, \text{ for all } 1 \le i \le j \le n.$$



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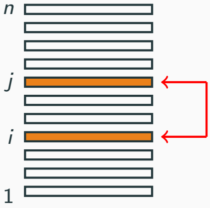
This differs significantly from previously studied shuffles which have been analysed using group representation theory:

- dependence between Left and Right hands
- generating set is entire conjugacy class of transpositions, but P_n is far from uniform on this set



Our main results

Theorem

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Diaconis & Shahshahani (1981) showed that the *standard* transposition shuffle exhibits a cutoff at time $\frac{n}{2} \log n$.

By biasing the Right hand, we can recover this result as a special case of the following:

Theorem

Suppose that the Right hand chooses card j with probability proportional to j^α. Then we see a cutoff at time t_n:

α	$(-\infty, -1)$	-1	$(-1, 1]$	$(1, \infty)$
t_n	$\zeta(-\alpha) n^{-\alpha} \log n$	$n(\log n)^2$	$\frac{1}{1+\alpha} n \log n$	$\frac{\alpha}{1+\alpha} n \log n$

Upper bound

We use the classical ℓ^2 bound on total variation distance.

Lemma

Let the eigenvalues of P_n be $1 = \beta_1 > \beta_2 \geq \cdots \geq \beta_m > -1$. Then

$$d_n(t)^2 \leq \frac{1}{4} \sum_{i \neq 1} \beta_i^{2t} \,.$$

Our analysis is inspired by work of Dieker & Saliola (2018) and Bernstein & Nestoridi (2019) on the Random-to-Random shuffle.

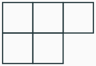
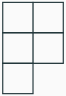
To get a handle on the eigenvalues of P_n we need to introduce the concept of **Young tableaux**.

Young tableaux

Definition

A **partition** of n is a decreasing tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $\sum_i \lambda_i = n$ and $\lambda_1 \geq \cdots \geq \lambda_l$. We denote this by $\lambda \vdash n$.

We may represent a partition using a **Young diagram**, e.g.

$(3, 2) =$  $(2, 2, 1) =$  $(5) =$ 

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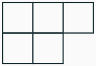
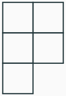
$(3,2) =$  $(2,2,1) =$  $(5) =$ 

A **standard Young tableau** (SYT) is an allocation of $1, \ldots, n$ to a Young diagram, such that rows and columns are increasing, e.g.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

The **dimension** of λ, d_λ, is the number of tableaux of shape λ.

Link to eigenvalues

Theorem

The eigenvalues of P_n are labelled by standard Young tableaux of size n, and the eigenvalue represented by a tableau of shape λ appears d_λ times.

Lemma

The eigenvalue corresponding to a tableau T is given by

$$\mathrm{eig}(T) = \frac{1}{n} \sum_{\substack{\text{boxes} \\ (i,j)\in T}} \frac{j-i+1}{T(i,j)} .$$

Example: if $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$ then $\mathrm{eig}(T) = \frac{1}{5}\left(\frac{1}{1} + \frac{2}{2} + \frac{3}{3} + \frac{0}{4} + \frac{1}{5}\right)$.

Main ideas:

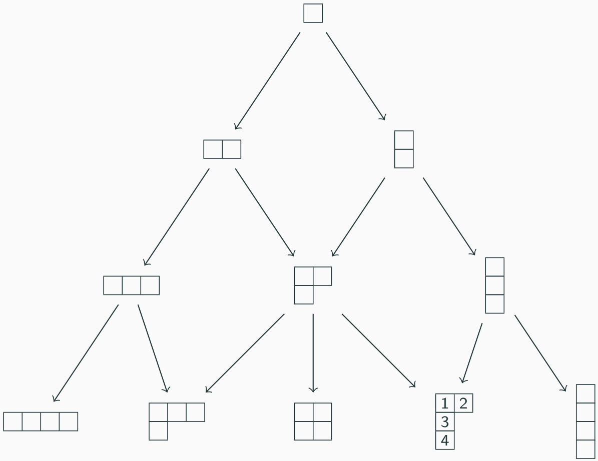
1. Natural recursive structure:
   - a deck of $(n+1)$ cards contains a deck labelled $1, \ldots, n$;
   - this corresponds to a natural embedding of S_n inside S_{n+1};
   - we can obtain Young diagrams for partitions of $(n+1)$ by adding boxes to diagrams representing partitions of n.
2. Commutation relation between the operator on n cards and the operator on $(n+1)$ cards:
   - arises when we consider the difference between

     adding a card and shuffling

     versus

     shuffling and then adding a card.

Upshot: we may **lift** the eigenvalues of P_n to obtain those of P_{n+1} by following paths through **Young's lattice**.



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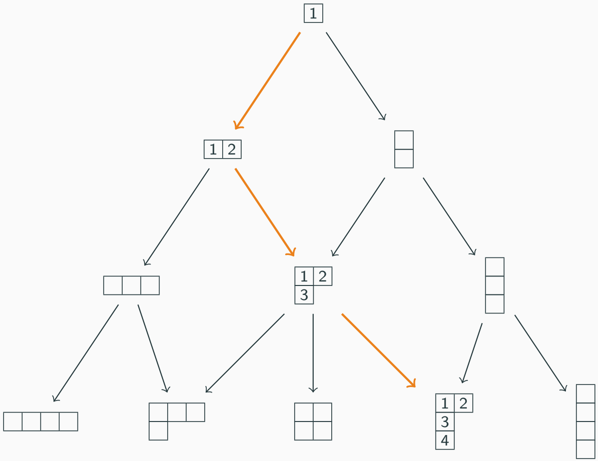

Upper bound on the mixing time

Combining these results we obtain the bound:

$$d_n(t)^2 \leq \frac{1}{4} \sum_{i \neq 1} \beta_i^{2t} = \frac{1}{4} \sum_{\substack{\lambda \vdash n \\ \lambda \neq (n)}} d_\lambda \sum_{T \in \mathsf{SYT}(\lambda)} \mathrm{eig}(T)^{2t}$$

To establish how large t must be to make this small, we need to understand how the dimensions and eigenvalues behave for large n.

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Theorem

For any $c > 0$,

$$\limsup_{n \to \infty} d_n(n \log n + cn) \leq \sqrt{2} e^{-c}.$$

Two helpful types of monotonicity.

1. Fixed λ, different tableaux.

Construct $T_\lambda^{\downarrow}$ by filling boxes of λ from top to bottom, and $T_\lambda^{\rightarrow}$ by filling boxes from left to right.

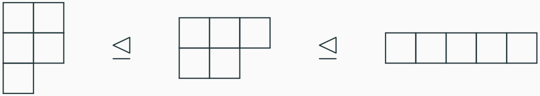
For any $T \in \mathrm{SYT}(\lambda)$,

$$\mathrm{eig}(T_\lambda^{\downarrow}) \leq \mathrm{eig}(T) \leq \mathrm{eig}(T_\lambda^{\rightarrow})$$

T	$\begin{array}{\|c\|c\|c\|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$
$\mathrm{eig}(T)$	0.503	0.523	0.57	0.59	0.64

2. Dominance ordering of partitions.

Write $\mu \trianglelefteq \lambda$ if we can form λ by moving boxes of μ up and right.



If $\mu \trianglelefteq \lambda$ then

$$\operatorname{eig}(T_\mu^{\rightarrow}) \leq \operatorname{eig}(T_\lambda^{\rightarrow}) \quad \text{and} \quad \operatorname{eig}(T_\mu^{\downarrow}) \leq \operatorname{eig}(T_\lambda^{\downarrow})$$

So it makes sense to deal with large ($\lambda_1 \geq 3n/4$) and small partitions separately, exploiting the above.

Upper bound insight: consider the partition $\lambda = (n-1, 1)$. This has dimension $(n-1)$ and there are $(n-1)$ tableaux with this shape, with the largest eigenvalue coming from the tableau

1	2	3	...	$n-2$	$n-1$
n					

The corresponding eigenvalue is $1 - \frac{1}{n}$, and so this partition makes a contribution to the upper bound of at most

$$d_\lambda \sum_{T \in \mathrm{SYT}(\lambda)} \mathrm{eig}(T)^{2t} \quad \leq \quad (n-1)^2 \left(1 - \frac{1}{n}\right)^{2t},$$

which at time $t = n \log n + cn$ is no greater than e^{-2c}.

Lower bound

Theorem

For any $c > 2$,

$$\liminf_{n\to\infty} \; d_n(n\log n - n\log\log n - cn) \;\geq\; 1 - \frac{\pi^2}{6(c-2)^2}\,.$$

Lower bound

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Sketch proof

For any set of permutations $B_n \subset S_n$,

$$d_n(t) \geq P_n^t(B_n) - \pi_n(B_n)\,.$$

Focus on cards near the top of the deck, since intuitively these should take longer to mix.

Let

$$B_n = \{\rho \in S_n : \rho \text{ has} \geq 1 \text{ fixed point in top } n/\log n \text{ cards}\}.$$

Then

- $\pi_n(B_n) \leq 1/\log n$
- $P_n^t(B_n) \geq \mathbb{P}(\text{not all top } n/\log n \text{ cards touched by time } t)$

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Now estimate how many shuffles it takes for all top $n/\log n$ cards to be touched, by coupling with a counting process.

This is similar to the standard coupon-collector problem, but:

- the Right and Left hands don't "collect" cards independently
- the counting process can increment by either one or two.

□

Final remarks

- Our analysis yields an exact formula for all of the eigenvalues of the one-sided transposition shuffle
- The results give both the cutoff time and a bound on the size of the cutoff window
- Weighting the distribution of the Right hand is possible, and shows that the fastest mixing time is obtained when Right and Left hands are independent

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