Stretching and Folding in the Configurational Quantum Cat Map

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ABSTRACT

A charged particle moving in a bounded region of the plane (with periodic boundary conditions) is subject to external periodic electromagnetic fields. Classically, they effect a hyperbolic mapping of the particle configuration space onto itself which leads to highly chaotic motion. A strong irregularity in the motion of the quantum system is discovered: the process of stretching and folding is found to occur in the system's configuration space. The formation of arbitrarily fine structures is reflected in the spectrum of the Floquet operator which turns out to be absolutely continuous.

INTRODUCTION

There are mainly two classes of works related to "quantum chaos".

On the one hand, the relation between classically chaotic Hamiltonian systems and their quantum-mechanical counterparts is investigated. Two aspects of this relation are emphasized: a) What can be learned from classical chaotic dynamics about the structure of the corresponding quantum-mechanical system? Basically, this question aims at an extension of semi-classical WKB methods to nonintegrable systems.[1)] b) Taking quantum mechanics as starting point, the inverse question reads: Are there unambiguous "precursors" of classical chaotic behaviour? Put another way: Is it possible

to "predict" - without explicitly taking the classical limit - the (regular or irregular) type of the classical motion? In this context one would like to understand how chaotic motion of the classical system "emerges" out of the quantum time-evolution.

On the other hand, one can ask whether "deterministic randomness" [2] is possible in the time evolution of the wave function. This would represent a *dynamical* source of unpredictability, fundamentally different from the probabilistic element which enters the theory through the statistical interpretation of quantum mechanics. Actually, the appearance of this phenomenon in generic classical systems has far-reaching *physical* consequences: the system's long-time behaviour becomes effectively inaccessible.

The present work belongs to the second class of problems. It will be shown that for a particular model the quantum-mechanical time-evolution is deterministically random, despite the appealing but irrelevant "linearity" of the Schrödinger equation with respect to the wave function. Consequently, in quantum mechanics again one faces the problem of extremely difficult long-time predictions.

The phase-space flow of chaotic systems is highly complicated: due to the continuous range of the canonical variables, arbitrarily fine structures on all scales are present. The formation of small structures, however, is limited in the coarse "phase space" of quantum systems. The noncommutativity of pairs of canonical variables does not allow to resolve structures smaller than h^N in $2N$-dimensional phase space. For example, the unit square, considered as the phase space of the one-dimensional quantum cat map,[3] is divided into n cells of size h. In this phase-space picture a state occupying one specific cell is the most precise information about the system one can have.

Yet, in quantum mechanics one is not restricted to phase-space considerations. Typically, the investigation of quantum systems is carried out by using a complete set of *commuting* observables. Their (possibly generalized) common eigenfunctions span the Hilbert space of the system. If these commuting observables have continuous spectra, the formation of ever finer structures in the associated basis becomes possible in principle. As an example, consider the position operators $\hat{x}_1, \hat{x}_2$ of a two-dimensional system: the wave function in the coordinate representation may develop arbitrarily small structures, if the dynamical evolution is chosen appropriately. In order to resolve the ever finer structures in configuration space, the momenta have to grow indefinitely in such a system. This was emphasized by Chirikov et al.[4] who analyzed an abstract model showing "configurational chaos".

In the next section the model to study is presented. Results which follow from the classical treatment are briefly discussed. Subsequently, the "kinematic" properties of the associated quantum system are investigated: the Floquet operator, its eigenfunctions, and the quasi-energy spectrum are given. Finally, the time evolution in the position basis is considered. It turns out that the coefficients of spatially localized states undergo in the course of time a transformation which is formally equivalent to "stretching and folding" known from classically chaotic systems.

THE CLASSICAL SYSTEM

The system under consideration[5] consists of a charged particle constrained to move in a unit square of the (x_1, x_2)-plane with periodic boundary conditions (period 1) under the influence of time-dependent electromagnetic fields. The Hamiltonian reads

$$H(\mathbf{x},\mathbf{p},t) = \frac{1}{2}\mathbf{p}^2 + \frac{1}{2}\left(\mathbf{p}\cdot\mathbf{V}\cdot\mathbf{x} + \mathbf{x}\cdot\tilde{\mathbf{V}}\cdot\mathbf{p}\right)\Delta_{T,\varepsilon}(t). \tag{1}$$

Here, $\Delta_{T,\varepsilon}(t)$ is a sequence of smooth kicks of period T, duration ε and height $1/\varepsilon$ with $\varepsilon \ll T$. $\mathbf{V}$ is a constant 2×2 matrix such that $\mathbf{C} = \exp[\mathbf{V}]$ is integer hyperbolic and has determinant 1. Comparing (1) with

$$H'(\mathbf{x},\mathbf{p},t) = \frac{1}{2}(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x},t))^2 + e\phi(\mathbf{x},t) \tag{2}$$

one can determine the vector potential $\mathbf{A}(\mathbf{x},t)$ and the scalar potential $\phi(\mathbf{x},t)$. It turns out[5] that the associated magnetic field $\mathbf{B}(\mathbf{x},t)$ is spatially uniform and directed along the x_3-axis, whereas the electric field $\mathbf{E}(\mathbf{x},t)$ has components in the (x_1,x_2)-plane only. Although it contains terms proportional to the *square* and the *derivative* of the kick function $\Delta_{T,\varepsilon}(t)$, such an electric field can in principle be realized for any *finite* kick width ε.

The important difference of this model to other kicked quantum systems[6] is the p-dependence of the kick amplitude which results in a qualitatively distinct time evolution. The bilinearity of the Hamiltonian (1) makes it possible to apply analytic methods throughout.

The classical equations of motion read

$$\begin{aligned} \dot{\mathbf{x}} &= \{\mathbf{x}, H\} = \ \mathbf{p} + \mathbf{V}\cdot\mathbf{x}\,\Delta_{T,\varepsilon}(t) \\ \dot{\mathbf{p}} &= \{\mathbf{p}, H\} = \ -\tilde{\mathbf{V}}\cdot\mathbf{p}\,\Delta_{T,\varepsilon}(t). \end{aligned} \tag{3}$$

The time evolution consists of two different elements. The particle moves freely between kicks. Coordinates and momenta directly before and after a kick, $(\mathbf{x}^-, \mathbf{p}^-)$ and $(\mathbf{x}^+, \mathbf{p}^+)$, are related to each other according to the symplectic transformation

$$\begin{aligned} \mathbf{x}^+ &= e^{\mathbf{V}}\mathbf{x}^- \equiv \mathbf{C}\cdot\mathbf{x}^- \\ \mathbf{p}^+ &= e^{-\tilde{\mathbf{V}}}\mathbf{p}^- \equiv \tilde{\mathbf{C}}^{-1}\mathbf{p}^-. \end{aligned} \tag{4}$$

In the derivation of (4), the limit $\varepsilon \to 0$ has been performed. Taking the periodic boundary conditions in the (x_1, x_2)-plane into account, it follows that the first part of (4) actually is a hyperbolic map of the unit square onto itself. Therefore the time evolution in *configuration space* enjoys all the properties which are known for such maps.[7] For example, orbits of a particle initially placed at $\mathbf{x}_0$ with zero momentum possess positive algorithmic complexity[8] for almost all $\mathbf{x}_0$. This simply follows from the fact that the subsequent particle positions in configuration space

$$\mathbf{x}((nT)^-) = (\mathbf{C}^n\cdot\mathbf{x}_0) \bmod 1 \tag{5}$$

coincide with the phase-space trajectories of a corresponding *one*-dimensional system with $q = x_1$, $p = x_2$.

The accessible phase space is unbounded for almost all trajectories with non-zero momenta. Strictly speaking, this prohibits a straightforward application of tools for the study of chaotic motion: they are designed for systems with a *bounded* energy surface.

QUANTUM KINEMATICS

The quantum mechanical description of the system is given by the same Hamiltonian (1). However, in this case $\mathbf{x}$ and $\mathbf{p}$ have to be considered as

pairs of canonically conjugate operators. They will be denoted by $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$, etc.

The time evolution of arbitrary states $|\psi\rangle$ is known if the time evolution of the elements of a basis in Hilbert space is known. Two bases prove to be particularly useful. On the one hand, there is the complete set of orthonormal (generalized) eigenvectors of the position operator

$$\hat{\mathbf{x}}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle \qquad x_1, x_2 \in [0,1) \tag{6}$$

with the property

$$|\mathbf{x}+\mathbf{z}\rangle = |\mathbf{x}\rangle \qquad z_1, z_2 \in \mathbb{Z} \tag{7}$$

because the (x_1, x_2)-plane is tiled into equivalent unit squares. On the other hand, there is an orthonormal basis associated with the momentum operator

$$\hat{\mathbf{p}}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle \equiv h\mathbf{k}|h\mathbf{k}\rangle \qquad k_1, k_2 \in \mathbb{Z}, \tag{8}$$

the discrete spectrum being a consequence of the spatial periodicity.

As a consequence of the bilinearity of the Hamiltonian (1), the equations of motion in the Heisenberg picture are formally identical with the classical equations of motion (3). Nevertheless, in order to investigate the long-time behaviour of a quantum system the time-evolution operator over one period, the Floquet operator $U(T)$, is more appropriate. Evaluating the formal expression (T is the time-ordering operator)

$$U(T) = \mathsf{T}\exp\left[-\frac{i}{\hbar}\int_T dt\hat{H}(t)\right] \tag{9}$$

yields in the limit $\varepsilon \to 0$

$$U(T) = \exp\left[-\frac{iT}{2\hbar}\hat{\mathbf{p}}^2\right]\exp\left[-\frac{i}{2\hbar}(\hat{\mathbf{x}}\cdot\tilde{\mathbf{V}}\cdot\hat{\mathbf{p}} + \hat{\mathbf{p}}\cdot\mathbf{V}\cdot\hat{\mathbf{x}})\right] \equiv U_F(T)\,U_K. \tag{10}$$

The period of integration has been chosen in such a way that the application of the kick U_K is followed by the free time-evolution according to $U_F(T)$. The transformation of the eigenstates of position and momentum under the kick is remarkably simple

$$U_K|\mathbf{x}\rangle = |(\mathbf{C}\cdot\mathbf{x}) \bmod 1\rangle \quad \text{and} \quad U_K|\mathbf{p}\rangle = |\tilde{\mathbf{C}}^{-1}\cdot\mathbf{p}\rangle. \tag{11}$$

Hence the electromagnetic fields effect a mapping of the states which is given by applying the classical kick transformation to the *labels* of the quantum states.

The eigenfunctions $|\psi\rangle$ of the time-evolution operator $U(T)$ can be found by solving the equation

$$U(T)\,|\psi\rangle = e^{-iET/\hbar}\,|\psi\rangle. \tag{12}$$

The set of all numbers E for which (12) holds is called the quasi-energy spectrum, and associated with every particular value of E there is a (possibly generalized) eigenstate.

An analysis of how the Floquet operator $U(T)$ acts on the momentum basis reveals the structure of its eigenstates. The operator U_K partitions this two-dimensional grid of states into "discrete hyperbolas" $S(\mathbf{P}) = \{|\tilde{\mathbf{C}}^s \cdot \mathbf{P}\rangle, s \in \mathbb{Z}\}$ labelled by $\mathbf{P}$, which are invariant under U_K: Application of this operator transforms the states on each $S(\mathbf{P})$ among themselves. The free time-evolution $U_F(T)$ only changes the phase of each momentum state. Therefore, superpositions of states on a single hyperbola $S(\mathbf{P})$ with appropriate phases turn out to be eigenstates of the total time evolution operator $U(T)$

$$|\mathbf{P},\alpha\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \exp[-\frac{iT}{2\hbar} f_n(\mathbf{P}) + i\alpha n]|\,\tilde{\mathbf{C}}^n \cdot \mathbf{P}\rangle \tag{13}$$

$$\text{with } f_n(\mathbf{p}) = \begin{cases} -\sum_{s=0}^{n-1} \tilde{\mathbf{p}} \cdot \mathbf{C}^s \cdot \tilde{\mathbf{C}}^s \cdot \mathbf{p} & n \geq 0 \\ \sum_{s=1}^{|n|} \tilde{\mathbf{p}} \cdot \mathbf{C}^{-s} \cdot \tilde{\mathbf{C}}^{-s} \cdot \mathbf{p} & n \leq 0 \end{cases} \tag{14}$$

where α is any real number in the interval $[0, 2\pi)$. Straightforward calculation shows that $\{|\mathbf{P},\alpha\rangle\}$ is a complete set of (generalized) orthonormal states,

$$\int_0^{2\pi} d\alpha \sum_{\mathbf{P}} |\mathbf{P},\,\alpha\rangle\langle\mathbf{P},\,\alpha| = 1 \tag{15}$$

and

$$\langle\mathbf{P},\,\alpha|\mathbf{P}',\,\alpha'\rangle \;=\; \delta(\mathbf{P},\mathbf{P}') \sum_{m=-\infty}^{\infty} \delta(\alpha - \alpha' + 2\pi m) \tag{16}$$

where only the $m = 0$ term is relevant. The Kronecker symbol in (16)

$$\delta(\mathbf{P},\mathbf{P}') = \begin{cases} 1 & \text{if } \mathbf{P} = \mathbf{P}' \\ 0 & \text{else} \end{cases} \tag{17}$$

expresses the fact that two hyperbolas labelled with different $\mathbf{P}$ and $\mathbf{P}'$ do not have any state in common.

From

$$U(T)\,|\mathbf{P},\alpha\rangle = e^{i\alpha}|\mathbf{P},\alpha\rangle, \tag{18}$$

it follows that the quasi-energy spectrum is absolutely continuous, and every value is countably infinite degenerate.

Having determined the properties of the Floquet operator $U(T)$ one can turn to the investigation of the system's long-time evolution. The continuous quasi-energy spectrum is a promising fact which, in combination with the bounded configuration space, may give rise to irregular quantum-mechanical motion.

QUANTUM DYNAMICS

The discrete and equidistant spectrum of the momentum operator has a consequence which is known as quantum resonance.[9] At the time $t_R = 2/h$ the free time-evolution operator U_F becomes the identity: $U_F(t_R) = 1$. Hence, if the external kick period T coincides with t_R (or an integer multiple of it) the Floquet operator over n periods T reduces to the n^{th} power of the kick operator U_K. The free time-evolution appears to be suppressed. It becomes possible to investigate exclusively the effect of the kicks which in the classical model generate the irregular behaviour. For simplicity, the matrix $\mathbf{C}$ is assumed to be symmetric from now on.

Taking a position eigenstate $|\mathbf{x}_0\rangle$ as initial state, the time evolution over n periods $T \equiv t_R$ immediately follows from (11)

$$U((nT)^-)\,|\mathbf{x}_0\rangle = U_K^n|\mathbf{x}_0\rangle = |\,\mathbf{C}^n \cdot \mathbf{x}_0\rangle. \tag{19}$$

Therefore the expectation value of the position operator $\hat{\mathbf{x}}$ becomes

$$\langle\psi(T_n^-(s))\,|\hat{\mathbf{x}}\,|\,\psi(T_n^-(s))\rangle = (\mathbf{C}^n \cdot \mathbf{x}_0) \bmod 1. \tag{20}$$

The last two equations show clearly the high irregularity of the exact quantum-mechanical time-evolution of the wave function and, consequently, of the position expectation-value. It is completely determined by the repeated application of a hyperbolic map **C** on some initial value. As an immediate consequence of (19) the quantum time-evolution in this model has positive algorithmic complexity.

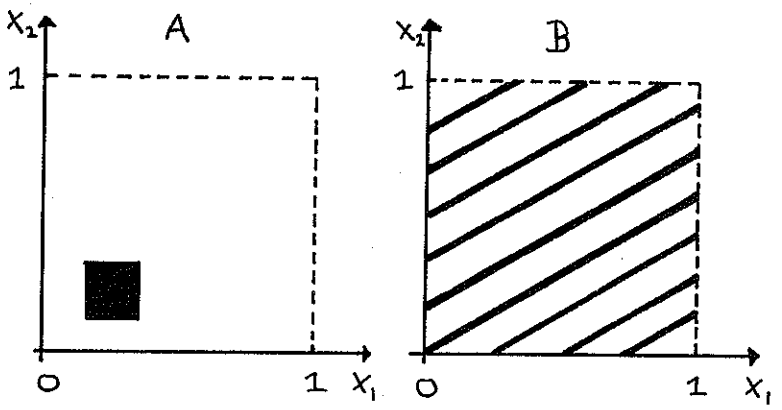

Figure 1: The distribution of non-zero coefficients of the state $|\psi\rangle$ over the configuration space at time $t = 0^-$ (A) and after n periods (B)

Let $|\psi\rangle$ be a state which initially is localized in a small region of configuration space (Fig. 1A). The non-zero coefficients of this state under the time evolution are spread exponentially quick over the total set of basis vectors $|\mathbf{x}\rangle$. After n periods T the coefficients vary strongly over the coordinate basis, as is depicted schematically in Fig. 1B. This process formally is equivalent to the transformation of a classical *phase-space* density of a one-dimensional system under the influence of a hyperbolic map. Classically, stretching and folding in phase space is the origin of "mixing" behaviour.[7] After introducing a coarse-graining in phase space one discovers classical observables to approach an "equilibrium state".[10]

As a consequence of stretching and folding of the wave function in configuration space, expectation values of various observables with respect to an initially localized state $|\psi\rangle$ do approach a limit - even without introducing a coarse-graining procedure. Let the initial state $|\psi\rangle$ give rise to a non-zero probability amplitude $|\psi(\mathbf{x})|^2 = \chi(\mathbf{x}, \mathbf{x}_0)/\Delta x_1 \Delta x_2$ only in a small rectangle centered about $\mathbf{x}_0$,

$$\chi(\mathbf{x}, \mathbf{x}_0) = \begin{cases} 1 & |x_{1,0} - x_1| \leq \Delta x_1/2 \text{ and } |x_{2,0} - x_2| \leq \Delta x_2/2 \\ 0 & \text{else.} \end{cases} \tag{21}$$

Then the estimate

$$|\langle\psi|\hat{x}_1(T_n^-)|\psi\rangle - \frac{1}{2}| \leq \frac{2}{\Delta x_1 \Delta x_2} e^{-n|\ln\lambda|} \tag{22}$$

can be derived, λ being an eigenvalue of the matrix **C**. Hence the expectation value of the position operator $\hat{\mathbf{x}}$ indeed approaches its limit

$$\lim_{n\to\infty} \langle\psi|\hat{\mathbf{x}}(T_n^-)|\psi\rangle = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} \tag{23}$$

exponentially quick. The same is true for the variance of the position operator. One finds ($\Delta x_1 \equiv \hat{x}_1 - \langle\psi|\hat{x}_1|\psi\rangle$)

$$|\langle\psi|(\Delta\hat{x}_1(T_n^-))^2|\psi\rangle - \frac{1}{12}| \leq c_1 e^{-n|\ln\lambda|} + c_2 e^{-2n|\ln\lambda|}, \tag{24}$$

where c_1, c_2 are positive real numbers. Furthermore, the position autocorrelation function containing information about the time evolution of spatial correlations also decays exponentially.[5]

Turning to expectation values of momentum operators, one generically encounters the expected growth of momenta $\hat{\mathbf{p}}(t)$ and, consequently, of the energy $\hat{H}(t)$. The explicit calculation shows that the increase of these quantities is exponential in time, and the rate is again determined by the eigenvalues of the matrix **C**. It is worth remarking that in this model the time dependence of the expectation value of the energy strictly parallels the classical one. The ubiquitous "quantum mechanical suppression of (diffusive) energy growth" [9] is not present here.

SUMMARY

It has been shown for a particular model that within the framework of quantum mechanics, the formation of arbitrarily small structures is possible

on the level of wave functions. The spectrum of the Floquet operator has been obtained from the complete set of analytically given eigenfunctions, and turned out to be absolutely continuous. This property is reflected in the time evolution of various expectation values which do not change quasi-periodically in time but approach limiting values at an exponential rate. Obviously for such a spectrum the method of level statistics is not applicable. It is important to note that with respect to long-time predictions in this model one encounters basically the same difficulties which are present in classical chaotic systems.

Further investigations will concentrate on the problem of why the behaviour observed here generically seems to be suppressed. Another question of interest is whether autonomous systems are able to mimic such behaviour for some finite but possibly long time.

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