Z. Phys. B – Condensed Matter 80, 3–4 (1990)

Condensed Matter
Zeitschrift für Physik B

The configurational quantum cat map

St. Weigert

Institut für Physik der Universität Basel, Switzerland

Received March 2, 1990

A charged particle moving in a bounded region of the plane (with periodic boundary conditions) is subject to external periodic electromagnetic fields. Classically, they effect a hyperbolic mapping of the particle configuration space to itself which leads to highly chaotic motion. It is shown that the quantum-mechanical time-evolution operator has an *absolutely continuous* spectrum of quasi-energies, indicating a strong irregularity in the motion of the quantum system. The quantum time evolution turns out to have nonvanishing algorithmic complexity.

In this paper, a quantum system is presented which in its time evolution clearly exhibits chaotic features and algorithmic complexity.

Linear hyperbolic maps of a bounded region to itself contain all the features which are characteristic for the chaotic behaviour of classical systems. Assuming the unit square to represent the (toroidal) phase space of a fictitious physical system with one degree of freedom, repeated application of e.g. "Arnold's cat map" generates discrete orbits into which a Bernoulli shift can be embedded, corresponding to the highest degree of irregularity possible in dynamical systems [1]. Quantized versions of such maps have been introduced [2, 3] in order to understand the relevance of the concept of chaos in quantum mechanics, and to clarify the relation between classically chaotic systems and their quantum-mechanical counterparts. The formation of ever finer structures, being a sine qua non in classical chaotic systems, is in these systems, however, prevented by the *discrete* spectrum of the operators of both, momentum and position. The present paper demonstrates that this is not a fundamental limitation.

Consider the unit square as configuration space of a classical physical system with *two* degrees of freedom. Then the application of a hyperbolic map will yield "configurational chaos": the irregular behaviour of "paths" in configuration space is sufficient to render the time evolution of phase-space orbits chaotic. Subsequent quantization of such systems does *not* impose a coarse-grained structure on the configuration space because the position operators commute. Chirikov et al. [4] analyzed an abstract autonomous model with at least three degrees of freedom, and showed that there are features of configurational chaos which indeed survive quantization.

The Hamiltonian

$$H(\mathbf{x},\mathbf{p},t)=\tfrac{1}{2}\tilde{\mathbf{p}}\cdot\mathbf{p}+\tfrac{1}{2}(\tilde{\mathbf{p}}\cdot\mathbf{A}(\mathbf{x},t)+\tilde{\mathbf{A}}(\mathbf{x},t)\cdot\mathbf{p}) \qquad (1)$$

describes a charged particle constrained to move in a unit square of the xy-plane with periodic boundary conditions (period 1) under the influence of time-dependent electromagnetic fields. The electric field $\mathbf{E}(\mathbf{x},t)$ associated with (1) has components in the xy-plane only, whereas the magnetic field $\mathbf{B}(\mathbf{x},t)$ is directed along the z-axis.

A linear and time-periodic vector field

$$\mathbf{A}(\mathbf{x},t)=\mathbf{V}\cdot\mathbf{x}\,\Delta_{T,\varepsilon}(t) \qquad (2)$$

yields *linear* equations of motion, allowing throughout for analytic treatment [8]. Here, $\Delta_{T,\varepsilon}(t)$ is a sequence of smooth kicks of period T, duration $\propto\varepsilon$ and height $\propto 1/\varepsilon$ with $\varepsilon\ll T$, and $\mathbf{V}$ is a 2×2 matrix such that $\mathbf{C}=\exp[\mathbf{V}]$ is hyperbolic and has integer entries only, e.g. Arnold's cat map. Stroboscopic observation of a particle with vanishing momentum $\mathbf{p}_0$ initially placed at $\mathbf{x}_0$ already reveals fully chaotic orbits with positive algorithmic complexity [5]. In the limit $\varepsilon\to 0$ Arnold's cat map of the unit square

$$\mathbf{x}((nT)^-)=(\mathbf{C}^n\cdot\mathbf{x}_0)\bmod 1 \qquad (3)$$

describes exactly the particle positions at times $t=(nT)^-$ just before the kicks. Nonvanishing initial momenta $\mathbf{p}_0$ always lead to an increase of energy exponential in time.

The time-evolution operator over one period T or Floquet operator $U(T)$ is the appropriate tool for investigating the long-time behaviour of time-periodic quantum systems [6]. In the limit $\varepsilon\to 0$ it becomes

$$U(T)=\exp\left[-\frac{\mathrm{i}T}{2\hbar}\tilde{\mathbf{p}}\cdot\mathbf{p}\right]\exp\left[-\frac{\mathrm{i}}{2\hbar}(\tilde{\mathbf{x}}\cdot\tilde{\mathbf{V}}\cdot\mathbf{p}+\tilde{\mathbf{p}}\cdot\mathbf{V}\cdot\mathbf{x})\right] \qquad (4)$$

where it has been assumed that the kick operator U_K acts before the free time-evolution operator $U_F(T)$. The transformation of the states of the position and momentum basis under the kick is remarkably simple

$$U_K|\mathbf{x}\rangle = |(\mathbf{C}\cdot\mathbf{x}) \bmod 1\rangle, \qquad U_K|\mathbf{p}\rangle = |\tilde{\mathbf{C}}^{-1}\cdot\mathbf{p}\rangle \tag{5}$$

where $x, y \in [0, 1)$ and $p_x/h, p_y/h \in \mathbb{Z}$. The *labels* of the quantum states are mapped according to the classical canonical kick transformation.

The operator U_K partitions the 2-dimensional grid of momentum eigenstates into "discrete hyperbolas" with label $\mathbf{P}$: each of the countably infinite number of sets $S(\mathbf{P}) = \{|\tilde{\mathbf{C}}^s\cdot\mathbf{P}\rangle, s\in\mathbb{Z}\}$ is invariant under the application of the operator U_K. Superpositions of states on one hyperbola with appropriate phases turn out to be eigenstates of the total time evolution operator $U(T)$

$$|\mathbf{P}, \alpha\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{\mathrm{i}T}{2\hbar} f_n(\mathbf{P}) + \mathrm{i}\alpha n\right] |\tilde{\mathbf{C}}^n\cdot\mathbf{P}\rangle, \tag{6}$$

$$f_n(\mathbf{p}) = \begin{cases} -\sum\limits_{s=0}^{n-1} \tilde{\mathbf{p}}\cdot\mathbf{C}^s\cdot\tilde{\mathbf{C}}^s\cdot\mathbf{p} & n \geqq 0 \\ \sum\limits_{s=1}^{|n|} \tilde{\mathbf{p}}\cdot\mathbf{C}^{-s}\cdot\tilde{\mathbf{C}}^{-s}\cdot\mathbf{p} & n \leqq 0 \end{cases} \tag{7}$$

where α is any real number in the interval $[0, 2\pi)$. Straightforward calculation shows that $\{|\mathbf{P}, \alpha\rangle\}$ is a complete set of (generalized) orthonormal states. From

$$U(T)|\mathbf{P}, \alpha\rangle = \exp[\mathrm{i}\alpha]\,|\mathbf{P}, \alpha\rangle, \tag{8}$$

it follows that the quasi-energy spectrum is absolutely continuous and that every value is countably infinite degenerate. Level statistics is not applicable. The expectation value of the energy is *not* bounded but grows at an exponential rate.

For certain values of T quantum resonances [7] occur: the operator $U_F(T)$ becomes the identity. In this case, the time evolution of an initial state $|\mathbf{x}_0\rangle$ exactly parallels the classical time evolution: $U((nT)^-)|\mathbf{x}_0\rangle = (U_K)^n|\mathbf{x}_0\rangle = |\mathbf{x}((nT)^-)\rangle$. Thus there exist "quantum orbits" of positive algorithmic complexity. The highly irregular quantum time-evolution shows up in various respects [8]. In particular, a wave packet, initially concentrated in a small region of configuration space gets exponentially stretched and folded such that it is quickly distributed over the coordinate basis.

In summary, quantization of the time-dependent classically chaotic system presented here yields a quantum system with an absolutely continuous quasi-energy spectrum. Consequently, many concepts known from the description of classical chaotic dynamics can be applied to the quantum time evolution.

References

1. Arnold, V.I., Avez, A.: Ergodic problems of classical mechanics. Reading, MA: Benjamin 1968
2. Hannay, J.H., Berry, M.V.: Physica **1** D, 267 (1980)
3. Ford, J.: Directions in chaos. Hao, B.-L. (ed.), Vol. 2, p. 128. Singapore: World Scientific 1988; Ford, J., Mantica, G., Ristow, G.: (to be published)
4. Chirikov, B.V., Izrailev, F.M., Shepelyansky, D.L.: Physica **33** D, 77 (1988)
5. Chaitin, G.J.: Algorithmic information theory. Cambridge: Cambridge University Press 1987
6. Casati, G., Molinari, L.: Prog. Theor. Phys. Suppl. **98**, 287 (1989)
7. Casati, G., Chirikov, B.V., Izraelev, F.M., Ford, J. In: Lecture Notes in Physics. Casati, G., Ford, J. (eds.), Vol. 93, p. 334. Berlin, Heidelberg, New York: Springer 1979
8. Thomas, H., Weigert, St.: (to be published)

St. Weigert
Institut für Physik
Universität Basel
Klingelbergstrasse 82
CH-4056 Basel
Switzerland