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Classical degeneracy and the existence of additional constants of motion

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The relation between degeneracy of classical frequencies in integrable systems and additional global constants of motion is investigated from a general point of view. It is shown that in autonomous systems with N degrees of freedom the existence of $\Lambda < N$ such additional invariants imply Λ global linear relations with commensurate coefficients between the frequencies, and vice versa. Furthermore, such a degeneracy gives rise to a specific structure of the Hamiltonian considered as a function of action variables. The general statements are illustrated by various examples.

I. INTRODUCTION

Two types of autonomous Hamiltonian systems may be distinguished, the trajectories of which show qualitatively different behavior. The phase space of *integrable* systems is foliated into lower-dimensional submanifolds, and, effectively, any trajectory is restricted to move on a torus for all times. *Nonintegrable* systems, on the other hand, have a complicated phase-space structure, and their time evolution is extremely difficult to determine for long times. The condition for a system with N degrees of freedom to be integrable requires the existence of precisely N appropriate constants of motion.

Occasionally, one encounters situations where there exist *more* functionally independent invariants than required by integrability. For example, the three-dimensional Kepler problem is known to possess two conserved quantities transforming as vectors, namely the angular momentum **l** and the Runge-Lenz vector **K**, and a scalar quantity, the energy H of the system. Out of these seven invariants *five* are functionally independent—consequently, there are two more invariants than required for the system to be integrable. Furthermore, one realizes that in some integrable systems the classical frequencies determining the time evolution (described in appropriately chosen generalized angle variables) satisfy one or more linear relations with integer coefficients. This is the case for the two-dimensional harmonic oscillator with commensurate frequencies but it is also true for the above example: since all orbits in the Kepler problem are closed the solutions of the equations of motion, effectively, depend on one single frequency only. Consequently, there must exist general relations allowing to eliminate two out of the set of three frequencies which usually are necessary to describe the time evolution of an integrable system with three degrees of freedom.

It is the purpose of the present paper to discuss the connection between the existence of additional invariants and of degenerate classical frequencies in integrable systems from a general point of view. Moreover, the structure of Hamiltonian functions expressed in action-angle variables will be related to the degree of degeneracy of classical frequencies.

Some results related to this problem have been derived in the context of early formulations of quantum mechanics. For example, Born[1] and Schwarzschild,[2] when discussing the quantization conditions introduced by Bohr and Sommerfeld, carefully distinguish between integrable systems with degenerate and nondegenerate classical frequencies, respectively. In fact, this is necessary in order to formulate a consistent quantization scheme based on associating a particular set of discrete values to the classical actions. Likewise, Goldstein's book[3] contains various remarks relevant to the present problem, as do many other textbooks on classical mechanics.[4] Perelomov, dedicating two volumes to integrable systems and Lie algebras,[5] does not discuss explicitly the problem under consideration; however, his collection of general theorems at least implicitly has some bearing on the work presented here.

In the present paper, emphasis is laid on establishing a coherent and general view of the topic. To this end, three statements, $\alpha - \gamma$, are enunciated in the following section which will be shown to be three manifestations of one unique physical situation. Having proved the equivalence of these statements, we illustrate them by various familiar systems, and we discuss them in detail for the N-dimensional harmonic oscillator.

II. THREE VIEWS ON Λ-FOLD DEGENERATE SYSTEMS

Autonomous integrable Hamiltonian systems with N degrees of freedom are studied most conveniently in canonical variables $\mathbf{I}=(I_1,I_2,...,I_N)$ and $\boldsymbol{\varphi}=(\varphi_1,\varphi_2,...,\varphi_N)$, each φ_n defined modulo 2π, with $\{I_n,I_{n'}\}=\{\varphi_n,\varphi_{n'}\}=0$ and $\{\varphi_n,I_{n'}\}=\delta_{nn'}(n,n'=1,2,...,N)$, representing actions and angles, respectively. The Hamiltonian is assumed to be a smooth function of the actions: $H=H(\mathbf{I})$, leading to the following equations of motion

$$\dot{\mathbf{I}}=\{\mathbf{I},H\}=0,\quad \dot{\boldsymbol{\varphi}}=\{\boldsymbol{\varphi},H\}=\frac{\partial H}{\partial \mathbf{I}}\equiv\boldsymbol{\omega}(\mathbf{I}).\tag{1}$$

The actions $\mathbf{I}$ represent a basic set of invariants sufficient to guarantee integrability of the system. The existence of action-angle variables is a consequence of Liouville's theorem[6] which implies a foliation of phase space in N-dimensional tori T^N.

Let Λ be the number of additional, functionally independent, globally defined and smooth constants of motion, collectively denoted by $\mathbf{J}=(J_1,...,J_\Lambda)$. Since for systems which show any motion at all, there exist at most $2N-1$ global invariants, the number Λ may take a value between 1 and $N-1$. If $\Lambda=N-1$, one deals with a *completely degenerate* system, otherwise the system at hand is called *Λ-fold degenerate.* Due to their functional independence of the invariants $\mathbf{I}$, the additional constants of motion $\mathbf{J}$, cannot be in involution with all of the N basic invariants $\mathbf{I}$.

The statements $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ to be shown equivalent read as follows:

($\boldsymbol{\alpha}$) There exist $N+\Lambda$ single-valued and functionally independent constants of motion, N of which are in involution.

($\boldsymbol{\beta}$) Globally there exist Λ linear relations with integer coefficients between the N frequencies $\boldsymbol{\omega}$ of the system:

$$\sum_{n=1}^{N} c_{\lambda n}\omega_n(\mathbf{I})=0,\quad c_{\lambda n}\in\mathbb{Z},\tag{2}$$

and $\lambda=1,2,...,\Lambda$.

($\boldsymbol{\gamma}$) The Hamiltonian of the system can be written as a function of $N-\Lambda$ actions $\mathbf{I}'=(I'_{\Lambda+1},I'_{\Lambda+2},...,I'_N)$ only, $H=H(\mathbf{I}')$. In general, each of its arguments is given as a linear combination of the original action variables $\mathbf{I}$ with integer coefficients.

The proof of the equivalence of the statements $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ is divided into two steps. First, we focus on the equivalence $\boldsymbol{\beta}\Leftrightarrow\boldsymbol{\gamma}$, the nontrivial part of which, $\boldsymbol{\beta}\Rightarrow\boldsymbol{\gamma}$, has been proved by Born. Second, we demonstrate $\boldsymbol{\alpha}\Leftrightarrow\boldsymbol{\beta}$; in particular, two independent proofs of the nontrivial implication $\boldsymbol{\alpha}\Rightarrow\boldsymbol{\beta}$ are presented.

A. $\boldsymbol{\beta}\Leftrightarrow\boldsymbol{\gamma}$

The Hamiltonian of a Λ-fold degenerate system, effectively, depends on $N-\Lambda$ actions $\mathbf{I}'$ only. The implication $\boldsymbol{\beta}\Rightarrow\boldsymbol{\gamma}$ is an immediate consequence of a result obtained by Born[7] (Chap. 2, Sec. 15) who proved the following statement:

"If between the ω_n a number of Λ conditions of commensurability

$$\sum_{n=1}^{N}\tau_n\omega_n=0\tag{3}$$

[with integer coefficients τ_n] exist we can apply a canonical transformation [...] such that Λ of the frequencies $\omega'_n=\partial H/\partial I'_n$ vanish and that between the remaining $N-\Lambda$ frequencies no relation of the type (3) holds [...]; we have

$$\omega'_\alpha \text{ incommensurate},\quad \alpha=\Lambda+1,\Lambda+2,...,N,$$
$$\omega'_\lambda=0,\quad \lambda=1,2,...,\Lambda,\tag{4}$$

and the Hamiltonian function has the form

$$H(I'_{\Lambda+1},I'_{\Lambda+2},...,I'_N)."\tag{5}$$

Born's result is based on the idea to introduce Λ new angle variables φ'_λ depending linearly on the original angles

$$\varphi'_\lambda=\sum_{n=1}^{N}c_{\lambda n}\varphi_n,\tag{6}$$

with the coefficients given by the conditions of commensurability, Eqs. (2). Obviously, these variables have the property

$$\frac{d}{dt}\varphi'_\lambda(t)=0,\quad \lambda=1,2,...,\Lambda,\tag{7}$$

as follows from Eq. (3). The actions $(I'_{\Lambda+1},I'_{\Lambda+2},...,I'_N)$ are not fixed unambiguously; according to Born,[8] they are "[...] determined only up to homogeneous integer linear transformations with determinant ± 1." In other words, there exist canonical transformations resulting in

$$\mathbf{I}'\to\mathbf{I}''(\mathbf{I}')=\mathbf{M}\cdot\mathbf{I}',\tag{8}$$

where $\mathbf{M}$ is an $(N-\Lambda)\times(N-\Lambda)$-matrix with integer elements and $\det\mathbf{M}=\pm 1$. In these variables, the Hamiltonian H still is a function of $N-\Lambda$ arguments each of which is a linear combination of the $N-\Lambda$ actions $\mathbf{I}'$ with integer coefficients.

In general, the introduction of action-angle variables in a Λ-fold degenerate system will not lead to a Hamiltonian depending on only $N-\Lambda$ actions. Usually, *all* actions $\mathbf{I}$ will occur: therefore, it is helpful to know what the Hamiltonian of a Λ-fold degenerate system generically looks like. The transformation (6) is generated by a function F being a bilinear expression in the variables $\boldsymbol{\varphi}$ and $\mathbf{I}'$ with integer coefficients. Necessarily, the new actions $\mathbf{I}'$ obtained from such a generating function depend *linearly* on the original actions $\mathbf{I}$ with rational coefficients. This implies that the second part of statement $\boldsymbol{\gamma}$ holds: any argument of the Hamiltonian H is a linear combination of the actions $\mathbf{I}$ with commensurate coefficients.

The existence of Λ conditions of commensurability is an immediate consequence if the Hamiltonian depends on $N-\Lambda$ arguments only ($\boldsymbol{\gamma}\Rightarrow\boldsymbol{\beta}$). The following argument starts from the second part of statement $\boldsymbol{\gamma}$ which includes the first part as a special case. In a Hamiltonian H with N actions $\mathbf{I}$ distributed over $N-\Lambda$ arguments in the form of linear combinations with integer coefficients, exactly $N-\Lambda$ out of all N partial derivatives $\omega_n=\partial H/\partial I_n$, $n=1,2,...,N$, are linearly independent. Consequently, at least Λ linear relations with integer coefficients do exist between the frequencies $\boldsymbol{\omega}$

$$\sum_{n=1}^{N}c_{\lambda n}\omega_n(\mathbf{I})=0,\quad c_{\lambda n}\in\mathbb{Z},\tag{9}$$

where $\lambda=1,2,...,\Lambda$, which was to be shown.

B. $\alpha \Leftrightarrow \beta$

The existence of $N+\Lambda$ constants of motion is a consequence of Λ conditions of commensurability ($\beta \Rightarrow \alpha$). Since

$$0=\sum_{n=1}^{N} c_{\lambda n}\omega_n(\mathbf{I})=\frac{d}{dt}\left(\sum_{n=1}^{N} c_{\lambda n}\varphi_n\right),\quad c_{\lambda n}\in\mathbb{Z}, \tag{10}$$

$\lambda=1,2,\ldots,\Lambda$, each nonconstant smooth 2π-periodic function

$$J_\lambda(\varphi)=J_\lambda(\varphi+2\pi), \tag{11}$$

defines a single-valued and global additional constant of motion

$$J_\lambda\left(\sum_{n=1}^{N} c_{\lambda n}\varphi_n\right),\quad \lambda=1,2,\ldots,\Lambda, \tag{12}$$

being functionally independent of the other $J_{\lambda'}(\lambda\neq\lambda')$ and of all actions $\mathbf{I}$, which represent the required set of N involutive invariants.

The demonstration that the existence of Λ additional invariants gives rise to Λ conditions of commensurability ($\alpha \Rightarrow \beta$) is more involved.[9] Two proofs of this implication are given. The first one refers explicitly to the phase-space foliation of integrable systems; the second one, given in the Appendix, starts from the conditions of time independence and single valuedness of the invariants in phase space, resulting in an explicit expression for the relation between the frequencies.

Consider, for simplicity, at first an arbitrary integrable system with two degrees of freedom only. If, apart from the two actions I_1 and I_2 a third, functionally independent constant of motion J exists, then the intersections of the three sets of hypersurfaces $I_1=$const, $I_2=$const, $J=$const define a set of truly *one*-dimensional manifolds in phase space which are invariant under the flow: a phase point starting on any one of these intersections is confined to it for all times.

Integrability guarantees the existence of the torus structure in phase space and the two frequencies $\omega_1(\mathbf{I})$ and $\omega_2(\mathbf{I})$ are continuous functions in phase space. The following arguments require either $\omega_1(\mathbf{I})\neq 0$ or $\omega_2(\mathbf{I})\neq 0$; this is true everywhere in phase space except on separatrices which are of measure zero, however. The assumption that the frequency ratio $\omega_1(\mathbf{I})/\omega_2(\mathbf{I})=\omega_{12}(\mathbf{I})$ is *not* a constant rational number for all values of $\mathbf{I}$ leads to a contradiction.

(i) The ratio $\omega_{12}(\mathbf{I})\equiv\omega_{12}^0$ is an irrational constant for all $\mathbf{I}$. The existence of the invariants $\mathbf{I}$ assures that the motion in phase space takes place on two-dimensional tori. For irrational ω_{12}^0, however, motion on a torus is ergodic, i.e., there does *not* exist a partition of the torus into invariant submanifolds of positive measure.[10] Hence, the trajectory would cover densely a two-dimensional manifold for arbitrary fixed values of $\mathbf{I}$ contradicting the previously mentioned fact that the motion is restricted to a *one*-dimensional phase-space manifold.

(ii) The ratio $\omega_{12}(\mathbf{I})$ is a continuous nonconstant function of the actions $\mathbf{I}$. In this case, the tori with irrational frequency ratio are dense in the set of all tori and have measure 1 whereas the rational tori—likewise distributed densely—have measure 0. For every irrational torus the argument given in (i) applies. Therefore, each trajectory fills a two-dimensional manifold for arbitrary fixed values of the actions $\mathbf{I}$ except a set of measure 0. Again, no third global smooth integral can exist in phase space.

Since the cases (i) and (ii) exhaust all possibilities, the frequency ratio $\omega_{12}(\mathbf{I})$ in systems with two degrees of freedom, $N=2$, necessarily is rational all over phase space if three global integrals exist.

For an *arbitrary* number N and Λ-fold ($\Lambda=1,2,\ldots,N-1$) degenerate systems the same reasoning applies: $N+\Lambda$ global smooth integrals restrict the motion to a manifold of dimension $N-\Lambda$. Commensurability between frequencies ω is expressed by the existence of equations of the type

$$\sum_{n=1}^{N} c_{\lambda n}\omega_n(\mathbf{I})=0,\quad c_{\lambda n}\in\mathbb{Z}, \tag{13}$$

where $\lambda=1,2,\ldots,\Lambda$. Suppose one would have less than Λ equations of this type, say $\Lambda'(\Lambda'=0,1,\ldots,\Lambda-1)$. Then the motion would fill densely a manifold of dimension $N-\Lambda'>N-\Lambda$. This would make it impossible to have more than $N+\Lambda'(<N+\Lambda)$ independent integrals—in contradiction to the assumption. Therefore Λ conditions of commensurability must exist between the frequencies $\omega(\mathbf{I})$.

For a different proof of this result the reader is referred to the Appendix.

III. EXAMPLES

In this chapter the general results obtained for integrable systems possessing additional constants of motion are illustrated with various examples. On the one hand, some well-known physical systems with a small number of degrees of freedom, $N=2$ or $N=3$, are listed, and their properties are seen to fall into the scheme presented above. On the other hand, the harmonic oscillator is considered which allows to exemplify the relevant features explicitly for an arbitrary number of degrees of freedom N and arbitrary degeneracy Λ.

The study of these systems reveals that statement α actually is most powerful: often it is possible to prove the existence of a specific number of invariants some of which are in involution. The transformation to action-angle variables (being necessary to verify statement γ), on the other hand, although always being possible in principle, can be given analytically in exceptional cases only. Nevertheless, one can use knowledge about the whole set of possible motions in order to determine the actual number of involved frequencies.

A particle moving in the attractive Coulomb field of two fixed centers is an example of an integrable nondegenerate system ($N=3,\Lambda=0$). The Hamilton–Jacobi equation of this system is separable in elliptic coordinates; in addition to the energy and the component of angular momentum parallel to the line joining the two centers one finds a third independent integral of motion.[11] Correspondingly, the time evolution of the system contains three independent frequencies associated with the one-dimensional motions obtained by the separation. The explicit transformation to action-angle variables, however, is not known.

For a particle moving in a central field, apart from the energy all three components of the angular momentum $\mathbf{l}$ are conserved, rendering the system singly degenerate ($N=3,\Lambda=1$); three involutive invariants are given by H, l_z, and $\mathbf{l}^2$. In fact, the motion is restricted to the plane perpendicular to the angular momentum $\mathbf{l}$, and, generally, the orbits in this plane do not close. Therefore, the time evolution contains two incommensurate frequencies, the third frequency being equal to zero on account of the fixed orientation of the plane of motion (cf. statement $\boldsymbol{\beta}$). In contrast, in the case of the Kepler problem and the three-dimensional isotropic harmonic oscillator, all orbits are closed, i.e., the two frequencies become commensurate, and one single frequency effectively determines the time evolution of the system.[12] Hence, these two systems are completely degenerate ($N=3,\Lambda=2$), and as a matter of fact, both systems are known to possess the required two additional, functionally independent constants of motion. Exceptionally, for the Kepler problem and the oscillator the action-angle representation is known analytically, confirming the statements $\boldsymbol{\beta}$ and γ: in both cases, the Hamiltonian is a function of a sum of three actions with equal and constant coefficients.[13]

The free asymmetric top is another singly degenerate system with three degrees of freedom: again the (kinetic) energy and the three components of the angular momentum l_x,l_y,l_z in the laboratory system represent four independent conserved quantities ($N=3,\Lambda=1$). The Poinsot construction[14] shows that the motion indeed depends on two independent frequencies. The free symmetric top has an additional constant of motion, the component L_Z of the angular momentum[15] along the symmetry axis of the top, and one might therefore expect that it is doubly degenerate. However, this is not the case: the motion still contains two independent frequencies, the angular velocity about the symmetry axis of the top and the precession frequency. In fact, the energy H can be written as a function of $\mathbf{L}^2=\mathbf{l}^2$ and L_Z, so that only four out of the five constants of motion are independent. The spherical top, finally, is an example of a completely degenerate system ($N=3$, $\Lambda=2$), since between its seven constants of motion $E,\mathbf{l},\mathbf{L}$, there exist two functional relations, leaving five independent integrals. In fact, its motion is a uniform rotation about a fixed axis.

For the heavy asymmetric top only two constants of motion H,l_z exist and, therefore, it is nonintegrable. In the case of a symmetry axis Z, the component L_Z of the angular momentum is also conserved, turning the symmetric heavy top into an integrable but nondegenerate system ($N=3,\Lambda=0$). Its motion is characterized by three independent frequencies associated with the rotation about its symmetry axis, the precession of the angular momentum about the vertical, and the nutation of the symmetry axis about the instantaneous direction of the angular momentum.

Spin systems provide a further class of examples. A pair of exchange-coupled spins with uniaxial exchange and single-site anisotropy has two constants of motion, the energy and the component of total spin $\mathbf{S}_{\text{tot}}=\mathbf{s}_1+\mathbf{s}_2$ along the symmetry axis, and it is therefore integrable, but nondegenerate ($N=2,\Lambda=0$). It has been shown that a second integral of motion also exists for a class of asymmetric two-spin systems, if the constants of anisotropy satisfy a certain relation.[16] Correspondingly, two independent frequencies are visible in the motion of these systems. If the coupling is fully isotropic, $H=J\mathbf{s}_1\cdot\mathbf{s}_2$, all three components of the total spin $\mathbf{S}_{\text{tot}}$ are conserved. Since the energy can be expressed as a function of $\mathbf{S}_{\text{tot}}, H=\frac{1}{2}J\mathbf{S}_{\text{tot}}^2+\text{const}$, there are three independent constants of motion, making the system completely degenerate ($N=2,\Lambda=1$). In fact, the motion consists of a rigid uniform rotation of the array formed by the two spin vectors $\mathbf{s}_1,\mathbf{s}_2$ about the constant total spin $\mathbf{S}_{\text{tot}}$ with an angular velocity proportional to the vector product $\mathbf{s}_1\times\mathbf{s}_2$.

For two types of spin clusters, properties of integrability and degeneracy are known for an arbitrary number N of spins.[16,17] Let every spin be coupled to every other spin by a uniform isotropic exchange interaction: any such "complete cluster" possesses $2N-1$ independent integrals of motion and, consequently, it is completely degenerate ($\Lambda=N-1$); the motion consists of a rigid uniform rotation of the whole spin array about the constant total spin. The second type of spin system is composed of two subsystems A and B such that each spin of subsystem A interacts with all spins of subsystem B with a uniform anisotropic exchange interaction and vice versa, whereas spins belonging to the same subsystem are assumed not to interact. These systems possess at least $2N-4$ independent integrals of motion ($\Lambda\geqslant N-4$); the motion, therefore, is governed by at most four independent frequencies.

Finally, a Λ-fold degenerate harmonic oscillator in N-dimensions is considered; the particular feature of this system lies in the fact that all three statements $\boldsymbol{\alpha}-\gamma$ can be exhibited in detail.[18] Let the Hamiltonian be given by

$$H(\mathbf{p},\mathbf{q})=\sum_{\lambda=1}^{\Lambda+1}\frac{1}{2}(p_\lambda^2+\omega_\lambda^2q_\lambda^2)+\sum_{l=\Lambda+2}^{N}\frac{1}{2}(p_l^2+\omega_l^2q_l^2), \tag{14}$$

with frequencies $\{\omega_\lambda\}$ which, for convenience, are assumed to be integer numbers with no common divisor, in contrast to the $\{\omega_l\}$ being irrational incommensurate numbers. In action-angle variables the Hamiltonian reads

$$H(\mathbf{I})=\sum_{\lambda=1}^{\Lambda+1}\omega_\lambda I_\lambda+\sum_{l=\Lambda+2}^{N}\omega_l I_l. \tag{15}$$

($\boldsymbol{\alpha}$) Statement $\boldsymbol{\alpha}$ holds true for the Λ-fold degenerate N-oscillator since, on the one hand, the action variables $\mathbf{I}$ represent N global invariants in involution; on the other hand, Λ additional independent constants of motion are, for example, given by

$$J_\lambda=\sin(\omega_{\lambda+1}\varphi_\lambda-\omega_\lambda\varphi_{\lambda+1}),\quad \lambda=1,2,\ldots,\Lambda, \tag{16}$$

and none of the J_λ is in involution with all the actions $\mathbf{I}$.

($\boldsymbol{\beta}$) The second statement, $\boldsymbol{\beta}$, follows from Eq. (15): since the frequencies $\boldsymbol{\omega}$ are given by

$$\omega_n=\frac{\partial H}{\partial I_n}=\begin{cases}\omega_\lambda, & \lambda=1,\ldots,\Lambda+1,\\ \omega_l, & l=\Lambda+2,\ldots,N,\end{cases} \tag{17}$$

there exist Λ linear nontrivial independent relations of the type Eq. (2); choose, for example,

$$c_{\lambda n}\equiv\omega_{\lambda+1}\delta_{n\lambda}-\omega_\lambda\delta_{n\lambda+1}\in\mathbb{Z},\quad \lambda=1,2,\ldots,\Lambda. \tag{18}$$

(γ) The first part of this statement can be obtained by performing a canonical transformation to a different set of action-angle variables $(\mathbf{I}',\boldsymbol{\varphi}')$ such that the $(\Lambda+1)$st action reads

$$I'_{\Lambda+1}=\sum_{\lambda=1}^{\Lambda+1}\omega_\lambda I_\lambda=-\frac{\partial F(\mathbf{I},\boldsymbol{\varphi}')}{\partial\varphi'_{\Lambda+1}}. \tag{19}$$

This is achieved by a generating function $F(\mathbf{I},\boldsymbol{\varphi}')$ given by

$$F(\mathbf{I},\boldsymbol{\varphi}') = -\left(\sum_{\lambda=1}^{\Lambda+1}\sum_{\lambda'=1}^{\Lambda+1}\mu_{\lambda\lambda'}I_\lambda\varphi'_{\lambda'} + \sum_{l=\Lambda+2}^{N}\varphi'_l I_l\right), \tag{20}$$

with

$$\mu_{\lambda\Lambda+1} = \omega_\lambda, \tag{21}$$

where the $(\Lambda+1)\times(\Lambda+1)$-matrix $\boldsymbol{\mu}$ with integer entries $\mu_{\lambda\lambda'} \in \mathbb{Z}$ has determinant 1. The $(\Lambda+1)$ new actions I'_λ, as well as the new angle variables φ'_λ are given by linear combinations of the original variables, and the remaining new variables coincide with the original ones.

$$I'_l = I_l, \quad \varphi'_l = \varphi_l, \quad l = \Lambda+2, \Lambda+3, \ldots, N. \tag{22}$$

The properties of the matrix $\boldsymbol{\mu}$ guarantee that the angles φ'_λ, in fact, are 2π-periodic coordinates again. As a result, the Hamiltonian becomes a function of $N-\Lambda$ arguments only

$$H(\mathbf{I}) = H'(\mathbf{I}') \equiv H'(I'_{\Lambda+1},\ldots,I'_N) = I'_{\Lambda+1} + \sum_{l=\Lambda+2}^{N}\omega_l I'_l, \tag{23}$$

which was to be shown.

The second part of statement γ also is true for the oscillator at hand, due to the transformations generated by the function $F(\mathbf{I},\boldsymbol{\varphi}')$ of Eq. (20).

In summary, a number of physically different systems are seen to exhibit the general features of degenerate integrable systems introduced in Sec. II. The Λ-fold degenerate harmonic oscillator is particularly suited to demonstrate the equivalence of three manifestations of classical degeneracy of frequencies in arbitrary integrable systems: first, the existence of $N+\Lambda$ global smooth invariants (α), second, the existence of Λ linear relations between the classical frequencies with integer coefficients only (β), and third, the possibility to eliminate Λ actions of the Hamiltonian such that it becomes a function of $N-\Lambda$ arguments only (γ).

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APPENDIX

In this appendix a second proof of the implication $\boldsymbol{\alpha} \Rightarrow \boldsymbol{\beta}$ is given.

Take the N action variables $\mathbf{I}$ and the corresponding angle variables $\boldsymbol{\varphi}$ as canonical variables such that any additional constant of motion becomes a function $J(\mathbf{I},\boldsymbol{\varphi})$ of the $\mathbf{I}$ and $\boldsymbol{\varphi}$. The condition that J is invariant under the motion of the system,

$$\dot{J} = \sum_{n=1}^{N}\omega_n(\mathbf{I})\frac{\partial J}{\partial\varphi_n} = \boldsymbol{\omega}(\mathbf{I})\cdot\nabla_{\varphi}J = 0, \quad \omega_n(\mathbf{I}) = \frac{\partial H}{\partial I_n}, \tag{A1}$$

has the form of a linear partial differential equation in $\boldsymbol{\varphi}$. Its solutions are constant along the characteristics which are straight lines in $\boldsymbol{\varphi}$ space parallel to the vector $\boldsymbol{\omega}(\mathbf{I})$. In fact, it will usually be the case that the given J is also constant along other directions in $\boldsymbol{\varphi}$ space. In the remaining part of $\boldsymbol{\varphi}$ space of dimension $R \leqslant N-1$ containing no direction along which J is constant, one can introduce R coordinates

$$\chi_r = \sum_{n=1}^{N}\alpha_{rn}\varphi_n, \tag{A2}$$

parallel to R linearly independent directions α_r perpendicular to $\boldsymbol{\omega}(\mathbf{I})$,

$$\sum_{n=1}^{N}\alpha_{rn}\omega_n(\mathbf{I}) = 0 \quad (r=1,\ldots,R), \tag{A3}$$

such that

$$J = J(\mathbf{I},\chi_1,\ldots,\chi_R). \tag{A4}$$

Since J is assumed to be independent of the constants of motion $\mathbf{I}$, the set χ is not empty $(R \geqslant 1)$. It will turn out that additional constants of motion can be chosen such that each of them actually depends on a single coordinate χ only.

Consider, for the moment, one of the coordinates χ_r only. From the requirement that $J(\mathbf{I},\chi_r)$ is a single-valued phase-space function, i.e., that it is invariant against changes of the angles φ_n by arbitrary multiples of 2π,

$$J\left(\mathbf{I},\chi_r + 2\pi\sum_{n=1}^{N}k_n\alpha_{rn}\right) = J(\mathbf{I},\chi_r), \quad \forall k_n \in \mathbb{Z}, \tag{A5}$$

it follows that the α_{rn} must be commensurate,

$$\alpha_{rn} = \alpha_r c_{rn}, \quad c_{rn} \in \mathbb{Z}, \ \alpha_r \in \mathbb{R}, \tag{A6}$$

such that

$$\sum_{n=1}^{N}k_n\alpha_{rn} = \alpha_r c_r, \quad c_r = \sum_{n=1}^{N}k_n c_{rn} \in \mathbb{Z}, \tag{A7}$$

and $J(\mathbf{I},\chi_r)$ is periodic with period α_r

$$J(\mathbf{I},\chi_r + 2\pi\alpha_r) = J(\mathbf{I},\chi_r). \tag{A8}$$

For incommensurate α_{rn}, the values of $\Sigma_n k_n\alpha_{rn}$ form a dense set of points, and a continuous function $J(\mathbf{I},\chi_r)$ would be a constant.

Inserting (A6) into (A3) yields a relation between the frequencies

$$\sum_{n=1}^{N}c_{rn}\omega_n(\mathbf{I}) = 0, \quad c_{rn} \in \mathbb{Z}. \tag{A9}$$

This is a global relation, since it holds for all $\mathbf{I}$, and the c_{rn} cannot depend on $\mathbf{I}$ because of the continuity of the functions $\omega_n(\mathbf{I})$.

Thus it has been proved that the existence of a constant of motion depending on R variables χ_r gives rise to $R \geqslant 1$ global rational relations between the frequencies ω_n. If $R > 1$, the implication $\boldsymbol{\beta} \Rightarrow \boldsymbol{\alpha}$ shows that there exist $R-1$ further independent constants of motion.

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[7]Reference 1, pp. 104–105. Translated and notation adapted by the authors.
[8]Reference 1, p. 108.
[9]S. Weigert, "On Classical Symmetry and Deterministic Randomness in Quantum Mechanics," Ph. D. thesis, Basel, 1990, pp. 17–19.
[10]A. I. Khinchin, *Mathematical Foundations of Statistical Mechanics* (Dover, New York, 1949), Chap. II, Sec. 6 and Chap. III, Sec. 12.
[11]Reference 4, Chap. VII, Sec. 48, p. 154.
[12]Bertrand's theorem (see, e.g., Ref. 6, Chap. 2, Sec. 8, Pt. D) states that these two potentials are the only ones producing closed orbits. Numerous separable potentials in three spatial dimensions can be found in N. W. Evans, "Superintegrability in Classical Mechanics," Phys. Rev. A **41**, 5666–5676 (1990).
[13]Reference 3, Chap. 10, Sec. 7.
[14]Reference 3, Chap. 5, Sec. 6, pp. 205–208.
[15]Uppercase letters denote vector components and coordinates with respect to the body frame.
[16]E. Magyari, H. Thomas, R. Weber, C. Kaufmann, and G. Müller, "Integrable and Nonintegrable Classical Spin Clusters," Z. Phys. B **65**, 363–374 (1987).
[17]N. Srivastava, C. Kaufmann, G. Müller, R. Weber, and H. Thomas, "Integrable and Nonintegrable Classical Spin Clusters," Z. Phys. B **70**, 251–268 (1988).
[18]Reference 3, Chap. 10, Sec. 6, pp. 469–471.

THE ULTIMATE LAGRANGIAN

To be sure, existing formulations are conceded to be flawed; currently fashionable Lagrangians are only approximations to "the correct Lagrangian of the world." The assumption is that as the unification theories progress, they will more and more closely image "the real Theory of Everything," which by implication must somehow already exist abstractly. The history of science is thus seen as a sequence of successively better approximations, or fits, of "reality." In his much quoted inaugural lecture to the Lucasian Chair in Cambridge, Stephen Hawking declared that "the end of theoretical physics" is in sight. Hawking envisaged as imminent the correct identification of "the" Lagrangian, thus making physics a closed subject.

Paul Davies, in *Doing Science: The Reality Club, 2,* edited by John Brockman (Prentice-Hall, New York, 1991), p. 54.