Quantum Particle on a Rotating Loop: Topological Quenching due to a Coriolis-Aharonov-Bohm Effect

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A particle is assumed to move along a one-dimensional loop such as an ellipse that rotates in a plane. Because of the centrifugal force the particle is subjected to a symmetric double-well potential. Classically, the Coriolis force does not affect the motion of the particle, whereas the corresponding term in the Lagrangian influences the properties of the quantum system: its ground state turns out to be degenerate for a discrete set of angular velocities. The analogy between a constant magnetic field and a uniform rotation is used to propose, in addition, a variant of the Aharonov-Bohm experiment, which can be performed also with neutral particles.

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Gauge theories exhibit specific features if the configuration space of a system is multiply connected. This property is at the core of the Aharonov-Bohm effect [1] and of topological field theories (cf. [2]). The purpose of this Letter is to present two simple systems for which the interplay between a topologically nontrivial space and a gauge theory leads to surprising phenomena. In the first case a particle on a one-dimensional rotating loop is considered. It is shown that the quantum-mechanical energy spectrum depends on the angular velocity of the system in such a way that the ground state may degenerate for a discrete set of velocities. In the second case an Aharonov-Bohm type experiment—set up on a *rotating* table—will be described, which can be performed with neutral particles. The interference pattern produced on the screen by the split beams will turn out to be a function of the angular velocity.

These phenomena occur because angular and translational velocities of the moving frame with respect to an inertial one act as gauge fields: the motion of a classical particle moving freely in three-dimensional space can be considered as a gauge theory [3]. In this context, evaluating quantum-mechanical propagators in terms of path integrals has proved to be particularly useful: appropriately superposing classical solutions is often sufficient for a qualitative discussion of the salient features of the quantum system.

To begin with, a particle with mass m is assumed to move freely on a one-dimensional planar loop resting in the laboratory frame of reference, K_0. This situation can be obtained as a limit of a more realistic model in which the particle moves in a narrow two-dimensional gully [4]. The shape of the loop traces out a smooth curve $\partial\Gamma_0$ that does not intersect itself. Later on C_2 loops will be considered only, i.e., loops that are transformed into themselves under a rotation by π about an axis perpendicular to the plane [5]. The classical Lagrangian of the particle is given by

$$L_0(z_0, \dot{z}_0) = \tfrac{1}{2} m |\dot{z}_0|^2, \qquad z_0 \in \partial\Gamma_0, \tag{1}$$

where Cartesian coordinates x_0, y_0 of the plane are considered as real and imaginary parts of the complex variable z_0; the loop is described by a smooth function $G_0(z_0)$, i.e.,

$$\partial\Gamma_0 = \{z_0 | G_0(z_0) = 0\}. \tag{2}$$

The constraint that the particle move on the loop $\partial\Gamma_0$ only is taken into account conveniently by using an appropriate system of orthogonal coordinates $w_0 = \xi_0 + i\eta_0$. It is defined in such a way that the loop coincides with one of the coordinate lines leading to, say, $\xi_0 = \xi_0^* \equiv \text{const}$, during the motion. The Riemann mapping theorem [6] guarantees the existence of an appropriate conformal mapping

$$z_0 = g(w_0), \tag{3}$$

$g(w)$ being analytic. The Lagrangian turns into

$$L_0(w_0, \dot{w}_0)|_{w_0 \in \partial\Gamma_0} = L_0(\eta_0, \dot{\eta}_0) = \tfrac{1}{2} M(\eta_0) \dot{\eta}_0^2, \tag{4}$$

where $M(\eta_0) = m |dg(w_0)/dw_0|^2_{\xi_0 = \xi_0^*}$. The position-dependent mass M cannot become equal to zero, $M(\eta_0) \neq 0$ for all η_0, as follows from the conservation of energy.

Suppose the loop rotates rigidly with a constant angular velocity $\Omega = |\mathbf{\Omega}|$ about an axis $\mathbf{\Omega}/\Omega$ that is oriented perpendicularly to the plane and that pierces the center of the ellipse. The constraint becomes explicitly time dependent:

$$\partial\Gamma_t = \{z_0 | G_0(z_0, t) = 0\}. \tag{5}$$

With respect to a noninertial frame of reference, K, rotating with $\mathbf{\Omega}$ relative to K_0, the Lagrangian reads

$$L(z, \dot{z}) = \frac{1}{2} m |\dot{z}|^2 + \frac{1}{2i} m\Omega (z^* \dot{z} - z \dot{z}^*) + \frac{1}{2} m \Omega^2 z z^*, \qquad z \in \partial\Gamma, \tag{6}$$

where $\partial\Gamma$ describes the constraint in the moving frame. Introducing coordinates adapted to the loop in analogy to

(3), the effectively one-dimensional motion on the loop is described by

$$L(\eta, \dot{\eta}) = \tfrac{1}{2}M(\eta)\dot{\eta}^2 + \alpha(\eta)\dot{\eta} - W(\eta), \qquad (7)$$

and the expressions

$$M(\eta) = m \left| \frac{dg(w)}{dw} \right|^2_{\xi=\xi^*}, \qquad (8)$$

$$\alpha(\eta) = \frac{1}{2} m\Omega \left[\frac{\partial |g(w)|^2}{\partial \xi} \right]_{\xi=\xi^*}, \qquad (9)$$

$$W(\eta) = -\tfrac{1}{2} m\Omega^2 |g(w)|^2_{\xi=\xi^*} \qquad (10)$$

are interpreted as mass, a vector potential, and a scalar potential, respectively. As before, the particle acquires a position-dependent mass, $M(\eta)$; in addition, it is subjected to a scalar potential $W(\eta)$ due to the centrifugal force. It is interesting to see how the centrifugal force acting on the classical particle shows up on the quantum level. Effectively, the centrifugal potential $\propto -(\mathbf{\Omega} \times \mathbf{x})^2$ along the loop has a form such that the wave function is squeezed out of the regions close to the axis of rotation ($\sim$ high potential energy) into the more distant parts of the loop ($\sim$ low potential energy) where the particle prefers to stay classically. The centrifugal potential being proportional to Ω^2, it is obvious that this phenomenon becomes more and more pronounced for higher angular velocities.

The force associated with the Coriolis term $m\dot{\mathbf{x}} \cdot (\mathbf{\Omega} \times \mathbf{x})$ does not influence the *classical* motion since it points always perpendicular to the loop. For the Lagrangian L in (7) this follows from the fact that the term containing the vector potential, $a(\eta)$, is a total derivative,

$$\alpha(\eta)\dot{\eta} = \frac{d}{dt}\left(\int_{\tilde{\eta}}^{\eta} d\overline{\eta}\, a(\overline{\eta}) \right), \qquad (11)$$

representing thus a (classically irrelevant) gauge transformation of the Lagrangian. Nevertheless, this term must not be dropped in a quantum-mechanical description of the particle moving on the loop. As a matter of fact, on a multiply connected phase space such a term turns out to have physical meaning: In models of topological field theory [7] it can be used to generate a mass, and in the theory of anyons [8] it provides the interaction term resulting in braid statistics.

Also, a gauge term has been found to cause an unexpected quenching of the tunnel splitting in a spin system [9] possessing two classically equivalent equilibrium positions. The separation of the two lowest energy eigenvalues turns out to be a function of an externally applied magnetic field B, and for a discrete set of field strengths, B_n, the splitting drops to zero. An analogous phenomenon has been found to occur for a charged particle moving on a one-dimensional planar loop, along which a π-periodic double-well potential $V(\eta)$ is present [10]. Furthermore, a uniform magnetic field acts perpendicularly to the plane of the loop. If the loop is invariant under twofold rotations, the system has two equivalent minima. The tunnel splitting is found to be a function of the magnetic field, although classically the Lorentz force has no influence at all on the motion of the particle. Again, the tunnel splitting is quenched for a discrete set of field strengths B_n, leading to a degenerate ground state of the quantum system. The Lagrangian of this system has, as a matter of fact, exactly the form given in Eq. (7). Consequently, a (neutral) particle on a rotating loop is (with respect to the rotating frame) equivalent to a charged particle moving on a (nonrotating) identically shaped loop with uniform magnetic field if only the external potential $V(\eta)$ is chosen appropriately. This last condition is due to the fact that the form of the centrifugal potential $W(\eta)$ is determined completely by the shape of the rotating loop, whereas the shape of the double-well potential $V(\eta)$ acting along the loop at rest is arbitrary, apart from its symmetry.

Based on these observations it is straightforward to conceive a situation in which the gauge term present in the rotating system leads to an observable effect. Consider a loop invariant under twofold rotations such that the distance of the points on the loop from the origin, $|z(\eta)|$, has just two maxima, as well as the symmetries

$$|z(\eta + \pi)| = |z(\eta)| \quad \text{and} \quad |z(-\eta)| = |z(\eta)|. \qquad (12)$$

An ellipse provides an example of such a loop.

Applying the instanton method as presented in Ref. [10] one obtains a semiclassical expression for the separation of the two lowest energy eigenvalues, the tunnel splitting $\Delta E \equiv E_1 - E_0$, as a function of the angular velocity Ω,

$$\Delta E(\Omega) = 2(2\Delta)\cos[\sigma_0(\Omega)/\hbar]\exp(-S_e^0/\hbar), \qquad (13)$$

where the Euclidean action of a single instanton, S_e^0, is associated with a path connecting the maxima of the inverted potential, $-W(\eta)$; the nonzero prefactor Δ takes into account the contributions of the quadratic fluctuations about the instanton path [10]. Contrary to systems with simply connected configuration space an additional cosine factor arises in the formula for the energy splitting. It is due to the nontrivial topology of the loop: there are *two* distinct paths connecting the maxima of the inverted potential. Application of the instanton method requires one to superpose contributions of all topologically inequivalent paths connecting one maximum with the other, which are characterized by different winding numbers. The summation over the multi-instanton contributions finally collapses into the multiplicative cosine function. Its argument, $\sigma_0(\Omega)$, is given by the integral of the gauge term $\alpha(\eta)$ from one minimum to the other, i.e., along single (shortest) instanton path,

$$\sigma_0(\Omega) = \int_{\text{inst}} d\eta\, \alpha(\eta), \qquad (14)$$

and thus corresponds to an extra phase. The splitting $\Delta E(\Omega)$ given in (13) is equal to zero if

$$\sigma_0(\Omega) = (k + \tfrac{1}{2})\pi\hbar, \qquad k \in \mathbb{Z}, \qquad (15)$$

which by using (9) turns into a condition on the angular velocity,

$$\Omega_k = (k + \tfrac{1}{2})h/2mF_{\text{loop}}, \qquad k \in \mathbb{Z}, \tag{16}$$

where the area enclosed by the loop is given by

$$F_{\text{loop}} = \frac{1}{2}\int_{\text{inst}} d\eta [\partial_\xi |g(w)|^2]_{\xi=\xi^*}. \tag{17}$$

Consequently, there is a discrete set of frequencies $\Omega_k, k \in \mathbb{Z}$, for which the ground state of the system is degenerate. This result derived with respect to the rotating frame K also holds in the rest frame K_0 since the transition from K to K_0 does not lift the degeneracy of the energy levels. Previously [10], condition (15) was interpreted naturally in terms of the flux quantum ϕ_0, implying that the tunnel splitting would vanish whenever the magnetic field through the loop leads to a flux ϕ being an odd-integer multiple of the flux quantum ϕ_0. Formally, a "quantum of rotation" could be introduced here that would allow for an analogous statement. Imagine an electron constrained to an elliptic loop enclosing an area of about 2×10^{-12} m^{-2} [11]; then the quantum of rotation would be of the order of $\Omega_0 \sim 5 \times 10^7$ s^{-1}.

There is a hand-waving argument that makes it plausible that a discrete set of frequencies Ω_k is singled out from all rotation velocities. Imagine that a wave emanates from a given point on the resting loop (an "aphelion," for example): it travels along the two branches and interferes at the other aphelion. If the loop rotates then the two traveling waves are subjected to different histories since one is "comoving" and the other is "countermoving" with respect to the sense of the rotation. This effect leads to a change in the "interference pattern" from constructive to destructive interference, and, *a fortiori*, to different stationary states.

The analogy between an external magnetic field and a constant rotation can be strengthened from a theoretical point of view by considering a double-slit experiment such as it is used for discussion of the Aharonov-Bohm effect [1]. At the same time possible technical difficulties due to the necessity to suspend and rotate a mesoscopic loop at high velocity can be circumvented in this way. Consider an experimental setup appropriate for a measurement of the Aharonov-Bohm effect such as described in [12]. However, the magnetic flux through the tiny fiber (or coil) immediately behind the slit is zero, but the total apparatus is assumed to rotate about an axis located at the position of the fiber. Clearly, the interference pattern on the screen where the split beams meet will show a sequence of maxima and minima as a function of the angular velocity Ω, much in the same way as is known from the Aharonov-Bohm experiment when the magnetic flux through the coil is varied. Again, this experiment is in striking contrast to the experiment performed with classical particles: the doubly peaked intensity pattern observed on the screen in the absence of rotation would only be subjected to an overall shift for nonzero angular velocity. For electrons (they can be considered as neutral particles since no magnetic field is present) one obtains a quantum of rotation Ω_0 of the order 50 s^{-1}; this value is much smaller than that for the mesoscopic ellipse because the area enclosed by the "loop" is larger by about a factor of 10^6. Even smaller angular velocities were sufficient in an experiment using neutrons; however, the size of the optical bench involved [13] is on the order of 10 m.

From a general point of view, it is illuminating to contrast the phenomena reported here to effects resulting from a geometric phase [14]. Consider a quantum system the Hamiltonian of which depends on an adjustable parameter Λ taking on values in a topologically nontrivial space. If, simultaneously with the regular time evolution, the system also traverses adiabatically a closed loop in parameter space, its wave function acquires an additional phase factor. It is possible to effectively describe this phenomenon by modifying the original Hamiltonian: appropriate vector and scalar potentials are introduced acting on a fictitious charge of the particle. This same structure is also seen to emerge when the Born-Oppenheimer approximation is made, leading to what is known now as the "gauge theory of molecular physics" [15]. In this case, a parameter-dependent Hamilton operator is obtained by assuming that, at each instant of time, the electrons of a molecule move in a slowly varying field due to the slowly changing position of the heavy nuclei. These effective (or reduced) descriptions have in common that they arise from division of one physical system into two parts with dynamics on different time scales. In the system studied here, however, the relevant Lagrangian does not depend on slowly varying parameters. The modification stemming from the transition to the rotating frame of reference does not involve an adiabatic procedure: the introduction of scalar and vector potentials represents an *exact* transformation.

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