A Mass Transference Principle and the Duffin-Schaeffer conjecture for Hausdorff measures

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Dedicated to Tatiana Beresnevich

Abstract

A Hausdorff measure version of the Duffin-Schaeffer conjecture in metric number theory is introduced and discussed. The general conjecture is established modulo the original conjecture. The key result is a Mass Transference Principle which allows us to transfer Lebesgue measure theoretic statements for lim sup subsets of $\mathbb{R}^k$ to Hausdorff measure theoretic statements. In view of this, the Lebesgue theory of lim sup sets is shown to underpin the general Hausdorff theory. This is rather surprising since the latter theory is viewed to be a subtle refinement of the former.

2000 *Mathematics Subject Classification*: Primary 11J83, 28A78; Secondary 11J13, 11K60

Keywords and phrases: Metric Diophantine approximation, the Duffin-Schaeffer conjecture and Hausdorff measures.

[*]Research supported by EPSRC GR/R90727/01
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1 Introduction

Throughout $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ will denote a real, positive function and will be referred to as an *approximating function.* Given an approximating function ψ, a point $\mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$ is called *simultaneously ψ–approximable* if there are infinitely many $q \in \mathbb{N}$ and $\mathbf{p} = (p_1, \ldots, p_k) \in \mathbb{Z}^k$ such that

$$\left| y_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q} \qquad (p_i, q) = 1\,, \quad 1 \leq i \leq k\,. \tag{1}$$

The set of simultaneously ψ–approximable points in $\mathbb{I}^k := [0,1]^k$ will be denoted by $\mathcal{S}_k(\psi)$. For convenience, we work within the unit cube $\mathbb{I}^k$ rather than $\mathbb{R}^k$ – it makes full measure results easier to state and avoids ambiguity. In fact, this is not at all restrictive as the set of simultaneously ψ-approximable points is invariant under translations by integer vectors.

The pairwise co-primeness condition imposed in the above definition clearly ensures that the rational points $(p_1/q, \ldots, p_k/q)$ are distinct. To some extent the approximation of points in $\mathbb{I}^k$ by *distinct* rational points should be the main feature when defining $\mathcal{S}_k(\psi)$ in which case pairwise co-primeness in (1) should be replaced by the condition that $(p_1, \ldots, p_k, q) = 1$. Clearly, both conditions coincide in the case $k = 1$. We shall return to this discussion in §6.2.

1.1 The Duffin-Schaeffer conjecture

On making use of the fact that $\mathcal{S}_k(\psi)$ is a lim sup set, a simple consequence of the Borel–Cantelli Lemma from probability theory is that

$$m(\mathcal{S}_k(\psi)) = 0 \qquad \text{if} \qquad \sum_{n=1}^{\infty} (\phi(n)\,\psi(n)/n)^k \ < \ \infty\,,$$

where m is k–dimensional Lebesgue measure and ϕ is the Euler function. In view of this, it is natural to ask: what happens if the above sum diverges? It is conjectured that $\mathcal{S}_k(\psi)$ is of full measure.

Conjecture 1 $$m(\mathcal{S}_k(\psi)) = 1 \qquad \text{if} \qquad \sum_{n=1}^{\infty} (\phi(n)\,\psi(n)/n)^k \ = \ \infty\,. \tag{2}$$

When $k = 1$, this is the famous Duffin-Schaeffer conjecture in metric number theory [2]. Although various partial results are know, it remains a major open problem and has attracted much attention (see [5] and references within). For $k \geq 2$, the conjecture was formally stated by Sprindžuk [9] and settled by Pollington & Vaughan [8].

Theorem PV *For $k \geq 2$, Conjecture 1 is true.*

If we assume that the approximating function ψ is monotonic, then we are in good shape thanks to Khintchine's fundamental result.

Khintchine's Theorem *If ψ is monotonic, then Conjecture 1 is true.*

Indeed, the whole point of Conjecture 1 is to remove the monotonicity condition on ψ from Khintchine's theorem. Note that in the case that ψ is monotonic, the convergence/divergence behavior of the sum in (2) is equivalent to that of $\sum \psi(n)^k$; i.e. the co-primeness condition imposed in (1) is irrelevant.

1.2 The Duffin-Schaeffer conjecture for Hausdorff measures

In this paper, we consider a generalization of Conjecture 1 which in our view is the 'real' problem and the truth of which yields a complete metric theory. Throughout, f is a dimension function and $\mathcal{H}^f$ denotes the Hausdorff f–measure – see §2.1. *Also, we assume that* $r^{-k}f(r)$ *is monotonic* – this is a natural condition which is not particularly restrictive. A straightforward covering argument making use of the lim sup nature of $\mathcal{S}_k(\psi)$ implies that

$$\mathcal{H}^f(\mathcal{S}_k(\psi)) = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} f(\psi(n)/n)\ \phi(n)^k \ < \ \infty\ . \tag{3}$$

In view of this, the following is a 'natural' generalization of Conjecture 1 and can be viewed as the Duffin-Schaeffer conjecture for Hausdorff measures.

Conjecture 2 $\quad \mathcal{H}^f(\mathcal{S}_k(\psi)) = \mathcal{H}^f(\mathbb{I}^k) \quad \text{if} \quad \sum_{n=1}^{\infty} f(\psi(n)/n)\ \phi(n)^k \ = \ \infty\ .$

Again, in the case that ψ is monotonic we are in good shape. This time, thanks to Jarník's fundamental result.

Jarník's Theorem *If ψ is monotonic, then Conjecture 2 is true.*

To be precise, the above theorem follows on combining Khintchine's theorem together with Jarník's theorem as stated in [1, §8.1] – the co-primeness condition imposed on the set $\mathcal{S}_k(\psi)$ is irrelevant since ψ is monotonic. The point is that in Jarník's original statement, various additional hypotheses on f and ψ were assumed and they would prevent us from stating the above clear cut version. Note that Jarník's theorem together with (3), imply precise Hausdorff dimension results for the sets $\mathcal{S}_k(\psi)$ – see [1, §1.2].

1.3 Statement of results

Regarding Conjecture 2, nothing seems to be known outside of Jarník's theorem which relies on ψ being monotonic. Of course, the whole point of Conjecture 2 is to remove the monotonicity condition from Jarník's theorem. Clearly, on taking $\mathcal{H}^f = m$ we have that

$$\textit{Conjecture 2} \quad \Longrightarrow \quad \textit{Conjecture 1}\ .$$

We shall prove the converse of this statement which turns out to have obvious but nevertheless rather unexpected consequences.

Theorem 1 *Conjecture 1* $\Longrightarrow$ *Conjecture 2.*

Theorem 1 together with Theorem PV gives:

Corollary 1 *For $k \geq 2$, Conjecture 2 is true.*

Theorem 1 gives:

Corollary 2 *Khintchine's Theorem* $\Longrightarrow$ *Jarník's Theorem.*

It is remarkable that Conjecture 1, which is only concerned with the metric theory of $\mathcal{S}_k(\psi)$ with respect to the ambient measure m, underpins the whole general metric theory. In particular, as a consequence of Corollary 2, if ψ is monotonic then Hausdorff dimension results for $\mathcal{S}_k(\psi)$ (i.e. the general form of the Jarník-Besicovitch Theorem) can in fact be obtained via Khintchine's Theorem. At first, this seems rather counterintuitive. In fact, the dimension results for monotonic ψ are a trivial consequence of Dirichlet's theorem (see §3.2).

The key to establishing Theorem 1 is the Mass Transference Principle of §3. In short, this allows us to transfer m-measure theoretic statements for lim sup subsets of $\mathbb{R}^k$ to $\mathcal{H}^f$-measure theoretic statements. In §6.1, we state a general Mass Transference Principle which allows us to obtain the analogue of Theorem 1 for lim sup subsets of locally compact metric spaces.

2 Preliminaries

Throughout (X, d) is a metric space such that for every $\rho > 0$ the space X can be covered by a countable collection of balls with diameters $< \rho$. A ball $B = B(x, r) := \{y \in X : d(x, y) \leqslant r\}$ is defined by a fixed centre and radius, although these in general are not uniquely determined by B as a set. By definition, B is a subset of X. For any $\lambda > 0$, we denote by λB the ball B scaled by a factor λ; i.e. $\lambda B(x, r) := B(x, \lambda r)$.

2.1 Hausdorff measures

In this section we give a brief account of Hausdorff measures. A *dimension function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotonic, non-decreasing function such that $f(r) \to 0$ as $r \to 0$. Given a ball $B = B(x, r)$, the quantity

$$V^f(B) := f(r) \tag{4}$$

will be referred to as the *f-volume of B*. If B is a ball in $\mathbb{R}^k$, m is k–dimensional Lebesgue measure and $f(x) = m(B(0,1))x^k$, then V^f is simply the volume of B in the usual geometric sense; i.e. $V^f(B) = m(B)$. In the case when $f(x) = x^s$ for some $s \geq 0$, we write V^s for V^f.

The Hausdorff f–measure with respect to the dimension function f will be denoted throughout by $\mathcal{H}^f$ and is defined as follows. Suppose F is a subset of (X, d). For $\rho > 0$, a countable collection $\{B_i\}$ of balls in X with $r(B_i) \leq \rho$ for each i such that $F \subset \bigcup_i B_i$ is called a *ρ-cover for F*. Clearly such a cover exists for every $\rho > 0$. For a dimension function f define

$$\mathcal{H}^f_\rho(F) \;=\; \inf \sum_i V^f(B_i),$$

where the infimum is taken over all ρ-covers of F. The *Hausdorff f–measure* $\mathcal{H}^f(F)$ of F with respect to the dimension function f is defined by

$$\mathcal{H}^f(F) := \lim_{\rho \to 0} \mathcal{H}^f_\rho(F) \;=\; \sup_{\rho > 0} \mathcal{H}^f_\rho(F)\ .$$

A simple consequence of the definition of $\mathcal{H}^f$ is the following useful fact.

Lemma 1 *If f and g are two dimension functions such that the ratio $f(r)/g(r) \to 0$ as $r \to 0$, then $\mathcal{H}^f(F) = 0$ whenever $\mathcal{H}^g(F) < \infty$.*

In the case that $f(r) = r^s$ $(s \geq 0)$, the measure $\mathcal{H}^f$ is the usual s–dimensional Hausdorff measure $\mathcal{H}^s$ and the Hausdorff dimension $\dim F$ of a set F is defined by

$$\dim\, F \;:=\; \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}\,.$$

In particular when s is an integer and $X = \mathbb{R}^s$, $\mathcal{H}^s$ is comparable to the s–dimensional Lebesgue measure. Actually, $\mathcal{H}^s$ is a constant multiple of the s–dimensional Lebesgue measure but we shall not need this stronger statement.

For further details see [3, 7]. A general and classical method for obtaining a lower bound for the Hausdorff f-measure of an arbitrary set F is the following mass distribution principle.

Lemma (Mass Distribution Principle). *Let μ be a probability measure supported on a subset F of (X, d). Suppose there are positive constants c and r_o such that*

$$\mu(B) \;\leq\; c\, V^f(B)$$

for any ball B with radius $r \leq r_o$. If E is a subset of F with $\mu(E) = \lambda > 0$ then $\mathcal{H}^f(E) \geq \lambda/c$.

Proof : If $\{B_i\}$ is a ρ–cover of E with $\rho \leq r_o$ then

$$\lambda = \mu(E) = \mu\left(\cup_i B_i\right) \leq \sum_i \mu\left(B_i\right) \leq c \sum_i V^f(B_i)\ .$$

It follows that $\mathcal{H}^f_\rho(E) \geq \lambda/c$ for any $\rho \leq r_o$. On letting $\rho \to 0$, the quantity $\mathcal{H}^f_\rho(E)$ increases and so we obtain the required result.

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The following basic covering lemma will be required at various stages [6, 7].

Lemma 2 (The $5r$ covering lemma) *Every family $\mathcal{F}$ of balls of uniformly bounded diameter in a metric space (X,d) contains a disjoint subfamily $\mathcal{G}$ such that*

$$\bigcup_{B\in\mathcal{F}} B \ \subset \ \bigcup_{B\in\mathcal{G}} 5B \quad .$$

2.2 Positive and full measure sets

Let μ be a finite measure supported on (X,d). The measure μ is said to be *doubling* if there exists a constant $\lambda > 1$ such that for $x \in X$

$$\mu(B(x,2r)) \ \leq \ \lambda\, \mu(B(x,r)) \ .$$

Clearly, the measure $\mathcal{H}^k$ is a doubling measure on $\mathbb{R}^k$. In this section we state two measure theoretic results which will be required during the course of the paper.

Lemma 3 *Let (X,d) be a metric space and let μ be a finite doubling measure on X such that any open set is μ measurable. Let E be a Borel subset of X. Assume that there are constants $r_0, c > 0$ such that for any ball B with $r(B) < r_0$ we have that $\mu(E\cap B) \geqslant c\ \mu(B)$. Then, for any ball B*

$$\mu(E\cap B) \ = \ \mu(B) \ .$$

Lemma 4 *Let (X,d) be a metric space and μ be a finite measure on X. Let B be a ball in X and E_n a sequence of μ–measurable sets. Suppose there exists a constant $c > 0$ such that $\limsup_{n\to\infty} \mu(B \cap E_n) \geqslant c\ \mu(B)$. Then*

$$\mu(B \cap \limsup_{n\to\infty} E_n) \ \geqslant \ c^2\, \mu(B) \quad .$$

For the details regarding these two lemmas see [1, §8].

3 A Mass Transference Principle

Given a dimension function f and a ball $B = B(x,r)$ in $\mathbb{R}^k$, we define another ball

$$B^f := B(x, f(r)^{1/k}) \ . \tag{5}$$

When $f(x) = x^s$ for some $s > 0$ we also adopt the notation B^s, *i.e.* $B^s := B^{(x\mapsto x^s)}$. It is readily verified that

$$B^k = B. \tag{6}$$

Next, given a collection K of balls in $\mathbb{R}^k$, denote by K^f the collection of balls obtained from K under the transformation (5); i.e. $K^f := \{B^f : B \in K\}$.

The following property immediately follows from (4), (5) and (6):

$$V^k(B^f) = V^f(B^k) \qquad \text{for any ball } B. \tag{7}$$

Note that (7) could have been taken to be a definition in which case (5) would follow.

Recall that $\mathcal{H}^k$ is comparable to the k-dimensional Lebesgue measure m. Trivially, for any ball B we have that $V^k(B)$ is comparable to $m(B)$. Thus there are constants $0 < c_1 < 1 < c_2 < \infty$ such that for any ball B

$$c_1\ V^k(B) \leqslant \mathcal{H}^k(B) \leqslant c_2\ V^k(B). \tag{8}$$

In fact, we have the stronger statement that $\mathcal{H}^k(B)$ is a constant multiple of $V^k(B)$. However, the analogue of this stronger statement is not necessarily true in the general framework considered in §6.1 whereas (8) is. Therefore, we have opted to work with (8) even in our current setup. Given a sequence of balls B_i, $i = 1, 2, 3, \ldots$, as usually its limsup set is

$$\limsup_{i\to\infty} B_i := \bigcap_{j=1}^{\infty} \bigcup_{i\geqslant j} B_i\ .$$

The following theorem is without doubt the main result of this paper. It is the key to establishing the Duffin-Schaeffer conjecture for Hausdorff measures.

Theorem 2 (Mass Transference Principle) *Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^k$ with $r(B_i) \to 0$ as $i \to \infty$. Let f be a dimension function such that $x^{-k}f(x)$ is monotonic and suppose that for any ball B in $\mathbb{R}^k$*

$$\mathcal{H}^k\big(B \cap \limsup_{i\to\infty} B_i^f\,\big) = \mathcal{H}^k(B)\ . \tag{9}$$

Then, for any ball B in $\mathbb{R}^k$

$$\mathcal{H}^f\big(B \cap \limsup_{i\to\infty} B_i^k\,\big) = \mathcal{H}^f(B)\ .$$

Remark 1. $\mathcal{H}^k$ is comparable to the Lebesgue measure m in $\mathbb{R}^k$. Thus (9) simply states that the set $\limsup B_i^f$ is of full m measure in $\mathbb{R}^k$, *i.e.* its complement in $\mathbb{R}^k$ is of m measure zero.

Remark 2. In the statement of Theorem 2 the condition $r(B_i) \to 0$ as $i \to \infty$ is redundant. However, it is included to avoid unnecessary further discussion.

Remark 3. If $x^{-k}f(x) \to l$ as $x \to 0$ and l is finite then the above statement is relatively straightforward to establish. The main substance of the Mass Transference Principle is when $x^{-k}f(x) \to \infty$ as $x \to 0$. In this case, it trivially follows via Lemma 1 that $\mathcal{H}^f(B) = \infty$.

3.1 Proof of Theorem 1

First of all let us dispose of the case that $\psi(r)/r \nrightarrow 0$ as $r \to \infty$. Then trivially, $\mathcal{S}_k(\psi) = \mathbb{I}^k$ and the result is trivial. Without loss of generality, assume that $\psi(r)/r \to 0$ as $r \to \infty$. We are given that $\sum f(\psi(n)/n)\ \phi(n)^k = \infty$. Let $\theta(r) := r\, f(\psi(r)/r)^{1/k}$. Then θ is an approximating function and $\sum(\phi(n)\,\theta(n)/n)^k = \infty$. Thus, on using the supremum norm, Conjecture 1 implies that $\mathcal{H}^k(B \cap \mathcal{S}_k(\theta)) = \mathcal{H}^k(B \cap \mathbb{I}^k)$ for any ball B in $\mathbb{R}^k$. It now follows via the Mass Transference Principle that $\mathcal{H}^f(\mathcal{S}_k(\psi)) = \mathcal{H}^f(\mathbb{I}^k)$ and this completes the proof of Theorem 1.

3.2 The Jarník-Besicovitch Theorem

In the case $k = 1$ and $\psi(x) := x^{-\tau}$, let us write $\mathcal{S}(\tau)$ for $\mathcal{S}_k(\psi)$. The Jarník-Besicovitch theorem states that $\dim \mathcal{S}(\tau) = d := 2/(1+\tau)$ for $\tau > 1$. This fundamental result is easily deduced on combining Dirichlet's theorem with the Mass Transference Principle.

Dirichlet's theorem states that for any irrational $y \in \mathbb{R}$, there exists infintely many reduced rationals p/q $(q > 0)$ such that $|y - p/q| \leq q^{-2}$. With $f(x) := x^d$, (9) is trivially satisfied and the Mass Transference Principle implies that $\mathcal{H}^d(\mathcal{S}(\tau)) = \infty$. Hence $\dim \mathcal{S}(\tau) \geq d$. The upper bound is trivial. Note that we have actually proved a lot more than simply the Jarník-Besicovitch theorem. We have proved that the s–dimensional Hausdorff measure $\mathcal{H}^s$ of $\mathcal{S}(\tau)$ at the critical exponent $s = d$ is infinite.

4 The $K_{G,B}$ covering lemma

Before establishing the Mass Transference Principle we state and prove the following covering lemma, which provides an equivalent description of the full measure property (9).

Lemma 5 (The $K_{G,B}$ lemma) *Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^k$ with $r(B_i) \to 0$ as $i \to \infty$. Let f be a dimension function and for any ball B in $\mathbb{R}^k$ suppose that (9) is satisfied. Then for any B and any $G > 1$ there is a finite sub-collection $K_{G,B} \subset \{B_i : i \geqslant G\}$ such that the corresponding balls in $K^f_{G,B}$ are disjoint, lie inside B and*

$$\mathcal{H}^k\Big(\bigcup_{L\in K^f_{G,B}}^{\circ} L\Big) \geqslant \kappa\, \mathcal{H}^k(B) \qquad \textit{with} \quad \kappa := \tfrac{1}{2}(\tfrac{c_1}{c_2})^2 10^{-k} \quad . \tag{10}$$

Proof of Lemma 5. Let $\mathcal{F} := \{B_i^f : B_i^f \cap \frac{1}{2}B \neq \emptyset,\ i \geqslant G\}$. Since, $f(x) \to 0$ as $x \to 0$ and $r(B_i) \to 0$ as $i \to \infty$ we can ensure that every ball in $\mathcal{F}$ is contained in B for i sufficiently large. In view of the $5r$ covering lemma (Lemma 2), there exists a disjoint sub-family $\mathcal{G}$ such that

$$\bigcup_{B_i^f\in\mathcal{F}} B_i^f \subset \bigcup_{B_i^f\in\mathcal{G}} 5B_i^f \quad .$$

It follows that

$$\mathcal{H}^k\left(\bigcup_{B_i^f\in\mathcal{G}} 5B_i^f\right) \;\geq\; \mathcal{H}^k\left(\tfrac{1}{2}B\cap\limsup_{i\to\infty}B_i^f\right) \;\overset{(9)}{=}\; \mathcal{H}^k\left(\tfrac{1}{2}B\right) \;\overset{(8)}{\geq}\; \frac{c_1}{c_2}\,2^{-k}\,\mathcal{H}^k(B)\ .$$

However, since $\mathcal{G}$ is a disjoint collection of balls we have that

$$\mathcal{H}^k\left(\bigcup_{B_i^f\in\mathcal{G}} 5B_i^f\right) \;\overset{(8)}{\leq}\; \frac{c_2}{c_1}\,5^k\;\;\mathcal{H}^k\left(\mathop{\dot{\bigcup}}_{B_i^f\in\mathcal{G}} B_i^f\right)\ .$$

Thus,

$$\mathcal{H}^k\left(\mathop{\dot{\bigcup}}_{B_i^f\in\mathcal{G}} B_i^f\right) \;\geq\; \left(\frac{c_1}{c_2}\right)^2\;10^{-k}\;\;\mathcal{H}^k(B)\ . \tag{11}$$

The balls $B_i^f\in\mathcal{G}$ are disjoint, and since $r(B_i^f)\to 0$ as $i\to\infty$ we have that

$$\mathcal{H}^k\left(\mathop{\dot{\bigcup}}_{B_i^f\in\mathcal{G}\,:\,i\geq j} B_i^f\right) \;\to\; 0 \qquad \text{as} \quad j\to\infty\ .$$

Thus, there exists some $j_0>G$ for which

$$\mathcal{H}^k\left(\mathop{\dot{\bigcup}}_{B_i^f\in\mathcal{G}\,:\,i\geq j_0} B_i^f\right) \;<\; \frac{1}{2}\left(\frac{c_1}{c_2}\right)^2\;10^{-k}\;\;\mathcal{H}^k(B)\ . \tag{12}$$

Now let $K_{G,B}:=\{B_i : B_i^f\in\mathcal{G}, i<j_0\}$. Clearly, this is a finite sub-collection of $\{B_i\,:\,i\geqslant G\}$. Moreover, in view of (11) and (12) the collection $K_{G,B}^f$ satisfies the desired properties.

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Lemma 5 shows that the full measure property (9) of the Mass Transference Principle implies the existence of the collection $K_{G,B}^f$ satisfying (10) of the $K_{G,B}$ Lemma. For completeness, we prove that the converse is also true.

Lemma 6 *Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^k$ with $r(B_i)\to 0$ as $i\to\infty$. Let f be a dimension function and for any ball B and any $G>1$, assume that there is a collection $K_{G,B}^f$ of balls satisfying (10) of Lemma 5. Then, for any ball B the full measure property (9) of the Mass Transference Principle is satisfied.*

Proof of Lemma 6. For any ball B and any $G\in\mathbb{N}$, the collection $K_{G,B}^f$ is contained in B and is a finite sub-collection of $\{B_i^f\}$ with $i\geqslant G$. We define

$$E_G\;:=\;\bigcup_{L\in K_{G,B}^f} L\ .$$

Since $K^f_{G,B}$ is finite, we have that

$$\limsup_{G\to\infty} E_G \ \subset\ B\cap\limsup_{i\to\infty} B_i^f \ .$$

It follows from (10) that $\mathcal{H}^k(E_G)\geqslant\kappa\,\mathcal{H}^k(B)$ which together with Lemma 4 implies that $\mathcal{H}^k(\limsup_{G\to\infty}E_G)\geqslant\kappa^2\,\mathcal{H}^k(B)$. Hence, $\mathcal{H}^k(B\cap\limsup_{i\to\infty}B_i^f)\geqslant\kappa^2\,\mathcal{H}^k(B)$. The measure $\mathcal{H}^k$ is doubling and so the statement of the lemma follows on applying Lemma 3.

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In short, Lemmas 5 and 6 establish the equivalence: (9) $\iff$ (10).

5 Proof of Theorem 2 (Mass Transference Principle)

We start by considering the case that $x^{-k}f(x)\to l$ as $x\to 0$ and l is finite. If $l=0$, then Lemma 1 implies that $\mathcal{H}^f(B)=0$ and since $B\cap\limsup B_i^k\subset B$ the result follows. If $l\neq 0$ and is finite then $\mathcal{H}^f$ is comparable to $\mathcal{H}^k$ (in fact, $\mathcal{H}^f=l\,\mathcal{H}^k$). Therefore the required statement follows on showing that $\mathcal{H}^k\big(B\cap\limsup_{i\to\infty}B_i^k\big)=\mathcal{H}^k(B)$. This can be established by first noting that the ratio of the radii of the balls B_i^k and B_i^f are uniformly bounded between positive constants and then adapting the proof of Lemma 6 in the obvious manner.

In view of the above discussion, we can assume without loss of generality that

$$x^{-k}f(x)\ \to\ \infty \qquad \text{as} \qquad x\to 0\ .$$

Note that in this case, it trivially follows via Lemma 1 that $\mathcal{H}^f(B)=\infty$. Fix some arbitrary bounded ball B_0 of $\mathbb{R}^k$. The statement of the Mass Transference Principle will therefore follow on showing that

$$\mathcal{H}^f(B_0\cap\limsup B_i)=\infty\ .$$

To achieve this we proceed as follows. For any constant $\eta>1$, our aim is to construct a Cantor subset $\mathbb{K}_\eta$ of $B_0\cap\limsup B_i$ and a probability measure μ supported on $\mathbb{K}_\eta$ satisfying the condition that for an arbitrary ball A of sufficiently small radius $r(A)$

$$\mu(A)\ \ll\ \frac{V^f(A)}{\eta}\ , \tag{13}$$

where the implied constant in the Vinogradov symbol ($\ll$) is absolute. By the Mass Distribution Principle, the above inequality implies that

$$\mathcal{H}^f(\mathbb{K}_\eta)\ \gg\ \eta\ .$$

Since $\mathbb{K}_\eta\subset B_0\cap\limsup B_i$, we obtain that $\mathcal{H}^f\left(B_0\cap\limsup B_i\right)\gg\eta$. However, η can be made arbitrarily large whence $\mathcal{H}^f\left(B_0\cap\limsup B_i\right)=\infty$ and this proves Theorem 2.

In view of the above outline, the whole strategy of our proof is centred around the construction of a 'right type' of Cantor set $\mathbb{K}_\eta$ which supports a measure μ with the desired property.

5.1 The desired properties of $\mathbb{K}_\eta$

In this section we summarize the desired properties of the Cantor set $\mathbb{K}_\eta$. The existence of $\mathbb{K}_\eta$ will be established in the next section. Let

$$\mathbb{K}_\eta := \bigcap_{n=1}^{\infty} \mathbb{K}(n) \ ,$$

where each *level* $\mathbb{K}(n)$ is a finite union of disjoint balls such that

$$\mathbb{K}(1) \supset \mathbb{K}(2) \supset \mathbb{K}(3) \supset \ldots \ .$$

Thus, the levels are nested. Moreover, if $K(n)$ denotes the collection of balls which constitute level n, then $K(n) \subset \{B_i : i \in \mathbb{N}\}$ for each $n \geq 2$. We will define $K(1) := B_0$. It is then clear that $\mathbb{K}_\eta$ is a subset of $B_0 \cap \limsup B_i$. It will be convenient to also refer to the collection $K(n)$ as the n'th level. Strictly speaking, $\mathbb{K}(n) = \bigcup_{B \in K(n)} B$ is the n'th level. However, from the context it will be clear what we mean and no ambiguity should arise.

The construction is inductive and the general idea is as follows. Suppose the $(n-1)$'th level $\mathbb{K}(n-1)$ has been constructed. The next level is constructed by 'looking' locally at each ball from the previous level. More precisely, for every ball $B \in K(n-1)$ we construct the (n, B)–*local level* denoted by $K(n, B)$ consisting of balls contained in B. Thus

$$K(n) \ := \bigcup_{B \in K(n-1)} K(n, B) \qquad \text{and} \qquad \mathbb{K}(n) \ := \bigcup_{B \in K(n-1)} \mathbb{K}(n, B) \ ,$$

where

$$\mathbb{K}(n, B) \ := \bigcup_{L \in K(n,B)} L \ = \ B \cap \mathbb{K}(n) \ .$$

As mentioned above, the balls in each level will be disjoint. Moreover, we ensure that balls in each level scaled by a factor of three are disjoint. This is property (P1) below. This alone is not sufficient to obtain the required lower bound for $\mathcal{H}^f(\mathbb{K}_\eta)$. For this purpose, every local level will be defined as a union of *local sub-levels.* The (n, B)–local level will take on the following form

$$K(n, B) := \bigcup_{i=1}^{l_B} K(n, B, i) \ ,$$

where l_B is the number of local sub-levels (see property (P5) below) and $K(n, B, i)$ is the i-th local sub-level. Within each local sub-level $K(n, B, i)$, the separation of balls is much more demanding than simply property (P1) and is given by property (P2) below.

To achieve our main objective – the lower bound for $\mathcal{H}^f(\mathbb{K}_\eta)$ – we will require a controlled build up of 'mass' on the balls in every sub-level. The mass is related to the f-volume V^f of the balls in the construction and the overall number of sub-levels. These are governed by properties (P3) and (P5) below.

Finally, we will require that the f-volume of balls from one sub-level to the next decreases sufficiently fast. This is property (P4) below. However, the total f-volume within any one sub-level remains about the same. This is a consequence of property (P3) below.

We now formally state the properties (P1) – (P5) discussed above together with a trivial property (P0).

The properties of levels and sub-levels of $\mathbb{K}_\eta$

(P0) $K(1)$ consists of one ball, namely B_0.

(P1) For any $n \geqslant 2$ and any $B \in K(n-1)$ the balls

$$\{3L \ : \ L \in K(n,B)\}$$

are disjoint and contained in B and $3L \subset L^f$.

(P2) For any $n \geqslant 2$, $B \in K(n-1)$ and any $i \in \{1 \ldots, l_B\}$ the balls

$$\{L^f \ : \ L \in K(n,B,i)\}$$

are disjoint and contained in B.

(P3) For any $n \geqslant 2$, $B \in K(n-1)$ and $i \in \{1 \ldots, l_B\}$

$$\sum_{L \in K(n,B,i)} V^k(L^f) \ \geqslant \ c_3 \, V^k(B),$$

where $c_3 := \frac{\kappa \, c_1^2}{2 \, c_2^2 \, 10^k} > 0$ is an absolute constant.

(P4) For any $n \geqslant 2$, $B \in K(n-1)$, any $i \in \{1 \ldots, l_B - 1\}$ and any $L \in K(n,B,i)$ and $M \in K(n,B,i+1)$

$$V^f(M) \leqslant \frac{1}{2} \, V^f(L).$$

(P5) The number of local sub-levels is defined by

$$l_B := \begin{cases} \left[\dfrac{c_2 \, \eta}{c_3 \, \mathcal{H}^k(B)} \right] + 1 \quad , & \text{if } B = B_0 := \mathbb{K}(1), \\[2ex] \left[\dfrac{V^f(B)}{c_3 \, V^k(B)} \right] + 1 \quad , & \text{if } B \in K(n) \text{ with } n \geqslant 2 \end{cases}$$

and satisfies $l_B \geqslant 2$ for $B \in K(n)$ with $n \geqslant 2$.

5.2 The existence of $\mathbb{K}_\eta$

In this section we show that it is indeed possible to construct a Cantor set $\mathbb{K}_\eta$ with the desired properties as discussed in the previous section. We will use the notation

$$K_l(n,B) \ := \ \bigcup_{i=1}^{l} K(n,B,i) \ .$$

Thus, $K(n,B)$ is simply $K_{l_B}(n,B)$.

LEVEL 1. This is defined by taking the arbitrary ball B_0. Thus, $\mathbb{K}(1) := B_0$ and property (P0) is trivially satisfied.

We proceed by induction. Assume that the first $(n-1)$ levels $\mathbb{K}(1), \mathbb{K}(2), \ldots, \mathbb{K}(n-1)$ have been constructed. We now construct the n'th level $\mathbb{K}(n)$.

LEVEL n. To construct this level we construct local levels $K(n,B)$ for each $B \in K(n-1)$. Recall, that each local level $K(n,B)$ will consist of sub-levels $K(n,B,i)$ where $1 \leq i \leq l_B$ and l_B is given by property (P5). Therefore, fix some ball $B \in K(n-1)$ and a sufficiently small constant $\varepsilon = \varepsilon(B) > 0$ which will be determined later. Let G be sufficiently large so that

$$r(3B_i) < r(B_i^f) \qquad \text{whenever} \qquad i \geqslant G \tag{14}$$

$$\frac{V^k(B_i)}{V^f(B_i)} < \varepsilon \, \frac{V^k(B)}{V^f(B)} \qquad \text{whenever} \qquad i \geqslant G \tag{15}$$

and

$$\left[\frac{V^f(B_i)}{c_3 \, V^k(B_i)}\right] \; \geq \; 1 \qquad \text{whenever} \qquad i \geqslant G \; , \tag{16}$$

where c_3 is the constant appearing in property (P3) above. This is possible since $x^k/f(x) \to 0$ as $x \to 0$. Now let $\mathcal{C}_G := \{B_i : i \geq G\}$. The local level $K(n,B)$ will be constructed to be a finite, disjoint sub-collection of $\mathcal{C}_G$. Thus, (14) – (16) are satisfied for any ball B_i in $K(n,B)$. In particular, (16) implies that $l_{B_i} \geq 2$ and so property (P5) will automatically be satisfied for balls in $K(n,B)$.

Sub-level 1. With B and G as above, let $K_{G,B}$ denote the collection of balls arising from Lemma 5. Note, that in view of (14) the collection $K_{G,B}$ is a disjoint collection of balls. Define the first sub-level of $K(n,B)$ to be $K_{G,B}$; that is

$$K(n,B,1) \; := \; K_{G,B} \; .$$

By Lemma 5, it is clear that (P2) and (P3) are fulfilled for $i=1$. By (14) and the fact that the balls in $K^f_{G,B}$ are disjoint, we also have that (P1) is satisfied within this first sub-level. Clearly, $K(n,B,1) \subset \mathcal{C}_G$.

Higher sub-levels. To construct higher sub-levels we argue by induction. For $l < l_B$, assume that we have constructed the sub-levels $K(n,B,1), \ldots, K(n,B,l)$ satisfying properties (P1) – (P4) with l_B replaced by l and such that $K_l(n,B) \subset \mathcal{C}_G$. In view of the latter, (14) – (16) are satisfied for any ball L in $K_l(n,B)$. In particular, in view of (16), for any ball L in $K_l(n,B)$ property (P5) is trivially satisfied; i.e. $l_L \geq 2$. We now construct the next sub-level $K(n,B,l+1)$.

As every sub-level of the construction has to be well separated from the previous ones, we first verify that there is enough 'space' left over in B once we have removed the sub-levels

$K(n,B,1),\ldots,K(n,B,l)$ from B. More precisely, let

$$A^{(l)} \ :=\ \tfrac{1}{2}B \ \setminus \bigcup_{L\in K_l(n,B)} 4L\ .$$

We show that

$$\mathcal{H}^k\big(A^{(l)}\big) \geqslant \frac{1}{2}\,\mathcal{H}^k(\tfrac{1}{2}B)\ . \tag{17}$$

By construction and the fact that $l<l_B$,

$$\begin{aligned}
\mathcal{H}^k\Big(\bigcup_{L\in K_l(n,B)} 4L\Big) \ &\leqslant \sum_{L\in K_l(n,B)} \mathcal{H}^k(4L) \\
&\overset{(8)}{\leqslant}\ 4^k c_2 \sum_{L\in K_l(n,B)} V^k(L) \ =\ 4^k c_2 \sum_{L\in K_l(n,B)} V^f(L)\,\frac{V^k(L)}{V^f(L)} \\
&\overset{(15)}{\leqslant}\ 4^k c_2 \sum_{L\in K_l(n,B)} V^f(L)\ \varepsilon\ \frac{V^k(B)}{V^f(B)} \\
&\overset{(7)}{=}\ 4^k c_2\ \varepsilon\ \frac{V^k(B)}{V^f(B)} \sum_{i=1}^{l}\ \sum_{L\in K(n,B,i)} V^k(L^f) \\
&\overset{(8)}{\leqslant}\ \frac{4^k c_2\ \varepsilon}{c_1}\ \frac{V^k(B)}{V^f(B)} \sum_{i=1}^{l}\ \sum_{L\in K(n,B,i)} \mathcal{H}^k(L^f) \\
&\overset{(P2)}{\leqslant}\ \frac{4^k c_2\ \varepsilon}{c_1}\ \frac{V^k(B)}{V^f(B)}\ (l_B-1)\ \mathcal{H}^k(B)\ .
\end{aligned} \tag{18}$$

Now, if $B=B_0$ let

$$\varepsilon=\varepsilon(B_o)\ :=\ \frac{1}{2}\left(\frac{c_1}{c_2}\right)^2 \frac{c_3}{2^k\,4^k}\ \frac{V^f(B_0)}{\eta}\ .$$

If $B\neq B_0$, so that $B\in K(n)$ for some $n\geq 2$, let $\varepsilon := \varepsilon(B_0)\times(\eta/V^f(B_0))$ – an absolute constant independent of B, B_0 and η. It then follows from (18), (P5) and (8) that

$$\mathcal{H}^k\Big(\bigcup_{L\in K_l(n,B)} 4L\Big)\ \leqslant\ \frac{1}{2}\,\mathcal{H}^k(\tfrac{1}{2}B)\ ,$$

and this clearly establishes (17).

By construction, $K_l(n,B)$ is a finite collection of balls and so $d_{\min} := \min\{r(L) : L\in K_l(n,B)\}$ is well defined. Let $B^{(l)}$ denote a generic ball of diameter $d_{\min}$. At each point of $A^{(l)}$ place a ball $B^{(l)}$ and denote this collection by $\mathcal{A}^{(l)}$. By the $5r$–covering lemma (Lemma 2), there exists a disjoint sub-collection $\mathcal{G}^{(l)}$ such that

$$A^{(l)}\ \subset \bigcup_{B^{(l)}\in\mathcal{A}^{(l)}} B^{(l)}\ \subset \bigcup_{B^{(l)}\in\mathcal{G}^{(l)}} 5B^{(l)}\quad .$$

The collection $\mathcal{G}^{(l)}$ is clearly contained within B and it is finite – the balls are disjoint and all of the same size. Moreover, by construction

$$B^{(l)} \cap \bigcup_{L \in K_l(n,B)} 3L = \emptyset \qquad \text{for any } B^{(l)} \in \mathcal{G}^{(l)} \; ; \tag{19}$$

i.e. the balls in $\mathcal{G}^{(l)}$ do not intersect any of the $3L$ balls from the previous sub-levels. It follows that

$$\mathcal{H}^k\big(\bigcup_{B^{(l)} \in \mathcal{G}^{(l)}} 5B^{(l)}\big) \;\geq\; \mathcal{H}^k(A^{(l)}) \;\overset{(17)}{\geq}\; \tfrac{1}{2}\, \mathcal{H}^k(\tfrac{1}{2}B) \, .$$

On the other hand, since $\mathcal{G}^{(l)}$ is a disjoint collection of balls we have that

$$\mathcal{H}^k\big(\bigcup_{B^{(l)} \in \mathcal{G}^{(l)}} 5B^{(l)}\big) \;\overset{(8)}{\leq}\; \frac{c_2}{c_1}\, 5^k \;\; \mathcal{H}^k\big(\mathop{\dot{\bigcup}}_{B^{(l)} \in \mathcal{G}^{(l)}} B^{(l)}\big) \, ,$$

and so

$$\mathcal{H}^k\big(\mathop{\dot{\bigcup}}_{B^{(l)} \in \mathcal{G}^{(l)}} B^{(l)}\big) \;\geqslant\; \frac{c_1}{2c_2 5^k} \;\; \mathcal{H}^k(\tfrac{1}{2}B) \, . \tag{20}$$

We are now in the position to construct the $(l+1)$'th sub-level $K(n,B,l+1)$. To this end, let $G' \geq G$ be sufficiently large so that for every $i \geqslant G'$

$$V^f(B_i) \;\leqslant\; \frac{1}{2} \min_{L \in K_l(n,B)} V^f(L) \, . \tag{21}$$

We recall that $\{B_i\}$ is the original sequence of balls in Theorem 2. The number on the right of (21) is well defined and positive as there are only finitely many balls in $K_l(n,B)$. Furthermore, (21) is possible since $\lim_{i\to\infty} r(B_i) = 0$ and $\lim_{x\to 0} f(x) = 0$. Now to each ball $B^{(l)} \in \mathcal{G}^{(l)}$ we apply Lemma 5 to obtain a collection $K_{G',B^{(l)}}$ and define

$$K(n,B,l+1) := \bigcup_{B^{(l)} \in \mathcal{G}^{(l)}} K_{G',B^{(l)}} \; .$$

Note that since $G' \geq G$, (14) – (16) remain valid and $K(n,B,l+1) \subset \mathcal{C}_G$. We now verify properties (P1)– (P5) for this sub-level.

In view of Lemma 5, for any $B^{(l)}$ in $\mathcal{G}^{(l)}$ the collection $K^f_{G',B^{(l)}}$ is disjoint and contained within $B^{(l)}$. This together with (14) establishes property (P1) for balls L in $K_{G',B^{(l)}}$. Since the balls $B^{(l)}$ in $\mathcal{G}^{(l)}$ are disjoint and contained within B, we have that (P1) is satisfied for balls L in $K(n,B,l+1)$. In turn, this together with (19) implies property (P1) for balls L in $K_{l+1}(n,B)$. Clearly, the above argument also verifies property (P2) for balls L in $K(n,B,l+1)$. The following establishes property (P3) for $i = l+1$:

$$\sum_{L \in K(n,B,l+1)} V^k(L^f) \;=\; \sum_{B^{(l)} \in \mathcal{G}^{(l)}} \;\; \sum_{L \in K_{G',B^{(l)}}} V^k(L^f) \;\overset{(8)}{\geqslant}\; \frac{1}{c_2} \sum_{B^{(l)} \in \mathcal{G}^{(l)}} \;\; \sum_{L \in K_{G',B^{(l)}}} \mathcal{H}^k(L^f) \;\geqslant$$

$$\overset{(10)}{\geqslant} \frac{\kappa}{c_2} \sum_{B^{(l)} \in \mathcal{G}^{(l)}} \mathcal{H}^k(B^{(l)}) \overset{(20)}{\geqslant} \frac{\kappa}{c_2} \frac{c_1}{2c_2 5^k} \ \mathcal{H}^k(\tfrac{1}{2}B) \overset{(8)}{\geqslant} \frac{\kappa c_1^2}{2c_2^2 10^k} \ V^k(B) \ := \ c_3 \ V^k(B) \ .$$

Property (P4) is trivially satisfied as we have imposed condition (21). Finally, in view of (16), for any ball L in $K(n, B, l+1)$ property (P5) is satisfied; i.e. $l_L \geq 2$.

The upshot is that (P1) – (P5) are satisfied up to the local sub-level $K(n, l+1, B)$ and so completes the inductive step. This establishes the existence of the local level $K(n, B) := K_{l_B}(n, B)$ for each $B \in K(n-1)$ and thereby the existence of the n'th level $K(n)$.

5.3 The measure μ on $\mathbb{K}_\eta$

In this section, we define a probability measure μ supported on $\mathbb{K}_\eta$. We will eventually show that the measure satisfies (13). For any ball $L \in K(n)$, we attach a weight $\mu(L)$ defined recursively as follows.

For $n = 1$, we have that $L = B_0 := \mathbb{K}(1)$ and we set $\mu(L) := 1$.

For $n \geq 2$, let L be a ball in $K(n)$. By construction, there is a unique ball $B \in K(n-1)$ such that $L \subset B$. We set

$$\mu(L) \ := \ \frac{V^f(L)}{\sum\limits_{M \in K(n,B)} V^f(M)} \ \times \ \mu(B) \ .$$

This procedure thus defines inductively a mass on any ball appearing in the construction of $\mathbb{K}_\eta$. In fact a lot more is true — μ can be further extended to all Borel subsets F of $\mathbb{R}^k$ to determine $\mu(F)$ so that μ constructed as above actually defines a measure supported on $\mathbb{K}_\eta$; see Proposition 1.7 [3]. We state this formally as a

Fact. The probability measure μ constructed above is supported on $\mathbb{K}_\eta$ and for any Borel subset F of $\mathbb{R}^k$

$$\mu(F) := \mu(F \cap \mathbb{K}_\eta) \ = \ \inf \sum_{L \in \mathcal{C}(F)} \mu(L) \ ,$$

where the infimum is taken over all coverings $\mathcal{C}(F)$ of $F \cap \mathbb{K}_\eta$ by balls $L \in \bigcup_{n \in \mathbb{N}} K(n)$.

5.4 The measure of a ball in the Cantor construction

With $n \geq 2$, the aim of this section is to show that for any ball L in $K(n)$ we have that

$$\mu(L) \ \leqslant \ \frac{V^f(L)}{\eta} \ ; \qquad (22)$$

i.e. (13) is satisfied for balls in the Cantor construction. We start with level $n = 2$ and fix a ball $L \in K(2) = K(2, B_0)$ – recall that $B_0 = \mathbb{K}(1)$. Also, recall that $B = B^k$ for any ball B –

see (6). By definition,

$$\mu(L) \;:=\; \frac{V^f(L)}{\sum\limits_{M\in K(2,B_0)} V^f(M)} \;\times\; \mu(B_0) \;=\; \frac{V^f(L^k)}{\sum\limits_{i=1}^{l_{B_0}} \sum\limits_{M\in K(2,B_0,i)} V^f(M^k)}\ .$$

However,

$$\sum_{M\in K(2,B_0,i)} V^f(M^k) \overset{(7)}{=} \sum_{M\in K(2,B_0,i)} V^k(M^f) \overset{(P3)}{\geq} c_3\, V^k(B_0) \overset{(8)}{\geq} \frac{c_3}{c_2}\, \mathcal{H}^k(B_0)\,.$$

It now follows from the definition of l_{B_0} – see (P5), that

$$\mu(L) \;\leqslant\; \frac{c_2\, V^f(L)}{c_3\, \mathcal{H}^k(B_0)\, l_{B_0}} \;\leqslant\; \frac{V^f(L)}{\eta}\ .$$

To establish (22) for general n, we proceed by induction. For $n>2$, assume that (22) holds for balls in $K(n-1)$. Consider an arbitrary ball L in $K(n)$. Then, $L\in K(n,B)$ for some $B\in K(n-1)$. By definition and our induction hypothesis,

$$\mu(L) \;:=\; \frac{V^f(L)}{\sum\limits_{M\in K(n,B)} V^f(M)} \;\times\; \mu(B) \;\leqslant\; \frac{V^f(L)}{\sum\limits_{M\in K(n,B)} V^f(M)} \times \frac{V^f(B)}{\eta}\ .$$

Thus, (22) follows on showing that

$$\sum_{M\in K(n,B)} V^f(M) \;=\; \sum_{M\in K(n,B)} V^f(M^k) \geqslant V^f(B)\ .$$

Well,

$$\sum_{M\in K(n,B)} V^f(M^k) = \sum_{i=1}^{l_B} \sum_{M\in K(n,B,i)} V^f(M^k) \overset{(7)}{=} \sum_{i=1}^{l_B} \sum_{M\in K(n,B,i)} V^k(M^f) \geqslant$$

$$\overset{(P3)}{\geqslant} c_3 \sum_{i=1}^{l_B} V^k(B) \overset{(P5)}{\geqslant} c_3\, V^k(B)\, \frac{V^f(B)}{c_3\, V^k(B)} \;=\; V^f(B)$$

and so we are done. This completes the inductive step and thereby establishes (22) for any L in $K(n)$ with $n\geq 2$.

5.5 The measure of an arbitrary ball

Set $r_o := \min\{r(B) : B\in K(2)\}$. Take an arbitrary ball A in $\mathbb{R}^k$ with $r(A)<r_o$. The aim of this section is to establish (13) for A; that is

$$\mu(A) \;\ll\; \frac{V^f(A)}{\eta}\ ,$$

were the implied constant is independent of both A and η. This will then complete the proof of the Mass Transference Principle.

We begin by establishing the following geometric lemma.

Lemma 7 *Let $A = B(x_A, r_A)$ and $M = B(x_M, r_M)$ be arbitrary balls such that $A \cap M \neq \emptyset$ and $A \setminus (cM) \neq \emptyset$ for some $c \geqslant 3$. Then $r_M \leqslant r_A$ and $cM \subset 5A$.*

Proof. Let $z \in A \cap M$. Then $d(x_A, x_M) \leqslant d(x_A, z) + d(z, x_M) \leqslant r_A + r_M$. Here $d(.,.)$ is the standard Euclidean metric in $\mathbb{R}^k$. Now take $z \in A \setminus (cM)$. Then

$$c r_M \ \leqslant\ d(x_M, z) \ \leqslant\ d(x_M, x_A) + d(x_A, z) \ <\ r_A + r_M + r_A \ .$$

Hence, $r_M \leqslant \frac{2}{c-1}\, r_A$ and since $c \geq 3$ we have that $r_M \leqslant r_A$. Now for any $z \in cM$, we have that

$$d(x_A, z) \ \leqslant\ d(x_A, x_M) + d(x_M, z) \ \leqslant\ r_A + r_M + c r_M \ =\ r_A + (1+c) r_M \ \leq$$

$$\leqslant\ r_A + \frac{2(1+c)}{c-1} r_A \ =\ \left(3 + \frac{4}{c-1}\right) r_A \ \leqslant\ 5\, r_A \ .$$

♠

The measure μ is supported on $\mathbb{K}_\eta$. Thus, without loss of generality we can assume that $A \cap \mathbb{K}_\eta \neq \emptyset$; otherwise $\mu(A) = 0$ and there is nothing to prove.

We can also assume that for every n large enough A intersects at least two balls in $K(n)$; since if B is the only ball in $K(n)$ which has non-empty intersection with A, then

$$\mu(A) \ \leq\ \mu(B) \ \stackrel{(22)}{\leqslant}\ \frac{V^f(B)}{\eta} \ \to\ 0 \qquad \text{as} \qquad n \to \infty$$

$(r(B) \to 0$ as $n \to \infty)$ and again there is nothing to prove. Thus we may assume that there exists a unique integer n such that:

$$A \text{ intersects at least 2 balls from } K(n) \tag{23}$$

and

$$A \text{ intersects only one ball } B \text{ from } K(n-1).$$

In view of our choice of r_0 and the fact that $r(A) < r_0$, we have that $n > 2$. Note that since B is the only ball from $K(n-1)$ which has non-empty intersection with A, we trivially have that $\mu(A) \leqslant \mu(B)$. It follows that we can also assume that

$$r(A) \ <\ r(B) \ . \tag{24}$$

Otherwise, since f is increasing

$$\mu(A) \ \leq\ \mu(B) \ \stackrel{(22)}{\leqslant}\ V^f(B)/\eta \ :=\ f(r(B))/\eta \ \leqslant\ f(r(A))/\eta \ :=\ V^f(A)/\eta$$

and we are done. Since $K(n,B)$ is a cover for $A\cap\mathbb{K}_\eta$, we have that

$$\mu(A) \;\leqslant\; \sum_{i=1}^{l_B}\ \sum_{L\in K(n,B,i),\ L\cap A\neq\emptyset} \mu(L) \;\overset{(22)}{\leqslant}\; \sum_{i=1}^{l_B}\ \sum_{L\in K(n,B,i),\ L\cap A\neq\emptyset} V^f(L)/\eta\ . \tag{25}$$

In order to estimate the r.h.s. of (25), we consider two cases:

<u>*Case*</u> (i): Sub-levels $K(n,B,i)$ for which $\#\{\,L\in K(n,B,i)\ :\ L\cap A\neq\emptyset\,\} \,=\, 1$.

<u>*Case*</u> (ii): Sub-levels $K(n,B,i)$ for which $\#\{\,L\in K(n,B,i)\ :\ L\cap A\neq\emptyset\,\} \,\geqslant\, 2$.

Formally, there is a third case corresponding to those sub-levels $K(n,B,i)$ for which $\#\{\,L\in K(n,B,i)\ :\ L\cap A\neq\emptyset\,\} \,=\, 0$. However, this case is irrelevant since the contribution to the r.h.s. of (25) from such sub-levels is zero.

Dealing with Case (i). Pick a ball $L\in K(n,B,i)$ such that $L\cap A\neq\emptyset$. By (23), there is another ball $M\in K(n,B)$ such that $A\cap M\neq\emptyset$. By property (P1), $3L$ and $3M$ are disjoint. It follows that $A\setminus 3L\neq\emptyset$. Therefore, by Lemma 7, $r(L)\leqslant r(A)$ and thus

$$V^f(L) \;\leqslant\; V^f(A)\ . \tag{26}$$

Now, let $K(n,B,i^*)$ denote the first sub-level which has non-empty intersection with A. Thus, $L\cap A=\emptyset$ for any $L\in K(n,B,i)$ with $i<i^*$ and there exists a unique ball L^* in $K(n,B,i^*)$ such that $L^*\cap A\neq\emptyset$. Since we are in case (i), the internal sum of (25) consists of just one summand. It follows, via property (P4) and (26), that

$$\begin{aligned}\sum_{i\,\in\,\mathrm{Case(i)}}\ \sum_{L\in K(n,B,i),\ L\cap A\neq\emptyset} V^f(L)/\eta \;&\leqslant\; \sum_{i\,\in\,\mathrm{Case(i)}} \frac{1}{2^{i-i^*}}\,\frac{V^f(L^*)}{\eta}\\ &\leqslant\; 2\,\frac{V^f(L^*)}{\eta} \leqslant 2\,\frac{V^f(A)}{\eta}\ .\end{aligned} \tag{27}$$

Dealing with Case (ii). Again pick a ball $L\in K(n,B,i)$ such that $L\cap A\neq\emptyset$. Since we are in case (ii), there is another ball $M\in K(n,B,i)$ such that $A\cap M\neq\emptyset$. By property (P2), the balls L^f and M^f are disjoint. It follows that $A\setminus L^f\neq\emptyset$. Hence, by Lemma 7 and property (P1) we have that

$$L^f\subset 5A. \tag{28}$$

It follows that

$$\begin{aligned}\sum_{i\,\in\,\mathrm{Case(ii)}}\ \sum_{L\in K(n,B,i),\ L\cap A\neq\emptyset} \frac{V^f(L)}{\eta} \;&\overset{(7)}{=}\; \sum_{i\,\in\,\mathrm{Case(ii)}}\ \sum_{L\in K(n,B,i),\ L\cap A\neq\emptyset} \frac{V^k(L^f)}{\eta}\\ &\overset{(8)}{\leqslant}\; \frac{1}{c_1\,\eta}\sum_{i\,\in\,\mathrm{Case\ (ii)}}\ \sum_{L\in K(n,B,i),\ L\cap A\neq\emptyset} \mathcal{H}^k(L^f) \;\overset{(P2),(28)}{\leqslant}\; \frac{1}{c_1\,\eta}\sum_{i\,\in\,\mathrm{Case(ii)}}\mathcal{H}^k(5A)\\ &\overset{(8)}{\leqslant}\; \frac{5^k c_2\,V^k(A)\,l_B}{c_1\,\eta} \;\overset{(P5)}{\leqslant}\; \frac{5^k c_2\,V^k(A)}{c_1\,\eta}\times\frac{2\,V^f(B)}{c_3\,V^k(B)} \;\leq\; \frac{2\;5^k c_2}{c_1\,c_3}\times\frac{V^f(A)}{\eta}\ .\end{aligned} \tag{29}$$

The last inequality follows from (24) and the fact that the function $x^{-k}f(x)$ is decreasing.

On combining (25), (27) and (29) we attain our goal; i.e. $\mu(A) \ll V^f(A)/\eta$.

6 Final comments

6.1 A general Mass Transference Principle

We say that a function f is *doubling* if there exists a constant $\lambda > 1$ such that for $x > 0$

$$f(2x) \leq \lambda f(x) .$$

Let (X, d) be a locally compact metric space. Let g be a doubling, dimension function and suppose there exist constants $0 < c_1 < 1 < c_2 < \infty$ and $r_0 > 0$ such that

$$c_1 \; g(r(B)) \leqslant \mathcal{H}^g(B) \leqslant c_2 \; g(r(B)) ,$$

for any ball $B = B(x, r)$ with $x \in X$ and $r \leqslant r_0$. Since g is doubling, the measure $\mathcal{H}^g$ is doubling on X. Recall that $V^g(B) := g(r(B))$. Thus, the above condition corresponds to (8) in the $\mathbb{R}^k$ setup. Next, given a dimension function f and a ball $B = B(x, r)$ we define

$$B^f := B(x, g^{-1}f(r)) .$$

By definition, $B^g(x, r) = B(x, r)$ and

$$V^f(B^g) = V^g(B^f) \qquad \text{for any ball } B.$$

This is an analogue of (7). In the case $g(x) = x^k$, the current setup precisely coincides with that of §3 in which $X = \mathbb{R}^k$. The following result is a natural generalization of Theorem 2 – the Mass Transference Principle.

Theorem 3 (A general Mass Transference Principle) *Let (X, d) and g be as above and let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in X with $r(B_i) \to 0$ as $i \to \infty$. Let f be a dimension function such that $f(x)/g(x)$ is monotonic and suppose that for any ball B in X*

$$\mathcal{H}^g\big(B \cap \limsup_{i\to\infty} B_i^f\big) = \mathcal{H}^g(B) .$$

Then, for any ball B in X

$$\mathcal{H}^f\big(B \cap \limsup_{i\to\infty} B_i^g\big) = \mathcal{H}^f(B) .$$

The proof of the general Mass Transference Principle follows on adapting the proof of Theorem 2 in the obvious manner. The property that $\mathcal{H}^k$ is doubling is used repeatedly in the proof of Theorem 2. In establishing Theorem 3, this property is replaced by the assumption that $\mathcal{H}^g$ is doubling.

In short, the general Mass Transference Principle allows us to transfer $\mathcal{H}^g$-measure theoretic statements for lim sup subsets of X to general $\mathcal{H}^f$-measure theoretic statements. Thus, whenever we have a Duffin-Schaeffer type statement with respect to a measure μ comparable to $\mathcal{H}^g$, we obtain a general Hausdorff measure theory for free. For numerous examples of lim sup sets and associated Khintchine type theorems (the approximating function ψ is assumed to be monotonic) within the framework of this section, the reader is referred to [1].

6.2 The Duffin-Schaeffer conjecture revisited

Let $\mathcal{S}_k^*(\psi)$ denote the set of points $\mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{I}^k$ for which there exist infinitely many $q \in \mathbb{N}$ and $\mathbf{p} = (p_1, \ldots, p_k) \in \mathbb{Z}^k$ with $(p_1, \ldots, p_k, q) = 1$, such that

$$\left| y_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q} \qquad 1 \leq i \leq k \ .$$

Here, we simply ask that points in $\mathbb{I}^k$ are approximated by distinct rationals whereas in the definition of $\mathcal{S}_k(\psi)$ a pairwise co-primeness condition on the rationals is imposed. For $k = 1$, the two sets coincide. For $k \geq 2$, it is easy to verify that $m(\mathcal{S}_k^*(\psi)) = 0$ if $\sum \psi(n)^k < \infty$. The complementary divergent result is due to Gallagher [4].

Theorem G *For $k \geq 2$,* $\quad m(\mathcal{S}_k^*(\psi)) = 1 \quad$ if $\quad \sum_{n=1}^{\infty} \psi(n)^k = \infty$.

Notice that the Euler function ϕ plays no role in determining the measure of $\mathcal{S}_k^*(\psi)$ when $k \geq 2$. This is unlike the situation when considering the measure of the set $\mathcal{S}_k(\psi)$ – see Theorem PV (§1.1) and Corollary 1. It is worth mentioning that Gallagher actually obtains a quantative version of Theorem G.

The Mass Transference Principle together with Theorem G, implies the following general statement.

Theorem 3 For $k \geq 2$, $\quad \mathcal{H}^f(\mathcal{S}_k^*(\psi)) = \mathcal{H}^f(\mathbb{I}^k) \quad$ if $\quad \sum_{n=1}^{\infty} f(\psi(n)/n)n^k = \infty$.

It would be highly desirable to establish a version of the Mass Transference Principle which allows us to deduce a quantative Hausdorff measure statement from a quantative Lebesgue measure statement. We hope to investigate this sometime in the near future.

Acknowledgments: SV would like to thank Ayesha and Iona for making him appreciate once again all those wonderfully simple things around us: ants, pussycats, sticks, leaves and of course the many imaginary worlds that are often neglected in adulthood – especially the world of hobgoblins. VB would like to thank Tatiana for her help and patience during the difficult but nevertheless exciting time over the past nine months.

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