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Two-body relaxation in relativistic thermal plasmas

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Summary. An exact expression for the average energy exchange in a relativistic two-body collision between isotropic particle distributions is calculated. This is used to find the relaxation time-scale for particles with a Maxwellian distribution function. These time-scales are compared with important energy loss time-scales at temperatures $kT/m_e c^2 \gtrsim 1$.

1 Introduction

Relativistic thermal plasmas, where $kT_e \gtrsim m_e c^2$, are of relevance to high-energy astrophysics. Models of γ-ray burst sources require the existence of such high temperatures, as do some of the models of active galactic nuclei.

The question then arises, can the particles maintain a Maxwellian distribution, or is the cooling too great? The time-scale for two-body relaxation in the non-relativistic case is well known (Spitzer 1956). No relativistic generalization exists, although some approximate expressions have been given (Gould 1982a; Lightman & Band 1981).

Here I give an exact expression for the energy exchange rate in terms of an integral over the scattering cross-section (Section 3). This is evaluated for electron–proton, electron–electron and proton–proton relaxations (Section 4). The resulting time-scales are compared to the major energy-loss time-scales – those of bremsstrahlung, pair production and pion production – to find in what temperature ranges it is possible to consider the particle distributions as Maxwellian (Section 5).

2 Energy exchange in a two-body interaction

Consider two species, A and B, with masses m and M respectively, and isotropic distribution functions.

Before the collisions their four-momenta (in the plasma frame) are [putting $c = 1$, and using the metric $(+ - - -)$]

$$p_A^\mu = m\gamma_A(1, \boldsymbol{\beta}_A),$$
$$p_B^\mu = M\gamma_B(1, \boldsymbol{\beta}_B). \tag{2.1}$$

After the collision $p_{A2}^\mu = m\gamma_{A2}(1, \boldsymbol{\beta}_{A2})$ and the energy exchange is $m\Delta\gamma_A = m(\gamma_{A2} - \gamma_A)$.

The velocity of the centre-of-mass (CM) of the two-body system is

$$\mathbf{V} = \frac{m\gamma_A \boldsymbol{\beta}_A + M\gamma_B \boldsymbol{\beta}_B}{m\gamma_A + M\gamma_B}. \tag{2.2}$$

Let this direction define the x-axis, i.e.

$$\mathbf{V} = V(1,0,0).$$

Then

$$V\beta_{Ax} = \mathbf{V}\cdot\boldsymbol{\beta}_A.$$

Lorentz transforming p_A^μ to the CM frame

$$p_{A'}^\mu = m\gamma_A[\Gamma(1-V\beta_{Ax}),\ \Gamma(\beta_{Ax}-V),\ \beta_{Ay},\ \beta_{Az}]$$

where

$$\Gamma^2 = 1/(1-V^2).$$

Hence

$$\gamma_{A'} = \Gamma\gamma_A(1-V\beta_{Ax}), \tag{2.3}$$

$$\beta_{Ax'} = \frac{\beta_{Ax}-V}{1-V\beta_{Ax}}. \tag{2.4}$$

Let particle A be scattered through an angle 2α in the CM frame (Fig. 1). There is no energy exchange in this frame, so

$$\mathbf{p}_{A2'} = |\mathbf{p}_{A'}|\{\hat{\mathbf{r}}\cos 2\alpha + (\hat{\boldsymbol{\theta}}\cos\delta + \hat{\boldsymbol{\varphi}}\sin\delta)\sin 2\alpha\}$$

where δ is an angle related to the orientations of the orbital plane, and $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ are the usual spherical polar unit vectors. Their x components are

$$\hat{r}_x = x/r,$$

$$\hat{\theta}_x = xz/\rho r,$$

$$\hat{\phi}_x = -y/\rho$$

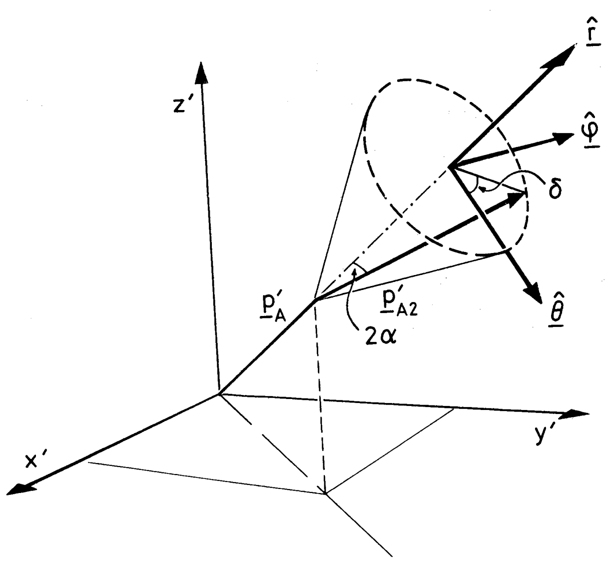

Figure 1. Scattering the centre-of-mass frame. The scattering angle is 2α. There is no energy exchange, hence $|\mathbf{p}_{A'}| = |\mathbf{p}_{A2'}|$.

where

$$\rho^2 = x^2 + y^2, \qquad r^2 = \rho^2 + z^2.$$

Because the distributions are isotropic

$$\langle p_{Ay} \rangle = \langle p_{Az} \rangle = 0.$$

Also

$$p_{Ay'} = p_{Ay}, \qquad p_{Az'} = p_{Az},$$

Hence

$$\langle \hat{\theta}_x \rangle = \langle \hat{\phi}_x \rangle = 0,$$

and so,

$$\langle p_{A2x'} \rangle = p_{Ax'} \cos 2\alpha.$$

Lorentz transforming $p^{\mu}_{Az'}$ back to the plasma frame gives

$$\langle \gamma_{A2} \rangle = \Gamma \gamma_{A2'} (1 + V\beta_{Ax'}).$$

Hence (replacing the factors of c)

$$\langle \Delta E_A \rangle = \frac{-2mMc^2 \sin^2 \alpha}{m^2 + M^2 + 2\gamma mM} [M\gamma_A - m\gamma_B + \gamma(m\gamma_A - M\gamma_B)] \qquad (2.5)$$

where $\gamma = \gamma_A \gamma_B (1 - \boldsymbol{\beta}_A \cdot \boldsymbol{\beta}_B)$ is the γ-factor of relative motion. The non-relativistic limit of equation (2.5) is

$$\langle \Delta E_A^{NR} \rangle = \frac{-Mm \sin^2 \alpha}{(M+m)^2} [(v_A^2 - v_B^2)(M+m) + v^2(m-M)]$$

where

$$v^2 = (\mathbf{v}_A - \mathbf{v}_B)^2.$$

Frankel, Hines & Dewar (1979) have calculated the energy loss in a single relativistic two-body collision.

3 Energy exchange rate

The relativistically invariant reaction rate (reactions per unit volume per unit time) is (Landau & Lifshitz 1975, section 12)

$$R_{12} = \int d\sigma\, v \frac{c^2 p_1 \cdot p_2}{E_1 E_2} \frac{dN_1\, dN_2}{1 + \delta_{12}} \qquad (3.1)$$

where $d\sigma$ is the differential cross-section, v is the relative velocity, p is the four-momentum, dN is the distribution function and δ_{12} is the Kronecker delta. The relativistic Maxwell distribution is

$$dN = \frac{N}{\theta K_2(1/\theta)} \gamma^2 \beta \exp(-\gamma/\theta)\, d\gamma \qquad (3.2)$$

where the dimensionless temperature $\theta = kT/Mc^2$ and K_2 is a modified Bessel function.

Substituting this distribution function into equation (3.1) and changing the order of integration gives (Weaver 1976)

$$R_{12} = \frac{N_1 N_2}{1+\delta_{12}} \frac{c}{\theta_1 \theta_2 K_2(1/\theta_1) K_2(1/\theta_2)} \int_{\gamma=1}^{\infty} \sigma(\gamma)\gamma^2\beta^2 \frac{K_1(z)}{z}\, d\gamma \tag{3.3}$$

where

$$z^2 = \frac{\theta_1^2 + \theta_2^2 + 2\gamma\theta_1\theta_2}{\theta_1^2\theta_2^2}.$$

In the case of the energy exchange rate, the quantity of interest is

$$\frac{dE_1}{dt} = \int \langle \Delta E_1 \rangle \, d\sigma(\gamma, \alpha)\, c\beta \frac{c^2 p_1 \cdot p_2}{E_1 E_2} \frac{dN_1\, dN_2}{1+\delta_{12}}. \tag{3.4}$$

Here the cross-section depends not only on the relative velocity, but also on the (centre-of-mass) scattering angle. $\langle \Delta E_1 \rangle$ is given by equation (2.5). Then, on changing the order of integration

$$\frac{dE_1}{dt} = -2Mmc \frac{N_1 N_2}{1+\delta_{12}} \frac{kT_1 - kT_2}{\theta_1^2\theta_2^2 K_2(1/\theta_1) K_2(1/\theta_2)} \times \int_{\gamma=1}^{\infty} \int_{\alpha_1}^{\alpha_2} \frac{(\gamma^2-1)^2 \sigma(\gamma,\alpha)\sin^2\alpha}{M^2+m^2+2\gamma m M} \frac{K_2(z)}{z^2}\, d\Omega\, d\gamma. \tag{3.5}$$

$\alpha_1 \equiv 1/\Lambda$ is the minimum scattering ½-angle. For divergent (Coulomb) cross-sections it is determined by the minimum momentum transfer, that of the excitation of a single plasmon in the forward direction. Hence $\alpha_1 \sim \hbar\omega_p/2\gamma mc^2$ and so $\Lambda \sim kT/\hbar\omega_p$. $\ln \Lambda$ is essentially a Coulomb logarithm. For a relativistic plasma $\omega_p^2 = 4\pi r_e cN/3T_*$. (Gould 1981). For convergent cross-sections $\alpha_1 = 0$.

α_2 is the maximum scattering ½-angle. For distinguishable particles $\alpha_2 = \pi/2$. For identical particles scattering with a symmetrized cross-section $\alpha_2 = \pi/4$.

4 Particle–particle relaxation

In a plasma heated by shocks or turbulence the protons will gain more energy than the electrons. It is, however, mainly the electrons which radiate (except for some pion production at very high temperatures, see Section 5.3). Hence the energy exchange rate from protons to electrons is a crucial factor in determining the temperature structure of the plasma. If the separate electron–electron and proton–proton thermalization time-scales are shorter than the electron heating time-scale, the electrons and protons will be able to achieve Maxwellian distributions with different temperatures, with $T_e < T_p < m_p T_e/m_e$.

The cross-sections for electron–proton, electron–electron and electron–positron scattering are well known, so these relaxation rates can be easily evaluated. Proton–proton relaxation is more difficult to calculate. Above energies of a few MeV elastic nuclear scattering becomes important, and a numerical fit must be made to the experimental cross-section.

4.1 ELECTRON–PROTON RELAXATION

Equation (3.5) is of the form

$$\frac{dE_e}{dt} = f(T_e, T_p)(kT_e - kT_p) \tag{4.1}$$

so defining the relaxation time-scale by

$$t_{ep} = \left| \frac{\kappa N_e (kT_e - kT_p)}{dE_e/dt} \right| \tag{4.2}$$

(Spitzer 1956), gives the dimensionless time-scale

$$t_* = \frac{t_{ep}}{t_0} = \frac{\kappa N_e}{|f(T_e, T_p)|} N_e \sigma_T c \tag{4.3}$$

where $t_0 = 1/N_e \sigma_T c$ is the Compton time-scale and $\langle E_e \rangle = \kappa N_e T_{*e} m_e c^2$ is the average energy per unit volume. $T_* = kT/m_e c^2$ and

$$\kappa = \begin{cases} 3/2 & T_* \ll 1 \\ 3 & T_* \gg 1. \end{cases}$$

Substituting the Rutherford cross-section into equation (3.5) and taking the non-relativistic limit ($z \gg 1$) gives

$$t_* = \sqrt{\frac{\pi}{2} \frac{m_p}{m_e}} \frac{N_e}{N_p \ln \Lambda} \left(T_{*e} + \frac{m_e}{m_p} T_{*p} \right)^{3/2}. \tag{4.4}$$

This is Spitzer's (1956) result.

The Rutherford formula for the cross-section assumes that both particles are pointlike and spinless. Including the effects of electron and proton spin, magnetic scattering and a non-pointlike proton results in the Rosenbluth cross-section (Perkins 1972). It is less than the Rutherford cross-section for all scattering angles. However, this difference is small for small scattering angles, which dominate in Coulomb scattering. For scattering angles (in the proton's rest frame) $\alpha_0 < m_p/\gamma m_e$, the Rutherford formula is sufficiently accurate. If $\alpha_0 \gg \alpha_{min} = 1/2\Lambda$ the total rate will be unaffected by replacing $\alpha_{max} = \pi/2$ by α_0. So if $\gamma \ll 2m_p \Lambda/m_e$ ($\gamma \ll 10^{12}$ for $\ln \Lambda \simeq 20$) the Rutherford formula may be used. This easily holds for the temperatures being considered here. Hence

$$d\sigma \simeq \frac{r_e^2}{4\gamma^2} \frac{d\Omega_{rest}}{\sin^4 \alpha_{rest}}. \tag{4.5}$$

These angles must be transformed to the centre-of-mass (see Appendix)

$$\frac{d\Omega_{rest}}{\sin^4 \alpha_{rest}} = \begin{cases} \dfrac{d\Omega}{\sin^4 \alpha}; & \gamma m_e \ll m_p \\[2ex] \dfrac{2\gamma m_e}{m_p} \dfrac{d\Omega}{\sin^4 \alpha}; & \gamma m_e \gg m_p. \end{cases} \tag{4.6}$$

For the case where the electrons and protons have the same temperature, $T_{*p} = T_{*e} \equiv T_*$, the electrons will be relativistic and the protons will be non-relativistic for

$1 \ll T_* \ll m_p/m_e$. Both species will be relativistic for $m_p/m_e \ll T_*$. Then substituting (4.5) and (4.6) into (3.5) gives

$$t_* = \begin{cases} \dfrac{2m_p}{m_e} \dfrac{T_*}{\ln\Lambda} \dfrac{N_e}{N_p}; & 1 \ll T_* \ll \dfrac{m_p}{m_e} \\ \dfrac{4\,T_*^2}{\ln\Lambda} \dfrac{N_e}{N_p} \quad ; & \dfrac{m_p}{m_e} \ll T_*. \end{cases} \tag{4.7}$$

For the case where $T_* = T_{*e} = T_{*p} m_e/m_p$ both species are relativistic if $T_* \gg 1$. Then

$$t_* = 4 \frac{m_p}{m_e} \frac{T_*^2}{\ln\Lambda} \frac{N_e}{N_p}. \tag{4.8}$$

These two cases bracket the actual proton temperature, so the electron–proton relaxation time-scale will lie between the limits given in equations (4.7) and (4.8). See Fig. 2.

4.2 ELECTRON–ELECTRON RELAXATION

The electron–electron interaction takes place via the Møller cross-section (Jauch & Rohrlich 1980, section 12). This leads to

$$t_* = \begin{cases} 4\sqrt{\pi}\, T_*^{3/2}/\ln\Lambda; & T_* \ll 1 \\ 8\, T_*^2/\ln\Lambda \quad ; & 1 \ll T_*. \end{cases} \tag{4.9}$$

This result disagrees with that of Gould (1982a) in that it includes a Coulomb logarithm. Gould's (1981) expression for the energy exchange is invalid at small scatting angles; i.e. those which dominate to produce the $\ln\Lambda$ term.

The non-relativistic limit can also be obtained from equation (4.4) on multiplying by a factor of 2 for identical particle effects.

Electron–positron scattering occurs via the Bhaba cross-section. The principal term (that $\propto 1/\sin^4\alpha$) has the same non-relativistic and ultra-relativistic limits as that of the Møller cross-section. Hence the electron–positron time-scale is just one-half the electron–electron time-scale, as there are no identical particle effects.

4.3 PROTON–PROTON RELAXATION

Pure Coulomb scattering in the non-relativistic case yields

$$t_* = 4\sqrt{\pi} \left(\frac{m_p}{m_e}\right)^{1/2} \frac{T_*^{3/2}}{\ln\Lambda} \frac{N_e}{N_p}. \tag{4.10}$$

However, nuclear scattering must also be considered. For relative kinetic energies below a few MeV only the $l = 0$ nuclear scattering term need be included (Schiff 1968, section 50). The modified scattering amplitude is

$$f(\theta) = f_c(\theta) + \exp\left[i(2\eta_0 + \delta_0)\right] \frac{\sin\delta_0}{k} \tag{4.11}$$

where $k = m_p\beta c/2\hbar$, δ_0 is the $l = 0$ phase shift and $f_c(\theta)$ is the Coulomb scattering amplitude

$$f_c(\theta) = \frac{\alpha_f}{2k\beta\sin^2\alpha} \exp i\left[-\frac{\alpha_f}{\beta} \ln\sin^2\alpha + \pi + 2\eta_0\right] \tag{4.12}$$

(Schiff 1968, section 20). The appropriate antisymmetrized cross-section for scattering identical spin-half particles is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 + |f(\pi-\theta)|^2 - Re\,[f(\theta)f^*(\pi-\theta)]. \tag{4.13}$$

Substituting (4.11) and (4.12) into (4.13) gives the cross-section

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega}\bigg|_c + \frac{\sin^2\delta_0}{k^2} - \frac{\alpha_f \sin\delta_0}{2\beta k^2}\left[\frac{\cos\{\alpha_f/\beta \ln \sin^2\alpha + \delta_0\}}{\sin^2\alpha} + \frac{\cos\{\alpha_f/\beta \ln\cos^2\alpha + \delta_0\}}{\cos^2\alpha}\right]. \tag{4.14}$$

The last term represents the interference between the nuclear and Coulomb scattering. The Coulomb cross-section

$$\frac{d\sigma}{d\Omega}\bigg|_c = \left(\frac{\alpha_f}{2k\beta}\right)^2\left[\frac{1}{\sin^4\alpha} + \frac{1}{\cos^4\alpha} - \frac{\cos\{\alpha_f/\beta \ln\tan^2\alpha\}}{\sin^2\alpha\cos^2\alpha}\right] \tag{4.15}$$

is the Mott scattering formula for identical particles. The last term in (4.15) represents the interference due to identical particle effects.

Defining the total cross-section to be (*cf.* equation 3.5)

$$\sigma = 4\int_{\alpha_{\min}}^{\pi/4} \frac{d\sigma}{d\Omega}\sin^2\alpha\, d\Omega = 32\pi\int_{\alpha_{\min}}^{\pi/4}\frac{d\sigma}{d\Omega}\sin^3\alpha\cos\alpha\, d\alpha$$

and substituting for $d\sigma/d\Omega$ gives

$$\sigma = \frac{8\pi\alpha_f^2}{k^2\beta^2}\left[\ln\Lambda - \frac{1}{4} - \frac{1}{2}\int_{1/4\Lambda^2}^{1}\cos\left(\frac{\alpha_f}{\beta}\ln y\right)\frac{dy}{1+y}\right] + 2\pi\frac{\sin^2\delta_0}{k^2} - 8\pi\frac{\alpha_f\sin\delta_0}{\beta k^2}$$

$$\times\left[\int_0^{1/2}\cos\left(\frac{\alpha_f}{\beta}\ln y + \delta_0\right)dy + \int_{1/2}^{1}\cos\left(\frac{\alpha_f}{\beta}\ln y + \delta_0\right)\frac{1-y}{y}\,dy\right] \tag{4.16}$$

$$= \sigma_c + \sigma_N - \sigma_I.$$

For $\beta/\alpha_f \ll 1$ ($E \ll 0.025$ MeV) the integrands oscillate rapidly and their contributions tend to zero (i.e. the interference terms disappear in the classical limit). This leaves

$$\sigma = \sigma_c + \sigma_N \simeq \frac{2\pi}{k^2}\left(\frac{4\alpha_f^2}{\beta^2}\ln\Lambda + \sin^2\delta_0\right).$$

However, since $\sin^2\delta_0 \ll 1$ and $\alpha_f^2/\beta^2 \ln\Lambda \gg 1$ the nuclear contribution is negligible at these energies.

More interesting is the case where $\beta/\alpha_f \gg 1$. The integrands in (4.16) are now slowly varying, and can be approximated by setting $\cos(\alpha_f/\beta \ln y) = 1$ and $\sin(\alpha_f/\beta \ln y) = 0$. This gives

$$\sigma \simeq \frac{8\pi\alpha_f^2}{k^2\beta^2}\ln\Lambda + 2\pi\frac{\sin^2\delta_0}{k^2} - 4\pi\ln 2\,\frac{\alpha_f}{\beta}\frac{\sin 2\delta_0}{k^2}. \tag{4.17}$$

The phase shift δ_0 can be calculated using effective range theory (Blatt & Weiskopf 1952, section 2.4). The interference and nuclear terms are equal (and hence cancel) when

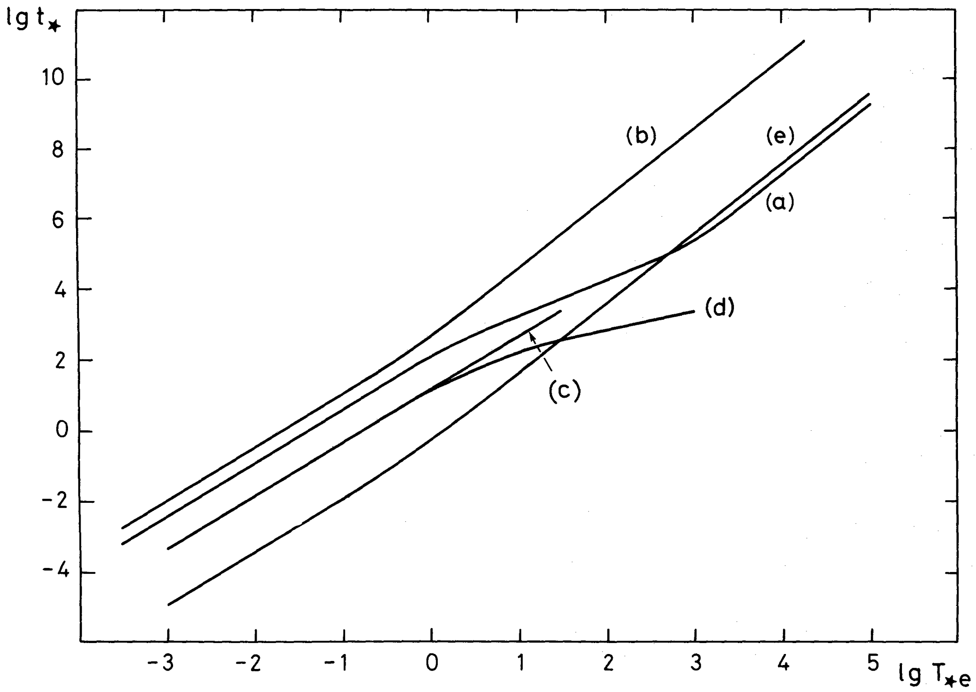

Figure 2. Two-body relaxation time-scales (putting $N_e = N_p$, ln $\Lambda = 20$). (a) Electron –proton relaxation, $T_p = T_e$. (b) Electron–proton relaxation, $T_p = m_p T_e/m_e$. (c) Pure Coulomb proton–proton relaxation. (d) Proton–proton relaxation including nuclear scattering. (e) Electron–electron relaxation.

$E \sim 0.8$ MeV. Using Gould's (1982b) calculated values of δ_0 for $E < 20$ MeV and the experimentally determined cross-section (Particle Data Group 1980) for higher energies gives the modified relaxation time-scale shown in Fig. 2. For $T_* \gtrsim 30$ proton–proton relaxation is faster than electron–electron relaxation.

5 Other time-scales

5.1 BREMSSTRAHLUNG

In the non-relativistic case only electron–proton bremsstrahlung need be considered. The electron–electron ($e^- e^-$) system has no dipole moment, and its quadrupole radiation is negligible. There are not enough positrons present to give significant electron–positron bremsstrahlung.

At relativistic temperatures the electron–electron bremsstrahlung cross-section is comparable to the electron–proton cross-section. Because there are pairs present, electron–positron and positron–positron bremsstrahlung must also be included. Baier, Fadin & Khoze (1967, 1968) have calculated the cross-sections for electron–electron ($e^{\pm} e^{(\pm)}$) bremsstrahlung in various reference frames. Alexanian (1968) gives the electron–electron ($e^- e^-$) energy loss rate from a Maxwellian distribution in the extreme relativistic limit.

Gould (1980) gives, for a plasma of electrons and protons

$$\frac{dE}{dt} = \frac{32}{3}\sqrt{\frac{2}{\pi}}\, mc^2 \alpha_f r_e^2 c N_e N_p T_*^{1/2} \left[1 + 6.8\, T_* \frac{N_e}{N_p} + \frac{5.6 \times 10^{-3}}{T_*^{1/2}}\right] \tag{5.1}$$

for the bremsstrahlung cooling rate. This expression is valid for $T_* \ll 1$, and includes relativistic corrections and electron–electron bremsstrahlung.

In the ultra-relativistic limit the electron–proton cross-section is (Jauch & Rohrlich 1980, section 15)

$$\omega\frac{d\sigma}{d\omega}=4\alpha_f r_e^2\left[1-\left(\frac{\gamma-\hbar\omega/mc^2}{\gamma}\right)^2-\frac{2(\gamma-\hbar\omega/mc^2)}{3\gamma}\right]\times\left[\ln\left(\frac{2mc^2\gamma(\gamma-\hbar\omega/mc^2)}{\hbar\omega}\right)-\frac{1}{2}\right]. \tag{5.2}$$

For $T_* \ll m_p/m_e$ the protons are very nearly at rest. The transformation of the photon energy from this (the proton's rest frame) to the plasma frame is thus trivial. The ultra-relativistic limit of equation (3.3) gives the energy loss rate as

$$\frac{dE}{dt}=\frac{N_e N_p c}{2\theta_e^3}\int_0^\infty\int_0^{\omega_{max}}\hbar\omega\, d\sigma\gamma^2\exp(-\gamma/\theta_e)\, d\gamma \tag{5.3}$$

where $\hbar\omega_{max}\simeq\gamma m_e c^2$. So

$$\frac{dE^{ep}}{dt}=N_e N_p mc^2\alpha_f r_e^2 c\; 12T_*(\ln 2T_*-\gamma_E+3/2) \tag{5.4}$$

where $\gamma_E\simeq 0.5772$ is Euler's constant. This is the result obtained by Stickforth (1961).

$$t_*=\frac{2\pi}{3\alpha_f}\frac{N_e}{N_p}\frac{1}{\ln(5.03\,T_*)}. \tag{5.5}$$

The transformation of photon energy to the plasma frame in electron–electron bremsstrahlung is non-trivial. The extreme relativistic limit found by Alexanian (1968) is

$$\frac{dE^{--}}{dt}=N_-^2 mc^2\alpha_f r_e^2 c\; 24T_*[\ln 2T_*-\gamma_E+5/4]. \tag{5.6}$$

Apart from a slightly different logarithmic factor, this is just twice the electron–proton bremsstrahlung rate, (7.4). Svensson (1981) shows why this is so.

In the ultra-relativistic limit the cross-sections for electron–electron and electron–proton bremsstrahlung are the same. However, in the electron–electron case, both particles radiate, giving a factor of 2 in the energy loss rate. Another factor of 2 comes from the leading term proportional to $\ln\gamma$ in the cross-section. γ is the γ-factor of relative motion, and so $\gamma\sim\gamma_e$ in the electron–proton case, and $\gamma\sim\gamma_e^2$ in the electron–electron case. Finally, there is a factor of ½, due to identical particle effect.

Now, the cross-section for electron–positron bremsstrahlung is the same as for electron–electron ($e^\pm e^\pm$) bremsstrahlung in this limit, except for some small differences at the hard photon end of the spectrum (Baier, Fadin & Khoze 1968). So using the above argument, the electron–positron rate will be twice the electron–electron rate (as there are no identical particle effects) and four times the electron–proton rate.

So

$$\frac{dE^{ee}}{dt}=24N_e^2 mc^2\alpha_f r_e^2 c\, T_*\ln(3.92\,T_*), \tag{5.7}$$

$$\frac{dE^{total}}{dt}=12\,T_*\, mc^2\alpha_f r_e^2 c\,[N_e N_p\ln(5.03\,T_*)+2N_e^2\ln(3.92\,T_*)], \tag{5.8}$$

$$t_*=\frac{2\pi}{3\alpha_f}\bigg/\left[2\ln(3.92\,T_*)+\frac{N_p}{N_e}\ln(5.03\,T_*)\right]. \tag{5.9}$$

5.2 PAIR PRODUCTION AND ANNIHILATION

In an optically thin plasma electron–positron pairs are produced only in particle–particle collisions. The cross-section is not well known near threshold, but for $\gamma \gtrsim 100$, Budnev *et al.* (1975) give

$$\sigma(\gamma) = \frac{28}{27\pi} \alpha_f^2 r_e^2 [\ln^3 2\gamma - A \ln^2 2\gamma + B \ln 2\gamma + C] \tag{5.10}$$

where γ is the γ-factor of relative motion and

	$ep \to e^+ e^- e p$	$ee \to e^+ e^- e e$
A	178/28	178/28
B	2.6	−11
C	~ 40	~ 100
threshold γ	3	7.

This cross-section holds for lower temperatures in the case of electron–electron collisions, where $\gamma \simeq 2\gamma_e^2$, than for electron–proton collisions, where $\gamma \simeq \gamma_e$. Substituting this cross-section into equation (3.3) gives the pair production rate as

$$\frac{dN}{dt} = \frac{28}{27\pi} \alpha_f^2 r_e^2 c\, [4N_e^2 f_1(T_*) + N_e N_p f_2(T_*)] \tag{5.11}$$

where

$$f_1(T_*) = \ln^3 T_* + 0.92 \ln^2 T_* - 5.1 \ln T_* + 5.5$$

$$f_2(T_*) = \begin{cases} 8\{\ln^3 T_* - 10.4 \ln^2 T_* + 34 \ln T_* - 31\} & m_p/m_e \ll T_* \\ \ln^3 T_* - 1.5 \ln^2 T_* - 8.93 \ln T_* + 31 & T_* \ll m_p/m_e \end{cases}$$

and

$$N_e = N_- + N_+ .$$

Defining the time-scale as

$$t_{\text{pair}} = \left| \frac{N_e}{dN/dt} \right|$$

gives

$$t_*^{\text{pair}} = \frac{18\pi^2}{7\alpha_f^2} \Big/ \left[4f_1(T_*) + \frac{N_p}{N_e} f_2(T_*) \right]. \tag{5.12}$$

The annihilation cross-section is (Landau & Lifshitz 1971, section 88)

$$\sigma(\beta) = \pi r_e^2/\beta; \qquad \beta \ll 1$$

$$\sigma(\gamma) = \frac{\pi r_e^2}{\gamma} (\ln 2\gamma - 1); \qquad 1 \ll \gamma.$$

Substituting these into equation (3.3) and defining the annihilation time-scale as

$$t_{\text{ann}} = \left| \frac{N_+}{dN^{\text{ann}}/dt} \right|$$

gives

$$t_* = \begin{cases} \dfrac{8N_e}{3N_-}; & T_* \ll 1 \\ \dfrac{16}{3}\dfrac{N_e}{N_-}\dfrac{T_*^2}{\ln(1.12T_*)}; & 1 \ll T_*. \end{cases} \quad (5.13)$$

Ramaty & Mészáros (1981) have calculated t_* in the region $T_* \simeq 1$ using a Monte Carlo technique.

In the optically thick case photon–particle and photon–photon collisions must also be considered. These serve to increase the pair production rate, and hence to decrease the time-scale. Svensson (1981) has considered these processes in some detail.

5.3 PION PRODUCTION

Pions can be produced in inelastic nuclear proton–proton collisions. The threshold energy is ~ 290 MeV, so for moderate temperatures ($T_{*p} \simeq 100$) the protons in the high-energy Maxwell tail can produce pions. This cools the protons without heating the electrons.

The important reactions are

$$p + p \rightarrow \begin{cases} p + n + \pi^+ \\ \quad \downarrow \quad \llcorner\!\rightarrow \mu^+ + \nu_\mu & \text{threshold 292 MeV,} \\ \quad e^- + p + \bar{\nu}_e \quad \llcorner\!\rightarrow e^+ + \nu_e + \bar{\nu}_\mu & \\ p + p + \pi^0 & \text{threshold 280 MeV,} \\ \qquad \llcorner\!\rightarrow 2\gamma & \\ d + \pi^+ & \text{threshold 287 MeV.} \end{cases}$$

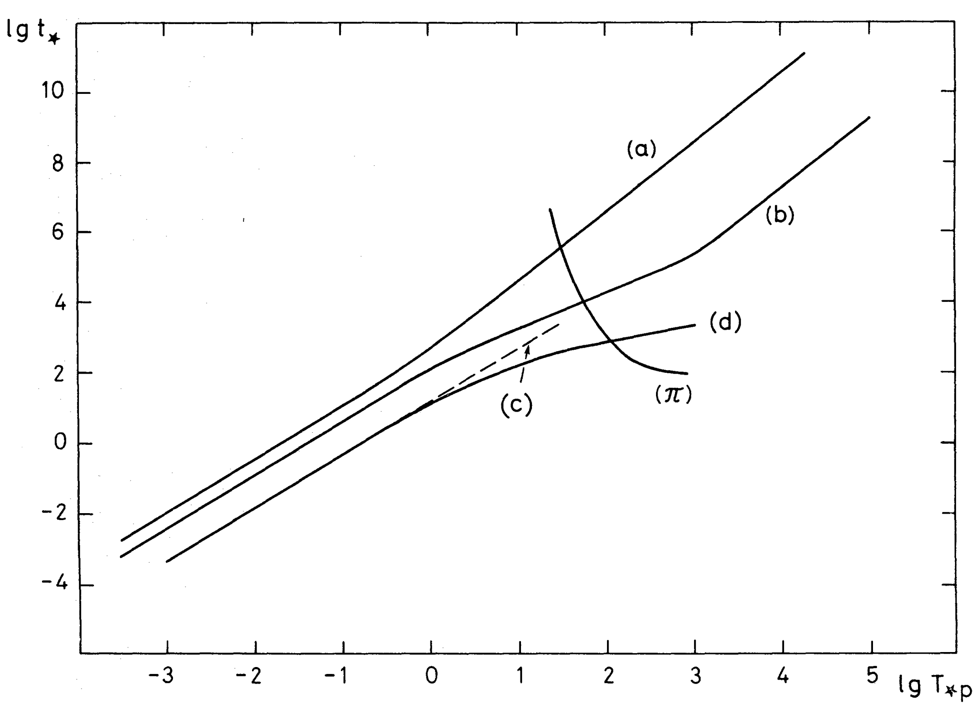

Figure 3. Pion production time-scale. The curve labelled π shows the time-scale for cooling protons by pion production. The other proton time-scales are shown for comparison, labelled as in Fig. 2.

If the pions are assumed to be produced at rest by the non-relativistic protons, then

$$\frac{dE}{dt} = -m_\pi c^2 \frac{N_p^2}{2} \langle \sigma v \rangle, \tag{5.14}$$

$$t_* = 3T_{*p} \frac{m_e}{m_\pi} \frac{N_e}{N_p} \frac{\sigma_T c}{\langle \sigma v \rangle}. \tag{5.15}$$

Kolykhalov & Syunyaev (1979) give a graph of $\langle \sigma v \rangle$ against temperature. t_* is plotted in Fig. 3.

The actual energy loss will be greater than this estimate, and hence the time-scale shorter, for two reasons: the pions are not produced at rest and some collisions produce more than one pion. However, these corrections are not large. For $T_{*p} \lesssim 100$ this cooling time-scale is much greater than the proton–proton thermalization time-scale. Hence the proton distribution can maintain the Maxwell tail necessary for pion production.

So for $T_{*p} \gtrsim 30$ ($kT_p \gtrsim 15$ MeV) the protons actually cool faster by pion production than they do by Coulomb collisions with the electrons.

6 Conclusions

The various time-scales are summarized in Fig. 4. The main processes which cool the protons are energy transfer to the electrons in Coulomb collisions, and pion production. The electrons cool by pair production and bremsstrahlung. Particle production and bremsstrahlung have cross-sections which are increasing functions of energy. Thus in a Maxwellian distribution it is the particles in the high-energy tail which cool fastest. If the thermalization

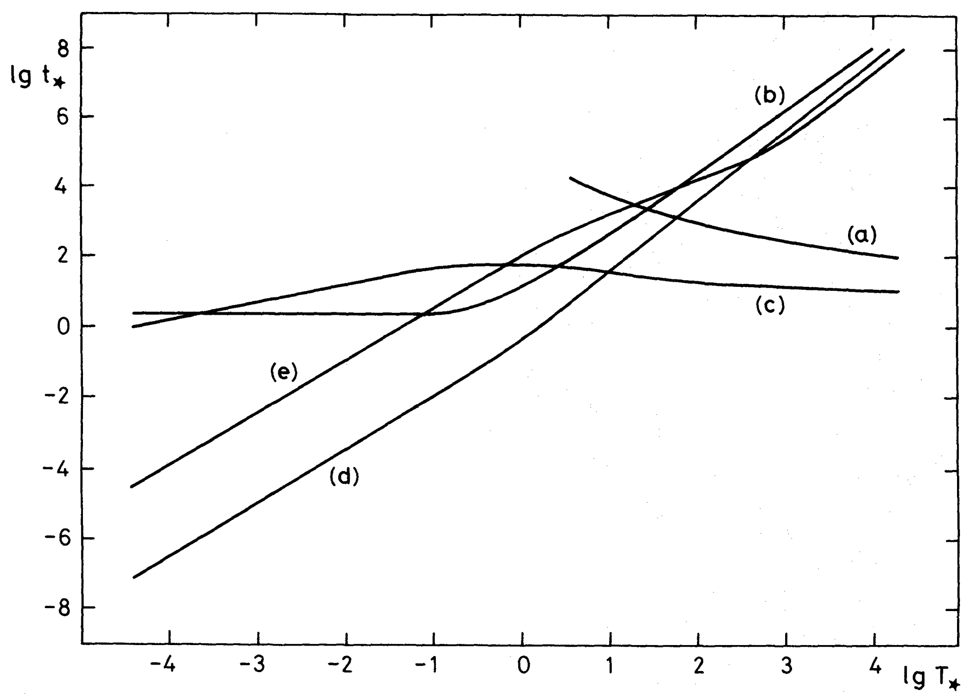

Figure 4. Electron interaction time-scales (putting $N_e = N_p$, $\ln \Lambda = 20$). (a) Electron–positron pair production. (b) Pair annihilation. (c) Bremsstrahlung. (d) Electron–electron relaxation. (e) Electron–positron relaxation, with $T_p = T_e$.

time-scale is too long, the tail will not be repopulated and the distribution function will no longer be Maxwellian. This in turn will increase the cooling time-scale.

Pion production was discussed in Section 5.3. The protons can maintain a Maxwell distribution for $T_{*p} \lesssim 100$.

If the dominant electron cooling mechanism is bremsstrahlung, the electrons will be able to maintain a Maxwell distribution for $T_{*e} \lesssim 10$. Above this the bremsstrahlung time-scale will be increased. The effect will not be large, however. The time-scale depends only weakly on γ; $t_* \propto 1/\ln \gamma$.

Pair production itself would not start to modify the electron distribution until $T_{*e} \simeq 60$. By then, however, it will already have been modified by bremsstrahlung. This increases the pair-production time-scale, though again only weakly; $t_* \propto 1/\ln^3 \gamma$.

Fig. 2 shows that, for $T_* \lesssim 10^3$ the separate electron–electron and proton–proton relaxation time-scales are less than the energy exchange time-scale. Thus it is possible for the electrons and protons to maintain Maxwellians at different temperatures, provided $T_{*e} \lesssim 10$, $T_{*p} \lesssim 100$.

If the particles do not have Maxwellian distributions the concept of temperature becomes less well defined. However, the distribution functions will still have a spread of energies about the mean, and will be isotropic, and so are still, in some sense, 'thermal'.

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References

Alexanian, M., 1968. *Phys. Rev.*, **165**, 253.
Baier, V. N., Fadin, V. S. & Khoze, V. A., 1967. *Sov. Phys. JETP*, **24**, 760.
Baier, V. N., Fadin, V. S. & Khoze, V. A., 1968. *Sov. Phys. JETP*, **26**, 1238.
Blatt, J. M. & Weisskopf, V. F., 1952. *Theoretical Nuclear Physics*, Wiley, New York.
Budnev, V. M., Ginzburg, I. F., Meledin, G. V. & Serbo, V. G., 1975. *Phys. Rep.*, **C15**, 181.
Frankel, N. E., Hines, K. C. & Dewar, R. L., 1979. *Phys. Rev. A.*, **20**, 2120.
Gould, R. J., 1980. *Astrophys. J.*, **238**, 1026 and 1981. **234**, 677(E).
Gould, R. J., 1981. *Phys. Fluids*, **24**, 102.
Gould, R. J., 1982a. *Astrophys. J.*, **254**, 755.
Gould, R. J., 1982b. *Astrophys. J.*, submitted.
Jauch, M. M. & Rohrlich, F., 1980. *The Theory of Photons and Electrons*, 2nd edn, Springer, Berlin.
Kolykhalov, P. I. & Syunyaev, R. A., 1979. *Sov. Astr.*, **23**, 189.
Landau, L. D. & Lifshitz, E. M., 1971. *Relativistic Quantum Theory*, part I, Pergamon, Oxford.
Landau, L. D. & Lifshitz, E. M., 1975. *The Classical Theory of Fields*, 4th edn, Pergamon, Oxford.
Lightman, A. P. & Band, D. L., 1981. *Astrophys. J.*, **251**, 713.
Particle Data Group, 1980. *Rev. Mod. Phys.*, **52**, SI.
Perkins, D. H., 1972. *Introduction to High Energy Physics*, Addison-Wesley, Reading, Massachusetts.
Ramaty, R. & Mészarós, P., 1981. *Astrophys. J.*, **250**, 384.
Schiff, L. I., 1968. *Quantum Mechanics*, 3rd edn, McGraw-Hill, New York.
Spitzer, L., 1956. *Physics of Fully Ionized Gases*, Wiley, New York.
Stickforth, J., 1961. *Z. Physik*, **164**, 1.
Svensson, R., 1981. *Astrophys. J.*, **258**, 335.
Weaver, T. A., 1976. *Phys. Rev. A.*, **13**, 1563.

Appendix: Transformation of scattering angles

The scattering angles and four-momenta in the rest frame and centre-of-mass (CM) frame are defined in Fig. 5.

The transformation velocity is, putting $c = 1$

$$V = \frac{\gamma m \beta}{\gamma m + M}.$$

Transforming $p^{\mu}_{\rm A}$ to the CM gives

$$p^{\mu}_{\rm A'} = \gamma m \Gamma (1 - V\beta, \beta - V, 0, 0)$$

where

$$\Gamma^2 = 1/(1 - V^2).$$

Hence

$$p^{\mu}_{\rm A2'} = \gamma \Gamma (1 - V\beta) m \left[1, \frac{\gamma M \beta}{m + \gamma M} \cos\theta, \frac{\gamma M \beta}{m + \gamma M} \sin\theta, 0\right].$$

Transforming back to the rest frame gives

$$p^{\mu}_{\rm A2} = \gamma \Gamma (1 - V\beta) m \left[\Gamma\left(1 + \frac{V \gamma M \beta \cos\theta}{m + \gamma M}\right), \Gamma\left(\frac{\gamma M \beta \cos\theta}{m + \gamma M} + V\right), \frac{\gamma M \beta \sin\theta}{m + \gamma M}, 0\right].$$

Putting

$$\cos\Theta = \frac{p_{{\rm A2}x}}{(p^2_{{\rm A2}x} + p^2_{{\rm A2}y})^{1/2}}$$

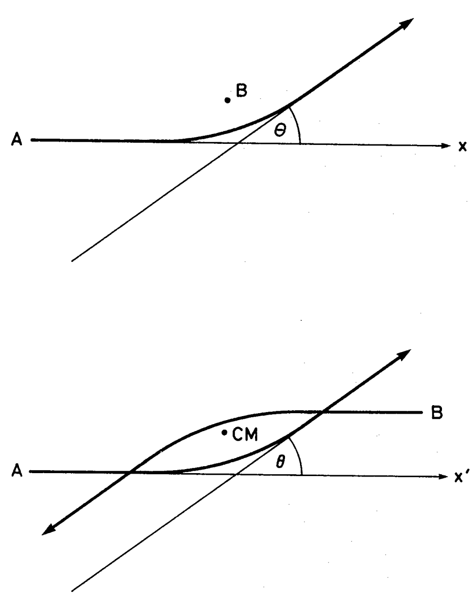

Figure 5. (a) Rest frame scattering angle, Θ. The four-momenta of particle A before and after the collision are $p^{\mu}_{\rm A} = \gamma m(1, \beta, 0, 0)$, $p^{\mu}_{\rm A2} = \gamma_2 m(1, \beta_2 \cos\Theta, \beta_2 \sin\Theta, 0)$. (b) Centre-of-mass frame scattering angle, θ. There is no energy exchange in this frame. Hence the four-momenta of particle A before and after the collision are $p^{\mu}_{\rm A'} = \gamma' m(1, \beta', 0, 0)$, $p^{\mu}_{\rm A2'} = \gamma' m(1, \beta' \cos\theta, \beta' \sin\theta, 0)$.

and substituting for V and Γ gives

$$\cos\Theta = \frac{M(\gamma m + M)\cos\theta + m(m+\gamma M)}{[M^2m^2(\gamma^2-1)\cos^2\theta + 2Mm(\gamma m + M)(m+\gamma M)\cos\theta + m^2(m+\gamma M)^2 + M^2(m^2+M^2+2\gamma mM)^2]^{1/2}}.$$

Hence

$$\cos\Theta \simeq \begin{cases} \cos\theta; & \gamma m \ll M, \theta \ll \pi \\ \dfrac{1-M(1-\cos\theta)}{\gamma m(1+\cos\theta)}; & M \ll \gamma m, \theta \ll \pi. \end{cases}$$