APPLICATIONS OF FINSLER GEOMETRY TO SPEED LIMITS TO QUANTUM INFORMATION PROCESSING

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We are interested in fundamental limits to computation imposed by physical constraints. In particular, the physical laws of motion constrain the speed at which a computer can transition between well-defined states. Here, we discuss speed limits in the context of quantum computing. We review some relevant parts of the theory of Finsler metrics on Lie groups and homogeneous spaces such as the special unitary groups and complex projective spaces. We show how these constructions can be applied to analysing the limit to the speed of quantum information processing operations in constrained quantum systems with finite dimensional Hilbert spaces of states. We demonstrate the approach applied to a spin chain system.

Keywords: Quantum Optimal Control, Finsler Geometric, Quantum Speed Limit, Quantum Computing

1. Problem and Motivation

The Margolus-Levitin (ML) bound [18] indicates a limit to the speed of dynamical evolution of any quantum system with a time-independent Hamiltonian in terms of the system's energy expectation. This and other such speed limit bounds have an interpretation in terms of the maximum information processing rate of any quantum system [17]. This bound complements the Mandelstam-Tamm inequality [25], a bound to the speed of dynamical evolution of a quantum system in terms of the energy *uncertainty*. These bounds have been combined into the 'unified bound' [16] which is, in a sense, tight for all systems. Bounds are also know for the optimal time in which a quantum gate can be implemented [8]. There is a geometric derivation of the ML bound [13].

In the application of quantum optimal control to quantum computing, a time-*dependent* Hamiltonian is more common [24], and so a more complete analysis of the limit to the speed of quantum computers needs to take into account the time dependence of the Hamiltonian. A notable result analogous to the ML bound, applicable to time dependent systems in the adiabatic regime, can be found in [1].

Both the Margolus-Levitin theorem and the Time-Energy uncertainty relation quantify the quantum information processing (QIP) capacity of a quantum system by counting or estimating the number of orthogonal states the system can pass between in a given amount of time [17]. The interpretation as a limit to QIP is justified, because pairs of orthogonal states $|\psi\rangle, |\phi\rangle$ are exactly the states that can be distinguished with certainty using a single measurement of a projection operator onto one of the states in question, such as $|\psi\rangle\langle\psi|$.

Our current programme of work aims to put facts about the quantum speed limit (QSL) on a firm geometric footing. We aim to do this in a fashion that is extensible to other physical theories such as quantum field theory so that in the future the limits to QIP presented by QFT can be analysed in a legitimately comparable manner. Further to this we aim to establish the effectiveness of the application of Finlser and related geometry to the investigation of speed limits to computation in more general analogue models of computation, especially those inspired by physics. As a step in this programme, in this paper:

- we review known speed limit theorems
- we review some relevant geometric quantum mechanics
- we review existing applications of geometry to the QSL
- we show how Finsler metrics on $SU(N)$ and CP^N can be used to derive speed limit theorems that apply to any quantum system with potentially time-*dependent* Hamiltonians
- we use this technique to re-derive some existing results and investigate its application to Heisenberg model spin chains in one dimension

2. Quantum Speed Limits

Here we review known QSL theorems.

2.1. *The Margolus Levitin Theorem*

The Margolus-Levitin theorem [18] states that for any quantum system such that:

(1) the Hamiltonian $\hat{H}$ is non-degenerate, has a discrete spectrum and has lowest eigenvalue 0

(2) the state $|\psi\rangle$ has energy expectation $\overline{E} = \langle\psi|\hat{H}|\psi\rangle$

then $t_\perp$, the least possible time in which the system can transition from $|\psi\rangle$ to a state orthogonal to $|\psi\rangle$, is bounded by:

$$t_\perp \geq \frac{\pi\hbar}{2\overline{E}} \tag{1}$$

Note that condition (1) can readily be imposed via the transformation:

$$\hat{H} \mapsto \hat{H} - E_0\hat{I} \tag{2}$$

which has no effect on the dynamics of any physical quantities. Here E_0 is the lowest eigenvalue of $\hat{H}$.

2.2. *The Time-Energy 'Uncertainty Relation'*

The Time-Energy 'Uncertainty Relation' is unlike other uncertainty relations in quantum mechanics: time is not an operator in quantum mechanics but instead a privileged, independent variable on which the state depends [7]. Although there are some situations in which time can be treated as an operator [7], that view is not taken here, and we employ the standard use of time as a real-valued independent variable throughout.

The Time-Energy 'Uncertainty Relation' states:

$$t_\perp \geq \frac{\pi\hbar}{2\Delta E} \tag{3}$$

where ΔE is the variance of energy, or 'energy uncertainty', $\Delta E = \sqrt{\langle\psi|\hat{H}^2|\psi\rangle - \langle\psi|\hat{H}|\psi\rangle^2}$.

3. Geometric Quantum Mechanics

Geometric quantum mechanics is an attempt to reformulate the theory of quantum mechanics in terms of differential geometry [6]. We find this a particularly fruitful formulation for investigating QSL questions.

In this section, we review the material needed to derive our results in following sections. We are concerned only with *finite* dimensional systems: systems for which the Hilbert space of states in the standard formulation of QM is finite dimensional.

3.1. *CP^N as the Space of States*

The space of states in quantum mechanics is typically formulated as a complex vector space $\mathbb{C}^{N+1}$ with a hermitian inner-product $\langle\cdot|\cdot\rangle : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ [10]. However, not every vector in the space of states corresponds to a truly different physical state, as physical states are normalised to have norm one (according to the norm $||\,|\psi\rangle\,|| = \sqrt{\langle\psi|\psi\rangle}$), and global phases are not physical quantities (i.e. they cannot be observed) [7]. The space of physically distinct states is given by CP^N, as shown below.

3.1.1. *Normalisation*

The first step in realising CP^N as the space of physically distinct states is treating the normalisation condition. Define an equivalence relation $\sim_1 \subseteq \mathbb{C}^{N+1} \times \mathbb{C}^{N+1}$ by: $|\psi\rangle \sim_1 \lambda|\psi\rangle \ \forall\lambda \in \mathbb{R}/\{0\}$. This yields a realisation of the sphere S^{2N+1} as a set of equivalence classes of points in $\mathbb{C}^{N+1}$. This construction can also be clarified by writing an arbitrary state as:

$$|\psi\rangle = \sum_{k}^{k=N} \alpha_k |A_k\rangle \tag{4}$$

where $|A_k\rangle$ are any orthonormal basis of $\mathbb{C}^N$. The condition that the state has norm one becomes:

$$\langle\psi|\psi\rangle = \sum_{k}^{k=N} |\alpha_k|^2 = \sum_{k}^{k=N} \Re(\alpha_k)^2 + \Im(\alpha_k)^2 = 1 \tag{5}$$

which is the formula defining a sphere of radius 1.

3.1.2. *Phase*

The next step is to associate states which differ only by a global phase $e^{i\theta}$. Take a quotient of S^{2N+1} into equivalence classes, where the equivalence relation $\sim_2 \subseteq S^{2N+1} \times S^{2N+1}$ is defined by $[|\psi\rangle]_{\sim_1} \sim_2 e^{i\theta}[\ |\psi\rangle\ ]_{\sim_1}\ \forall\theta \in \mathbb{R}$. The new space is:

$$S^{2N+1}/U(1) \cong CP^N \tag{6}$$

This quotienting process can be represented by:

$$\mathbb{C}^{N+1} \xrightarrow{\phi_1} S^{2N+1} \xrightarrow{\phi_2} CP^N, \qquad \phi := \phi_2 \circ \phi_1 : \mathbb{C}^{N+1} \to CP^N$$

where:

$$\phi_1 : |\psi\rangle \mapsto [|\psi\rangle]_{\sim_1} \tag{7}$$

$$\phi_2 : [|\psi\rangle]_{\sim_1} \mapsto [[|\psi\rangle]_{\sim_1}]_{\sim_2} \tag{8}$$

Here $\phi : \mathbb{C}^{N+1} \to CP^N$ and realises the quotient by the equivalence relation $\sim \subseteq \mathbb{C}^{N+1} \times \mathbb{C}^{N+1}$ defined by $|\psi_1\rangle \sim \lambda e^{i\theta}|\psi_1\rangle\ \forall\lambda \in \mathbb{R}/\{0\}, \theta \in \mathbb{R}$. The purpose of breaking the quotienting process into two steps is to illustrate the mathematical realisation of the two physical principals: that states are normalised and that global phases are unphysical.

3.2. *SU(N) as the Space of Quantum Time Evolution Operators*

Typically in quantum mechanics the set of all time evolution operators $\hat{U} : \mathbb{C}^N \to \mathbb{C}^N$ is taken to be the group of unitary operators. A condition (there are other equivalent ones) for unitary of an operator $\hat{U}$ is that all inner products are preserved by its action of $\mathbb{C}^N$:

$$\langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle = \langle\psi|\psi\rangle \qquad \forall\, |\psi\rangle \in \mathbb{C}^N \tag{9}$$

This is exactly the condition that the total probability of finding a system in any state is preserved to be 1.

The condition that the phase of the state is unchanged by time evolution is more difficult to analyse. Unitary operators can 're-phase' a state, for example the operator $e^{i\theta}\hat{I}$ is unitary. This operator has the action on a state $|\psi\rangle$:

$$e^{i\theta}\hat{I}|\psi\rangle = e^{i\theta}|\psi\rangle \tag{10}$$

In our approach to QSL, optimal *times* for implementing a gate should be related to *distances* between points on the group, for some appropriate notion of distance. So we do not want to include changes in unphysical parameters (such as the global phase of an operator) when devising notions of distance on the group, In order to avoid changes in global phase of $\hat{U}_t$ contributing to the length of its trajectory on the group, we study metric structure on the special unitary group, rather than on the unitary group.

$SU(N)$ is a linear algebraic group (LAG) [4][a]; that is, it is a subgroup of $GL(\mathbb{C}^N)$. These groups have a simplifying feature: the differentials of left and right translation (where $\hat{U}, \hat{V} \in SU(N)$),

$$dL_{\hat{V}}\big|_{\hat{U}} : T_{\hat{U}}SU(N) \to T_{L_{\hat{V}}(\hat{U})}SU(N) = T_{\hat{V}\hat{U}}SU(N) \tag{11}$$
$$dR_{\hat{V}}\big|_{\hat{U}} : T_{\hat{U}}SU(N) \to T_{R_{\hat{V}}(\hat{U})}SU(N) = T_{\hat{U}\hat{V}}SU(N)$$

act on tangent vectors $\hat{A} \in T_{\hat{U}SU(N)}$ in a particularly simply way:

$$dL_{\hat{V}}\big|_{\hat{U}}(\hat{A}) = \hat{U}\hat{A} \tag{12}$$
$$dR_{\hat{V}}\big|_{\hat{U}}(\hat{A}) = \hat{A}\hat{U}$$

This follows from the fact that a linear map is always equal to its own differential.

3.3. *Finsler Metrics on $SU(N)$ and CP^N*

3.3.1. *Finsler Metrics*

A Finsler metric is a generalisation of a Riemannian metric.

Definition 1 (Minkowski Norm [9]) *A* Minkowski Norm *on a real, finite dimensional vector space V is a map $\|\cdot\| : V \to \mathbb{R}$ such that the following hold:*

(1) $F(v) \geq 0$ $\forall v \in V$ and $F(v) = 0 \Leftrightarrow v = 0$ (Non-degeneracy)
(2) $F(\lambda v) = \lambda F(v)$, $\forall \lambda \in \mathbb{R}/\{0\}$, $\forall v \in V$ (Positive homogeneity)
(3) The Hessian of F^2 at each vector in V is positive definite. (Strong Convexity)

Condition (3) in the definition of a Minkowski Norm, expressed in full, reads:

$$g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F(y + su + tv)^2\bigg|_{s=t=0} \tag{13}$$

[a]It is often misstated that $U(N) \cong SU(N) \times U(1)$. This can be directly shown to be false by applying the well known first isomorphism theorem for groups to the homomorphism $\phi : SU(N) \times U(1) \to U(N)$ given by $\phi(\hat{U}, e^{i\theta}) = e^{i\theta}\hat{A}$. The correct statement is instead that $U(N)$ can be written as a semi-direct product of $SU(N)$ and a $U(1)$ subgroup of $SU(N)$. The object which behaves as $U(N)$ with operators identified up to a global phase is the projective unitary group $PSU(N)$. It is defined by the quotient $U(N)/Z(U(N))$ where $Z(U(N))$ is the centre of $U(N)$, well know to be the 'scalar matrices' $\{e^{i\theta}\hat{I} | \theta \in [0, 2\pi)\}$. However it is true that $U(N) \cong (SU(N) \times U(1))/\mathbb{Z}_N$, which is the conclusion of applying the first isomorphism theorem to the homomorphism ϕ. As the Lie algebras of $PSU(N)$ and $SU(N)$ are identical, working with $SU(N)$ rather than $PSU(N)$ does not affect anything that follows in this paper; this footnote is included for completeness.

Definition 2 (Finsler Metric [2,9]) *A* Finsler metric *is a map* $F_p : T_pM \to \mathbb{R}$ *such that* F *is a Minkowski Norm* $\forall p \in M$ *and* F *is smooth on the slit tangent bundle (often denoted* $TM/\{0\}$ *by a mild abuse of notation) of* M*.*

Definition 3 (Reversible Finsler metric [9]) *A Finsler metric* F *on a manifold* M *is said to be* Reversible *if it satisfies:* $F_p(v) = F_p(-v)$ $\forall p \in M$*,* $\forall v \in T_pM$*.*

Note that reversible Finsler metrics are exactly those for which the Minkowski norm is, point-wise, a norm in the usual sense of a normed vector space on each tangent space.

Example 4. *Every Riemannian metric* g *on* M *defines a Finsler metric* F *by:*

$$F_p(v) = \sqrt{g_p(v,v)} \tag{14}$$

It is in this sense that Finsler geometry generalises Riemannian geometry.

3.4. *Some Notions of Invariance*

There are several related notions of invariance of a Finsler metric on a Lie group.

Definition 5 (Left and Right Invariant Finsler Metric [9]) *A Finsler metric* F *on a Lie group* G *is a* Left Invariant Finsler metric *if,* $\forall g, h \in G$*,* $\forall v \in T_gG$*:*

$$F_{L_h(g)}\left(dL_h\big|_g(v)\right) = F_{hg}\left(dL_h\big|_g(v)\right) = F_g(v) \tag{15}$$

The corresponding condition for a Right Invariant *Finlser metric is:*

$$F_{R_h(g)}\left(dR_h\big|_g(v)\right) = F_{gh}\left(dR_h\big|_g(v)\right) = F_g(v) \tag{16}$$

These conditions simplify somewhat in the case of a LAG such as $SU(N)$, where the notions of left and right translation have a simple form. The left and right invariance conditions respectively take the form:

$$F_{hg}(hv) = F_g(v) \tag{17}$$

$$F_{gh}(vh) = F_g(v) \tag{18}$$

where the composition of a group element (such as g, h above) and a tangent vector (such as v above) is understood to be achieved by standard matrix multiplication.

3.5. *Existing Geometric Investigation into the Quantum Speed Limit*

Various authors have initiated a geometric investigation into the QSL with a variety of approaches. The earliest application of geometry to rationalisation times [5] applies the known formula for the geodesics lengths (of the Fubini-Study metric on CP^N) to pairs of orthogonal states in a two level system. Others [13,25] attempt a derivation of the ML theorem and MT inequality for finite dimensional systems with time independent Hamiltonians also based on the geodesic lengths of such curves on CP^N. See the original papers for the progress of this attempt. Most recently [15] initiates an investigation of the relationship between metric (in the finite sense of metric spaces, not the infinitesimal sense of differential geometry applied in this work) structures on $U(N)$ and the QSL.

4. Constrained Systems and Finsler Metrics

4.1. *General Manifolds*

Consider a "navigator" on a general manifold M (taken to be compact and connected), where there is some internal limitation restricting the motion on M that they can achieve. This can be represented, for each $p \in M$, by $\mathcal{A}_p$, a set of 'allowed' tangent vectors to any trajectory of the navigator at the point $p \in M$. For a more complete discussion of the type of navigation problem see [22]; for a discussion in the context specifically of quantum mechanics see [20].

We wish to achieve a geometric analysis of the limitation on the speed with which the navigator can traverse a desired specific trajectory. To do so, we define the following type of metric:

Definition 6 (Constraint Compatible Finsler Metric) *Given a manifold (compact and connected), and given a constraint in the form of a set of subsets $\mathcal{A}_p \subset T_pM$ (one for each $p \in M$), we say that a Finsler metric $F_p : T_pM \to \mathbb{R}$ such that $\forall p \in M, \forall x \in \mathcal{A}_p$, $F_p(x) \leq 1$ is* Constraint Compatible.

Our definition is justified by the following theorem, which shows how such a metric is related to the speed limit for the navigation problem in a constrained dynamical system on a manifold M:

Theorem 7. *Given:*

(1) A connected and compact smooth manifold M
(2) A constraint in the form of a collection of subsets $\mathcal{A}_p \subset T_pM$ $\forall p \in M$
(3) A constraint compatible Finsler metric F

then any parametrised curve $\gamma : [0, T] \to M$ satisfying:

$$\frac{d\gamma}{dt} \in \mathcal{A}_{\gamma(t)} \tag{19}$$

has the property

$$T \geq L[\gamma] = \int_0^T F_{\gamma(t)}\left(\frac{d\gamma}{dt}\right) dt \tag{20}$$

Proof. $L[\gamma] = \int_0^T F_{\gamma(t)}\left(\frac{d\gamma}{dt}\right) dt \leq \int_0^T 1\ dt = T$ □

4.2. *Speed Limits and Finsler Metrics on* $SU(N)$

We now apply theorem (7) to constrained quantum mechanical systems, and assess the minimum time in which they can implement a desired QIP task, from a geometric perspective. In order to implement a desired quantum operation $\hat{O}$ in a quantum system, one must have a system with the property that $\hat{U}_t = \hat{O}$, where $\hat{U}_t$ is the system's time evolution operator at some time t [24].

The time evolution operator $\hat{U}_t : \mathbb{C}^N \to \mathbb{C}^N$ in quantum mechanics obeys the Schrödinger equation [11]:

$$\frac{d}{dt}\hat{U}_t = -\frac{i}{\hbar}\hat{H}_t\hat{U}_t \tag{21}$$

Now suppose that the Hamiltonian $\hat{H}_t$ (possibly time dependent) is constrained (for all time), due to physical limitations in the lab, to be in some set of allowed Hamiltonians $\hat{H}_t \in \mathcal{H}$. This results in the tangent vector to the trajectory of $\hat{U}_t$, $\frac{d}{dt}\hat{U}_t$ being constrained to lie in the set $-\frac{i}{\hbar}\mathcal{H}\hat{U}_t$ at time t. The set $-\frac{i}{\hbar}\mathcal{H}\hat{U}_t$ is the right translation of the set of tangent vectors $-\frac{i}{\hbar}\mathcal{H} \subset T_{\hat{I}}(SU(N))$ to the identity. We have the following specific objects playing the role of the objects appearing in the premises of theorem (7):

(1) A compact, connected manifold $SU(N)$
(2) The time evolution operator playing the role of the navigator on the manifold
(3) The set of allowed tangent vectors at a point $\hat{U}$ on $SU(N)$ is $\mathcal{A}_{\hat{U}} = -\frac{i}{\hbar}\mathcal{H}\hat{U}$

As the set of allowed tangent vectors is right invariant (its construction by right translation guarantees this), and we seek metrics satisfying the premises of theorem (7), then it is correct to restrict to considering only right invariant metrics. This is justified since any non-right invariant metric can yield only speed limits that are not as tight as those yielded by some right invariant metric, i.e. the right translation of a metric for which the indicatrix is as small as possible at that point. The optimality of right invariant metrics follows from the right invariance of the set of allowed tangent vectors. Then, if a metric is optimal at a point, its right translation is optimal at *all* points.

It is also the case that bi-invariant metrics are simpler to calculate with, and thus to obtain explicit speed limit formulae, which are all right invariant.

Definition 8 (Shattan p-Norms [3]) *The* Shattan p-norm *of a matrix* $\hat{A} \in Mats_{N\times N}(\mathbb{C})$ *for* $p \in \mathbb{R}$*, such that* $1 \leq p$*, is given by:*

$$\|\hat{A}\|_p = \Big(\mathrm{Tr}(|\hat{A}|^p)\Big)^{1/p} \tag{22}$$

where $|\hat{A}| := (\hat{A}^\dagger\hat{A})^{1/2}$*. This matrix square root is taken to be the principal one, guaranteed to exist and be unique, as* $\hat{A}^\dagger\hat{A}$ *is positive semi-definite.*

We can readily observe by a simple calculation that the left or right translation of this norm from the tangent space at the identity of $SU(N)$ is a bi-invariant Finlser metric. These Finsler metrics are given by:

$$\begin{aligned} F_{\hat{U}}(\hat{A}\hat{U}) &= \mathrm{Tr}(|\hat{A}\hat{U}|^p)^{1/p} = \mathrm{Tr}((\hat{U}^\dagger \hat{A}^\dagger \hat{A}\hat{U})^{p/2})^{1/p} \\ &= \mathrm{Tr}(\hat{U}^\dagger (\hat{A}^\dagger \hat{A})^{p/2}\hat{U})^{1/p} = (\mathrm{Tr}(|\hat{A}|^p))^{1/p} = F_{\hat{I}}(\hat{A}) = \|\hat{A}\|_p \end{aligned} \tag{23}$$

The calculation checking left invariance is similar, and so omitted.

It is a theorem that a bi-Invariant Finlser metric on a compact, connected Lie group G has the property that its geodesics coincide with the one parameter subgroups [12, 14]. This fact is now exploited in order to derive speed limit theorems for constrained quantum systems.

Theorem 9. *Any quantum system (with potentially time dependant Hamiltonian) $\hat{H}_t$ such that $\alpha\|\frac{i}{\hbar}\hat{H}_t\|_p \leq 1\ \forall t \in \mathbb{R}$ cannot implement quantum gate $\hat{O}$ in a time less than $\alpha\,\mathrm{Tr}(|\log(\hat{O})|^p)^{1/p} = \alpha\|\log(\hat{O})\|_p$.*

Proof. We need the following facts:

(1) The metric, say F, defined as any positive multiple, say α, of the right translation of the Shatten p-norm at the identity on $SU(N)$ is bi-invariant and reversible.
(2) Bi-invariant Finlser metrics have the one parameter subgroups as their geodesics.
(3) Any physical trajectory of the time evolution operator is a solution to the Schrodinger equation.

We have, by assumption, that the system has $\alpha\|\frac{i}{\hbar}\hat{H}_t\|_p \leq 1$ for all times. This and (3) above entail that the time evolution operator has:

$$F_{\hat{U}_t}\left(\frac{d}{dt}\hat{U}_t\right) = F_{\hat{U}_t}\left(-\frac{i}{\hbar}\hat{H}_t\hat{U}_t\right) = \alpha\left\|-\frac{i}{\hbar}\hat{H}_t\right\|_p = \alpha\left\|\frac{i}{\hbar}\hat{H}_t\right\|_p \leq 1 \tag{24}$$

and thus we see that this metric is a constraint compatible metric for this constraint.

The final step uses the fact that the metrics formed by the right translation of the Shattan-p norm at the identity are manifestly reversible. In fact, the set of allowed tangent vectors at a point is exactly the indicatrix (Finsler analogue of the unit sphere) of this metric acting on the tangent space at that point.

Next we need that the length of the trajectory of $\hat{U}_t$ parametrised between $t = 0$ and $t = T$ on $SU(N)$ according to such an F is given by:

$$L[\hat{U}_t] = \int_{t=0}^{t=T} F_{\hat{U}_t}\left(\frac{d}{dt}\hat{U}_t\right) dt \geq T_{opt} \tag{25}$$

where the final inequality is due to an application of theorem (7) made possible by the observation that the metric is compatible with the constraint.

Now, if we wish to implement operation $\hat{O}$ time optimally using a system with a constrained Hamiltonian, as described in the premises, then we seek the least time T_{opt} such that $\hat{U}_t = \hat{O}$. We have (from basic properties of the time evolution operator) that $\hat{U}_0 = \hat{I}$. Next we seek the length of a geodesic of F connecting $\hat{I}$ to $\hat{O}$

in order to obtain the tightest bound available among all curves connecting the same end points. As stated above, such a geodesic must be a one parameter subgroup (in the sense that they share the same image on $SU(N)$, but perhaps not the same parametrisation). As the length of a curve is independent of the parametrisation of that curve, any parametrisation can be chosen.

As the desired geodesic is guaranteed to be a one parameter subgroup, we need consider only curves of the form:

$$\hat{U}_t = \exp\left(-\frac{i}{\hbar} t\hat{H}\right) \tag{26}$$

where $\hat{H}^\dagger = \hat{H}$ (so that $-\frac{i}{\hbar}\hat{H} \in \mathfrak{su}(N)$, the special unitary Lie algebra).

Such a geodesic can be found by supposing that $\hat{U}_T = \hat{O}$ holds, then taking matrix logarithms of both sides to yield:

$$\hat{H} = -\frac{\hbar}{iT}\log(\hat{O}) \tag{27}$$

Substituting this into equation (26), we find the desired curve to be:

$$\hat{U}_t = \exp\left(\frac{t}{T}\log(\hat{O})\right) \tag{28}$$

To obtain the final result we derive the length of this curve and apply theorem (7) as follows:

$$\begin{aligned} T_{opt} \geq L[\hat{U}_t] &= \int_{t=0}^{t=T} F_{\hat{U}_t}\left(\frac{d}{dt}\hat{U}_t\right) dt \qquad (29)\\ &= \int_{t=0}^{t=T} F_{\exp\left(\frac{t}{T}\log(\hat{O})\right)}\left(\frac{d}{dt}\exp\left(\frac{t}{T}\log(\hat{O})\right)\right) dt \\ &= \int_{t=0}^{t=T} F_{\exp\left(\frac{t}{T}\log(\hat{O})\right)}\left(\frac{1}{T}\log(\hat{O})\exp\left(\frac{t}{T}\log(\hat{O})\right)\right) dt \\ \text{(by right invariance of } F) &= \int_{t=0}^{t=T} F_{\hat{I}}\left(\frac{1}{T}\log(\hat{O})\right) dt \\ \text{(as } F \text{ is homogeneous)} &= \left(\frac{1}{T}\int_{t=0}^{t=T} dt\right) F_{\hat{I}}\left(\log(\hat{O})\right) \\ \text{(by definition of } F) &= \alpha \operatorname{Tr}\left(\left|\log(\hat{O})\right|^p\right)^{1/p} \end{aligned}$$

□

4.3. *Speed Limits and Finsler Metrics on* CP^N

There is a unique Riemannian metric, up to a constant multiple, on CP^N that is invariant under the natural choice of action of the unitary group on CP^N. That is to say it has the property that $U(N)$ consists only of isometries [6]. This metric is the Fubini-Study metric; there is more than one way to represent its metric tensor. One such way, well adapted for use in quantum mechanics, is to write it as a function on the tangent spaces to $\mathbb{C}^{N+1}$ that is constant on equivalence classes of the equivalence relation that allows CP^N to be constructed from $\mathbb{C}^{N+1}$ as described earlier. The formula is:

$$ds^2 = \frac{\langle\delta\psi|\delta\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\delta\psi|\psi\rangle\langle\psi|\delta\psi\rangle}{\langle\psi|\psi\rangle^2} \tag{30}$$

This metric can readily be used to prove the Time Energy uncertainty relation by knowing that the geodesic distance between orthogonal states in Hilbert space is $\frac{\pi}{2}$. This can be observed by noting that the finite form of this metric is [23]:

$$\gamma(|\psi\rangle, |\phi\rangle) = \arccos\sqrt{\frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle^2\langle\phi|\phi\rangle^2}} \tag{31}$$

The metric clearly has the same unitary invariance properties as its infinitesimal form, eqn.(30).

The following relationship between the time energy uncertainty relation is well known; we re-derive it to illustrate the usefulness of geometric constructions in proving quantum speed limit theorems.

In the case that $|\psi_t\rangle$ solves the Schrödinger equation for a time-independent Hamiltonian $\hat{H}$ we have:

$$|\delta\psi_t\rangle = \frac{d}{dt}|\psi_t\rangle = \frac{d}{dt}\exp\left(-it\hat{H}\right)|\psi_0\rangle = -i\hat{H}\exp(-it\hat{H})|\psi_0\rangle = -i\hat{H}|\psi_t\rangle \tag{32}$$

Substituting this into the definition of arc length corresponding to the metric at hand, we find:

$$\begin{aligned} L[|\psi_t\rangle] &= \int_{t=0}^{t=\tau} \sqrt{\langle\delta\psi_t|\delta\psi_t\rangle - \langle\delta\psi_t|\psi_t\rangle\langle\psi_t|\delta\psi_t\rangle}\,dt \\ &= \int_{t=0}^{t=\tau} \sqrt{\langle\psi_t|\hat{H}^2|\psi_t\rangle - \langle\psi_t|\hat{H}|\psi_t\rangle^2}\,dt \\ &= \int_{t=0}^{t=\tau} \Delta E_{|\psi_t\rangle}dt = \int_{t=0}^{t=\tau} \Delta E_{|\psi_0\rangle}dt = \tau\Delta E_{|\psi_0\rangle} \geq \frac{\pi}{2} \end{aligned} \tag{33}$$

$\Delta E_{|\psi_t\rangle}dt$ can be replaced by $\Delta E_{|\psi_0\rangle}$ in the last line, since here the Hamiltonian is time-independent, which implies that the energy uncertainty is also. From this follows the Mandelstam-Tamm inequality:

$$\tau \geq \frac{\pi}{2\Delta E_{|\psi_0\rangle}} \tag{34}$$

Compare this derivation to that in [13] (their eqns. 22-25; note that the 'Wootters distance' is simply the finite form of the Fubini-Study metric applied to normalised states). There the finite form of the metric is differentiated; here we use the differential form of the metric immediately.

At this point a relevant question is: can Finlser geometry be used to derive the ML theorem in a similar way to the above derivation of the MT inequality? We conjecture that it can.

4.4. *The Correspondence Between Speed Limits on $SU(N)$ and CP^N*

Suppose that we want to calculate the optimal time to implement a transformation from state $|\psi_I\rangle$ to some other state $|\phi_O\rangle$ using a system with a constrained Hamiltonian (so that there is any speed limit at all!). It is physically intuitive that the optimal time (T say) must be equal to the optimal time to drive the time evolution operator $\hat{U}_t$ from $\hat{I}$ at $t = 0$ to some $\hat{U}_T$ such that $\hat{U}_T|\psi_I\rangle = |\psi_O\rangle$. It is on this intuition that the following method builds:

Definition 10 (Homogenous Space [9]) *A manifold M which is equipped with a transitive action of a Lie group G on M is called a* Homogeneous Space.

Definition 11 (Stabilizer [9]) *Given a point $p \in M$, $G_p = \{g \in G \,|\, g \circ p = p\}$ is called the* Stabiliser Subgroup *of p.*

With the above definitions in mind the following relationship can be proved if $U(N)$ is taken to be the stabiliser of some fixed point $[|\psi^*\rangle]_\sim \in CP^N$:

Theorem 12 ([9], ch. 2) $CP^N \cong SU(N+1)/U(N)$

This equivalence is a taken to mean the existence of a diffeomorphism (invertible smooth map between manifolds with smooth inverse). The / symbol means quotienting into left cosets, that is $SU(N+1)/U(N) = \{\hat{U}U(N) \big| \hat{U} \in SU(N+1)\}$. There is no guarantee that this quotienting process yields a group structure, as $U(N)$ is not a normal subgroup of $SU(N+1)$. However, it is only an equivalence of geometries, not algebraic structures. $SU(N+1)$ has an action on $\mathbb{C}^{N+1}$ (by standard matrix multiplication). This can be used to define an action on CP^N given by $\hat{U}[|\psi\rangle]_\sim = [\hat{U}|\psi\rangle]_\sim$. One can readily check that this is equivalent to the action on CP^N.

The map $\gamma : SU(N+1) \to CP^N$ given by

$$\gamma(\hat{U}) := [\hat{U}|\psi^*\rangle]_\sim \tag{35}$$

is constant on cosets as, for any $\hat{V} \in U(N)$ the following holds:

$$\gamma(\hat{U}\hat{V}) = [\hat{U}\hat{V}|\psi^*\rangle]_\sim = \hat{U}(\hat{V}[|\psi^*\rangle]_\sim) = \hat{U}[|\psi^*\rangle]_\sim = [\hat{U}|\psi^*\rangle]_\sim = \gamma(\hat{U}) \tag{36}$$

The isomorphism $\beta : SU(N+1)/U(N) \to CP^N$ is given by [9]:

$$\beta(\hat{U}U(N)) := \gamma(\hat{U}) \tag{37}$$

from which it follows that:

$$\beta(\hat{U}U(N)) = [\hat{U}|\psi^*\rangle]_\sim \tag{38}$$

The differential of the map β,

$$d\beta\big|_{\hat{U}} : T_{(\hat{U}U(N))}SU(N+1)/U(N) \to T_{\beta(\hat{U}U(N))}CP^N = T_{[\hat{U}|\psi^*\rangle]}CP^N \tag{39}$$

can be used to study the relationship between Finsler metrics on CP^N that are invariant under the action of $SU(N+1)$, and Finsler metrics on $SU(N+1)$ itself satisfying a complicated invariance property (discussed at length in [9, ch.2]). That invariance property is being constant on cosets, or 'compatible with the quotient'; it must not depend on which representative of each equivalence class in $SU(N)$ is chosen: the 'same' tangent vectors at each point in a class must be assigned the same length, so as to avoid ambiguity when assigning lengths to tangent vectors on CP^N.

The push forward of a Finsler metric on $SU(N+1)$, $F_{\hat{U}} : T_{\hat{U}}SU(N+1) \to \mathbb{R}$ defines an invariant Finsler metric on $R_{\hat{U}U(N)} : T_{\hat{U}U(N)}CP^N \to \mathbb{R}$ (under the appropriate condition discussed above) via:

$$R_{\hat{U}U(N)}(d\beta\big|_{\hat{U}}(\hat{A})) := F_{\hat{U}}(\hat{A}) \tag{40}$$

We use this to prove a QSL theorem next. Its use proving more general QSL theorems will be studied in future work.

4.5. *Orthogonality Times*

One example of applying the relationship between metrics on the two spaces, $SU(N+1)$ and CP^N, can be used to derive bounds on orthogonality times, outlined below. We have yet to re-derive the known bound exactly using this method, but we conjecture that it is possible, and here derive some bounds on orthogonality times to illustrate the principle.

Any such $\hat{U}_t$ mapping some state $|\phi\rangle$ to an orthogonal states (for a possibly time-independent system) must have the following form for some unitary change of basis matrix $\hat{V}$, some unitary $\hat{A}$ and some $\theta \in [0, 2\pi]$:

$$\hat{U}_t = \exp(-it\hat{H}) = \hat{V}^\dagger \hat{B} \hat{V} \tag{41}$$

where

$$\hat{B} = \begin{pmatrix} 0 & -\exp(-i\theta) & 0\ldots 0 \\ \exp(i\theta) & 0 & 0\ldots 0 \\ 0 & 0 & \\ \vdots & \vdots & \hat{A} \\ 0 & 0 & \end{pmatrix} \tag{42}$$

Consider the norm on $\mathfrak{su}(N)$, $\|\cdot\|_p : SU(N) \to \mathbb{R}$ (for $p \geq 1$) defined as above. Taking the norm of both sides of eqn.(41) yields:

$$\mathrm{Tr}\left(\left|-it\hat{H}\right|^p\right) = \mathrm{Tr}\left(\left|\log(\hat{V}^\dagger \hat{B} \hat{V})\right|^p\right) \tag{43}$$

This implies:

$$\begin{aligned} t^p\,\mathrm{Tr}(|\hat{H}|^p) &= \mathrm{Tr}(|\log \hat{B}|^p) \\ &= \mathrm{Tr}(|\log \begin{pmatrix} 0 & -\exp(-i\theta) \\ \exp(i\theta) & 0 \end{pmatrix} |^p) + \mathrm{Tr}(|\log \hat{A}|^p) \\ &\leq \mathrm{Tr}(|\log \begin{pmatrix} 0 & -\exp(-i\theta) \\ \exp(i\theta) & 0 \end{pmatrix} |^p) = \frac{2\pi^p}{2^p} \end{aligned} \tag{44}$$

Taking the p^{th} root of each side yields our result, the bound:

$$t \geq \frac{2^{\frac{1}{p}}\pi}{2\mathrm{Tr}(|\hat{H}|^p)^{\frac{1}{p}}} = \frac{\pi}{2}\left(\frac{2}{\mathrm{Tr}(|\hat{H}|^p)}\right)^{\frac{1}{p}} \tag{45}$$

This is similar to, but not as strong as, the bounds given in [15].

This bound coincides, in the special case of a two level system, with the Margolus-Levitin bound if $p = 1$.

5. Example of Geometric Derivation of QSL on $SU(N)$: the Controlled Heisenberg Spin Chain

We apply theorem (9) to obtain results for a specific quantum system of relevance in quantum computing: a spin chain. A controlled Heisenberg spin chain of N spins (with coupling constants J_x, J_y, J_z) has Hamiltonian [19]:

$$\begin{aligned} \hat{H}_t = &\left(\sum_{k\in\{x,y,z\}} J_k \left(\sum_{n=0}^{N-2} \hat{I}_2^{\otimes^n} \otimes \sigma^k \otimes \sigma^k \otimes \hat{I}_2^{\otimes^{N-n-2}} \right) \right) \\ &+ \left(\sum_{n=0}^{N-1} f_n(t) \hat{I}_2^{\otimes^n} \otimes \sigma^z \otimes \hat{I}_2^{\otimes^{N-n-1}} \right) \end{aligned} \tag{46}$$

We impose the physical constraint that the total energy used to produce the control functions is less than κ^2, and obtain:

$$\sum_{k=0}^{N-1} f_n(t)^2 \leq \kappa^2 \tag{47}$$

We now find a constraint compatible, bi-invariant metric (in this example a Riemannian one) as follows:

Theorem 13. *The Finsler metric F_{opt} that is the largest multiple α of the metric $F_{\hat{U}}(\hat{A}\hat{U}) = \mathrm{Tr}(\hat{A}^\dagger\hat{A})^{1/2} = \|\hat{A}\|_2$ such that the metric is constraint compatible with the constraint in eqn(47) is given by:*

$$F_{opt}(\hat{A}) = \frac{\mathrm{Tr}(\hat{A}^\dagger\hat{A})^{1/2}}{\sqrt{(N-1)({J_x}^2 + {J_y}^2 + {J_z}^2) + \kappa^2}} \tag{48}$$

That is, the tightest speed limit (obtained from this specific one parameter family of metrics alone) is obtained by setting:

$$\alpha = \frac{1}{\sqrt{(N-1)({J_x}^2 + {J_y}^2 + {J_z}^2) + \kappa^2}} \tag{49}$$

Proof. Omitted: a simple but laborious substitution of $i\hat{H}_t\hat{U}$ into the formula for the prescribed metric at the point $\hat{U}$ on the special unitary group given by $F_{\hat{U}}(i\hat{H}_t\hat{U})$, followed by observation of the maximum value this quantity can take. □

Constraint compatible, explicitly non-Riemannian Finsler metrics can also be found, and will be part of further work.

We now conclude:

Theorem 14. *The optimal time to implement a gate $\hat{O}$ in a constrained (as in constraint given in formula (47)) Heisenberg spin chain system of N spins is bounded by:*

$$T_{opt} \geq \frac{\mathrm{Tr}(\log(\hat{O})^\dagger \log(\hat{O}))^{1/2}}{\sqrt{(N-1)({J_x}^2 + {J_y}^2 + {J_z}^2) + \kappa^2}} \tag{50}$$

Proof. by a direct application of theorem (9). □

6. Conclusions

We have demonstrated the use of geometric formalisms in deriving time bounds for QIP in constrained quantum systems. We have shown the promise of the application of Finsler geometry to the problem of the QSL in constrained quantum systems. More importantly, the new formulation allows us to exploit the approach for time dependent and constrained systems, as relevant to quantum computation. We have demonstrated this by deriving and proving a speed limit result, theorem (9). We have applied this to a spin chain system.

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