Geometric Methods for Analysing Quantum Speed Limits: time-dependent controlled quantum systems with constrained control functions

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Abstract. We are interested in fundamental limits to computation imposed by physical constraints. In particular, the physical laws of motion constrain the speed at which a computer can transition between well-defined states. Here, we discuss speed limits in the context of quantum computing. We derive some results in the familiar representation, then demonstrate that the same results may be derived more readily by transforming the problem description into an alternative representation. This transformed approach is more readily extended to time-dependent and constrained systems. We demonstrate the approach applied to a spin chain system.

1 Problem and Motivation

The Margolus-Levitin bound [11] defines the limit to the speed of dynamical evolution of a quantum system with time-independent Hamiltonian as imposed by the energy expectation. This and other such speed limit bounds have an interpretation in terms of the maximum information processing rate of a quantum systems [10]. This bound complements the Mandelstam-Tamm inequality [16], a bound to the speed of dynamical evolution of a quantum system in terms of the energy *uncertainty*.

However, in the application of quantum optimal control to quantum computing, a time-*dependent* Hamiltonian is more common [15], and a more complete analysis of the limit to the speed of quantum computers needs to take into account the time dependence of the Hamiltonian. A notable result analogous to the Margolus-Levitin bound, applicable to time dependent systems in the adiabatic regime, can be found in [1].

When considering the 'ultimate' physical limits to computation [10], one considers a time-independent system as a model for the fastest possible quantum computer. We can consider a time-dependent control system as a sub-system of a larger time-independent system as follows. Let system A be the computational system, and let another system B produce the control fields; these could be considered to be subsystems of the larger system $A \otimes B$. If B is chosen to include all of the environment of A that significantly affects A, then the combined system could be described by a time-independent Hamiltonian, subject to the 'ultimate' physical limits. Since the control functions must be implemented by some quantum system, the limits quantum mechanics places on the dynamics of the system producing the control fields places limits on the control fields themselves. Hence, in addition to time dependence, the control functions may be subject to further physical constraints.

Geometric derivations [6, 16, 9] have been used to determine bounds for time-independent systems. Specifically, [9] provides a strong connection between the quantum speed limit and metric structures on unitary groups. Here we extend the kind of analysis of [9] to time-dependent and constrained systems. We analyse the relationship between constraints on the control functions and the quantum speed limit using tools from Riemannian and related geometries.

The structure of the paper is as follows. First we summarise two complementary geometrical formalisms of quantum mechanics, and use these to rederive known time bounds (§2). Then we further develop the approach, to cover time-dependent and constrained systems (§3). We then apply these results to a specific spin chain system (§4).

2 The Geometry of Finite Dimensional Quantum Mechanics

Throughout this paper, $su(n)$ refers to the special unitary Lie algebra of $N \times N$ anti-hermitian matrices; $SU(N)$ refers to the group of $N \times N$ special unitary matrices.

Quantum mechanical states are typically formulated in term of a complex Hilbert space of states [4]. Quantum time evolution is then typically formulated by considering a unitary time evolution operator $\hat{U}_t$ acting on the state space. In the case of a Hamiltonian (always taken to be a Hermitian operator) with no explicit time dependence, $\hat{H}$, the time evolution is given by:

$$\hat{U}_t = \exp\left(-it\hat{H}\right) \tag{1}$$

This solves the Schrödinger equation for the state if one defines the state after time t to be $|\psi_t\rangle = \hat{U}_t|\psi_0\rangle$ for some given initial state $|\psi_0\rangle$. There are many different approaches to representing the time evolution operator for an explicitly time-dependent Hamiltonian; common methods include the Dyson series and the Magnus expansion [2].

There are, however, other, more geometrical ways to formulate quantum mechanics. The space of states, in the case of a finite dimensional complex Hilbert spaces, $\mathbb{C}^N$, can be formulated as a complex projective space [5]. This construction possesses no redundancy in its description of the state of a quantum system. In the Hilbert space construction, physically distinguishable states correspond to equivalence classes of vectors in Hilbert space (each class is comprised of a complex line or 'ray'). In contrast, a single point in the projective space description corresponds to a single physically distinct state.The price for this minimality is the increased difficulty of working with differentiable manifolds rather than with vector spaces.

The space of all unitary operators acting on a (finite dimensional) complex Hilbert space of states forms a Lie group [3], the Lie algebra of which consists of all anti-Hermitian operators. Both of these constructions introduce differential geometry into the picture of quantum dynamics.

2.1 Metrics on CP^n

We do not attempt a full account of the material discussed in this section; [8] is a clear and rigorous source for the mathematics relating to complex projective spaces, projective Hilbert spaces, and the Fubini-Study metric.

The projective structure corresponding to $\mathbb{C}^{N+1}$, considered as a vector space without any inner product or norm structure, is CP^N, a differentiable manifold. In quantum mechanics the standard inner product structure on $\mathbb{C}^{N+1}$ is employed to form an inner product space $\langle \mathbb{C}^{N+1}, \langle\cdot|\cdot\rangle\rangle$, where the standard inner product is given by:

$$\langle \boldsymbol{u}|\boldsymbol{v}\rangle = \sum_{k=0}^{N} \overline{u}_k v_k \tag{2}$$

After quotienting $\mathbb{C}^{N+1}/\{\mathbf{0}\}$ into equivalence classes to form CP^N, a natural choice of Riemannian metric arises for CP^N: the Fubini-Study metric. The infinitesimal form of this metric, the metric tensor, is given by [8]:

$$ds^2 = \frac{\langle \delta\psi|\delta\psi\rangle}{\langle \psi|\psi\rangle} - \frac{\langle \delta\psi|\psi\rangle\langle \psi|\delta\psi\rangle}{\langle \psi|\psi\rangle^2} \tag{3}$$

where $|\psi\rangle$ is the point on CP^N that corresponds to the ray in $\langle \mathbb{C}^{N+1}, \langle\cdot|\cdot\rangle\rangle$ to which $|\psi\rangle$ belongs; $|\delta\psi\rangle$ is an element of the tangent space at this point, $T_{|\psi\rangle}CP^N$. (Where no confusion arises, we use the common abuse of notation that does not distinguish between the vector $|\psi\rangle$ and the corresponding point in CP^N.) This is the unique metric tensor (up to overall scaling by a positive constant) on CP^N invariant under all unitary transformations of $|\psi\rangle$ [8].

Invariance under unitary transformations follows directly from the definition of a unitary operator as an operator the leaves all inner products of states in $\langle \mathbb{C}^{N+1}, \langle\cdot|\cdot\rangle\rangle$ invariant. This is the projective counterpart to the standard inner product on $\mathbb{C}^{N+1}$; that is, it is compatible with the quotient into rays of $\langle \mathbb{C}^{N+1}, \langle\cdot|\cdot\rangle\rangle$ rather than the quotient of just the vector space structure. This is readily verified: under the transformation $|\psi\rangle \mapsto Z|\psi\rangle$, for any $Z \in \mathbb{C}/\{0\}$, the metric is invariant:

$$
\begin{aligned}
ds^2 \mapsto & \frac{\langle\delta\psi|\overline{Z}Z|\delta\psi\rangle}{\langle\psi|\overline{Z}Z|\psi\rangle} - \frac{\langle\delta\psi|\overline{Z}Z|\psi\rangle\langle\psi|\overline{Z}Z|\delta\psi\rangle}{\langle\psi|\overline{Z}Z|\psi\rangle^2} \\
&= \frac{|Z|^2}{|Z|^2}\frac{\langle\delta\psi|\delta\psi\rangle}{\langle\psi|\psi\rangle} - \left(\frac{|Z|^2}{|Z|^2}\right)^2 \frac{\langle\delta\psi|\psi\rangle\langle\psi|\delta\psi\rangle}{\langle\psi|\psi\rangle^2} \\
&= ds^2
\end{aligned} \tag{4}
$$

The finite form of this metric is given by [14]:

$$
\gamma(|\psi\rangle, |\phi\rangle) = \arccos\sqrt{\frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle^2\langle\phi|\phi\rangle^2}} \tag{5}
$$

and in this metric, the manifold forms a metric space. The metric clearly has the same invariance and uniqueness properties as its infinitesimal form, eqn.(3).

2.2 Deriving the Mandelstam-Tamm inequality

Our following rederivation of the Mandelstam-Tamm inequality illustrates the usefulness of Riemannian geometry in the context of quantum speed limits. From eqn.(5) we have that $\forall|\psi\rangle, |\phi\rangle \in \mathbb{C}^N$ $\langle\psi|\phi\rangle = 0$ implies $\gamma(|\psi\rangle, |\phi\rangle) = \arccos(0) = \frac{\pi}{2}$. Hence (if all states involved are initially normalised), if $|\psi_t\rangle$ connects to orthogonal state in time τ, i.e. $\langle\psi_\tau|\psi_0\rangle = 0$, then:

$$
\begin{aligned}
L[|\psi_t\rangle] &= \int_{t=0}^{t=\tau} ds \\
&= \int_{t=0}^{t=\tau} \sqrt{\frac{\langle\delta\psi|\delta\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\delta\psi|\psi\rangle\langle\psi|\delta\psi\rangle}{\langle\psi|\psi\rangle^2}}\, dt \\
&= \int_{t=0}^{t=\tau} \sqrt{\langle\delta\psi|\delta\psi\rangle - \langle\delta\psi|\psi\rangle\langle\psi|\delta\psi\rangle}\, dt \geq \frac{\pi}{2}
\end{aligned} \tag{6}
$$

where L is the length functional for a curve on the projective space according to the Fubini-Study metric. Here the inequality follows from the definition of a geodesic.

In the case that $|\psi_t\rangle$ solves the Schrödinger equation for a time-independent Hamiltonian $\hat{H}$ we have:

$$
|\delta\psi_t\rangle = \frac{d}{dt}|\psi_t\rangle = \frac{d}{dt}\exp\left(-it\hat{H}\right)|\psi_0\rangle = -i\hat{H}\exp(-it\hat{H})|\psi_0\rangle = -i\hat{H}|\psi_t\rangle \tag{7}
$$

Substituting this into eqn.(6), we find:

$$
\begin{aligned}
L[|\psi_t\rangle] &= \int_{t=0}^{t=\tau} \sqrt{\langle\delta\psi_t|\delta\psi_t\rangle - \langle\delta\psi_t|\psi_t\rangle\langle\psi_t|\delta\psi_t\rangle}\, dt \\
&= \int_{t=0}^{t=\tau} \sqrt{\langle\psi_t|\hat{H}^2|\psi_t\rangle - \langle\psi_t|\hat{H}|\psi_t\rangle^2}\, dt \\
&= \int_{t=0}^{t=\tau} \Delta E_{|\psi_t\rangle} dt = \int_{t=0}^{t=\tau} \Delta E_{|\psi_0\rangle} dt = \tau\Delta E_{|\psi_0\rangle} \geq \frac{\pi}{2}
\end{aligned} \tag{8}
$$

Here $\Delta E_{|\psi_t\rangle} dt$ can be replaced by $\Delta E_{|\psi_0\rangle}$ in the last line since the Hamiltonian is time-independent, which implies that the energy uncertainty is also. From this follows the Mandelstam-Tamm inequality:

$$\tau \geq \frac{\pi}{2\Delta E_{|\psi_0\rangle}} \tag{9}$$

It is worth comparing our derivation to that in [6] (their eqns. 22-25; note that the 'Wooters distance' is simply the finite form of the Fubini-Study metric applied to normalised states). There the finite form of the metric is differentiated; here we use the differential form of the metric immediately.

2.3 Metrics on $SU(N)$

There is a natural choice of metric tensor on the Lie group of all special unitary operators acting on $\mathbb{C}^N$, $SU(N)$. This is due to a general result about symmetric bilinear forms on semi-simple Lie groups. ($U(N)$ is not semi-simple, and so we specialise from here on to $SU(N)$, which is in fact simple.) Again, we do not give a complete description of these constructions; specifically we do not discuss adjoint representations and the general definition of the Killing form, but instead specialise to $SU(N)$ immediately. A good source for this material is [7].

The Killing form (denoted by B) is the unique symmetric bilinear form (up to a positive constant multiple) on $su(N)$ (which consists of all traceless, anti-hermitian operators on $\mathbb{C}$) satisfying $\forall x, y, z \in su(N)$:

1. $B([x, y], z) = B(x, [y, z])$
2. $B(s(x), s(y)) = B(x, y)$ for any automorphism of s of $su(n)$.

For $su(n)$, the Killing form is given by:

$$B(x, y) = 2n \operatorname{Tr}(xy) \tag{10}$$

Then $g(x, y) = -B(x, y) = 2n \operatorname{Tr}(x^\dagger y)$ is a Riemannian metric on $SU(n)$. The length of a smooth curve (according to the metric g) on $\hat{U}_t$ $SU(n)$ is now given by:

$$L[\hat{U}_t] = \int_{t=0}^{t=\tau} \sqrt{2n \operatorname{Tr} \frac{d\hat{U}_t}{dt}^\dagger \frac{d\hat{U}_t}{dt}}\, dt \tag{11}$$

In the case that $\hat{U}_t$ solves the Schrödinger equation for some (possibly time-dependent) Hamiltonian $\hat{H}_t$

$$\frac{d}{dt}\hat{U}_t = -i\hat{H}_t\hat{U}_t \tag{12}$$

then the length of this curve can be written in terms of $\hat{H}_t$. This follows from the Schrödinger equation and the unitary invariance of the Killing form (due to the cyclic property of the trace):

$$\begin{aligned} L[\hat{U}_t] &= \int_{t=0}^{t=\tau} \sqrt{2n \operatorname{Tr} \frac{d\hat{U}_t}{dt}^{\dagger} \frac{d\hat{U}_t}{dt}}\, dt \\ &= \int_{t=0}^{t=\tau} \sqrt{2n \operatorname{Tr} \left(-i\hat{H}_t\hat{U}_t\right)^{\dagger} \left(-i\hat{H}_t\hat{U}_t\right)}\, dt \\ &= \int_{t=0}^{t=\tau} \sqrt{2n \operatorname{Tr} \left(\hat{U}_t^{-1}\hat{H}_t^2\hat{U}_t\right)}\, dt \\ &= \int_{t=0}^{t=\tau} \sqrt{2n \operatorname{Tr} \left(\hat{H}_t^2\hat{U}_t\hat{U}_t^{-1}\right)}\, dt \\ &= \int_{t=0}^{t=\tau} \sqrt{2n \operatorname{Tr} \left(\hat{H}_t^2\right)}\, dt \end{aligned} \tag{13}$$

Notice that the dependence on the operator $\hat{U}_t$ has disappeared. This reduces, in the case of time-independent $\hat{H}$, to:

$$L[\hat{U}_t] = \int_{t=0}^{t=\tau} \sqrt{2n \operatorname{Tr} \hat{H}^2}\, dt = \tau\sqrt{2n \operatorname{Tr} \hat{H}^2} \tag{14}$$

2.4 A bound on the orthogonality time

To illustrate the relationship between point to set distances on the special unitary group and speed limits for state transfer problems, we include our deviation of a bound on the orthogonality time (for time-independent systems) similar to the Margolus-Levitin bound.

Consider the shortest time that $\langle\psi_0|\psi_t\rangle = \langle\psi_0|\hat{U}_t|\psi_0\rangle = 0$ can be achieved. This is the same as the shortest time in which the time evolution operator can be driven from $\hat{I}$ to some $\hat{U}_t$ achieving this. Any such $\hat{U}_t$ achieving this, for a time-independent system, must have the following form for some unitary change of basis matrix $\hat{V}$, some unitary $\hat{A}$ and some $\theta \in [0, 2\pi]$:

$$\hat{U}_t = \exp(-it\hat{H}) = \hat{V}^{\dagger}\hat{B}\hat{V} \tag{15}$$

where

$$\hat{B} = \begin{pmatrix} 0 & -\exp(-i\theta) & 0\dots0 \\ \exp(i\theta) & 0 & 0\dots0 \\ 0 & 0 & \\ \vdots & \vdots & \hat{A} \\ 0 & 0 & \end{pmatrix} \tag{16}$$

Consider the function $f_p : SU(N) \to \mathbb{R}$ (for $p \geq 1$) defined as:

$$f_p(\hat{A}) \stackrel{\text{def}}{=} \operatorname{Tr}\left(\left|\log(\hat{A})\right|^p\right) \tag{17}$$

where $|\hat{A}| = \sqrt{\hat{A}^\dagger \hat{A}}$. Applying this function to both sides of eqn.(15) gives:

$$\mathrm{Tr}(|-it\hat{H}|^p) = \mathrm{Tr}(|\log(\hat{V}^\dagger \hat{B}\hat{V})|^p) \tag{18}$$

This implies:

$$\begin{aligned} t^p\, \mathrm{Tr}(|\hat{H}|^p) &= \mathrm{Tr}(|\log \hat{B}|^p) \\ &= \mathrm{Tr}(|\log \begin{pmatrix} 0 & -\exp(-i\theta) \\ \exp(i\theta) & 0 \end{pmatrix} |^p) + \mathrm{Tr}(|\log \hat{A}|^p) \\ &\leq \mathrm{Tr}(|\log \begin{pmatrix} 0 & -\exp(-i\theta) \\ \exp(i\theta) & 0 \end{pmatrix} |^p) = \frac{2\pi^p}{2^p} \end{aligned} \tag{19}$$

Taking the p^{th} root of each side yields the bound:

$$t \geq \frac{2^{\frac{1}{p}}\pi}{2\mathrm{Tr}(|\hat{H}|^p)^{\frac{1}{p}}} = \frac{\pi}{2}\left(\frac{2}{\mathrm{Tr}(|\hat{H}|^p)}\right)^{\frac{1}{p}} \tag{20}$$

This is similar to, but not as strong as, the bounds given in [9]. As in [9], it is possible to optimise this bound, to:

$$t \geq \min_{\epsilon\in\mathbb{R}} \frac{\pi}{2}\left(\frac{2}{\mathrm{Tr}(|\hat{H}+\epsilon\hat{I}|^p)}\right)^{\frac{1}{p}} \tag{21}$$

by reassigning a new ground state energy.

This bound coincides with the Margolus-Levitin bound for a two level system and $p = 1$. This bound with $p = 2$ corresponds to the bound arising from the metric induced by the Killing form.

3 Speed Limits For Time Dependent Controls With Constraints

A mathematical method for obtaining answers to the following questions about a time-dependent quantum system with constrained control functions is relevant to quantum computing:

1. Given two states, what is the least time the system can transfer between them (if the constraint permits this transformation) and which control functions cause this to happen?
2. Given a desired time evolution operator, what is the least time the system can transfer from the identity on the unitary group to it (if the constraint permits this transformation) and which control functions cause this to happen?

In many physically plausible cases, the time-dependent Hamiltonian for a candidate system for the implementation of quantum gates can be cast in the form:

$$\hat{H}(t) = \hat{H}_{int} + \sum_{n}^{M} f_n(t)\hat{H}_n \tag{22}$$

where $\hat{H}_{int}$ is the time-independent portion of the Hamiltonian; M is the number of control functions; f_n are control functions, and H_n is the n^{th} control Hamiltonian.

3.1 Constraints as Submanifolds of $su(N)$

Consider a system with Hamiltonian $\hat{H}(t)$ given as in eqn.(22). One can form a geometric interpretation of a constraint on the control functions f_n by considering the relationship between a constraint given (perhaps implicitly) by an equation of the form $F(f_1, \ldots, f_M) = c$ and submanifolds of the tangent spaces to $SU(n)$.

In cases where $F : \mathbb{R}^M \to \mathbb{R}$ is a sufficiently smooth function the level sets $\{(f_1, \ldots, f_M) \in \mathbb{R}^M \quad s.t. \quad F(f_1, \ldots, f_M) = c\}$ foliate $\mathbb{R}^M$. The intuitive picture of a level set in this context is given by imagining the set of all vectors in a vector space with the same length according to some norm. That is: each level set is a disjoint submanifold of $\mathbb{R}^M$; each level set has the same dimension $(M-1)$; the union of all levels sets is $\mathbb{R}^M$. These level sets can be carried over to the tangent spaces to $SU(N)$ at each point $\hat{A} \in SU(N)$, $T_{\hat{A}}SU(n)$, by considering the submanifold of tangent vectors compatible with the constraint given. This set can be expressed as:

$$\mathcal{A}_F \overset{\text{def}}{=} \left\{ i\hat{H} \in T_{\hat{A}}SU(n) \; s.t. \; \hat{H} = \hat{H}_{int} + \sum_{n=0}^{M} f_n \hat{H}_n, \; F(f_1, \ldots, f_M) = c \right\} \quad (23)$$

3.2 Sphere Bundles and Speed Limits

We now consider the relationship between speed limits arising from a constraint F on the control functions and a special type of Riemannian metric on $SU(n)$. For the purpose of simplifying the statement of the following theorem, define:

Definition 1. *A smooth parametrised curve $\hat{U}(t) \in SU(n)$ (parametrised by $t \in [0, \tau]$) is said to be* Constraint Compatible *for a constraint function F if $\forall t \in [0, \tau]$, $d\hat{U}/dt \in \mathcal{A}_F$.*

Theorem 1. *Each Riemannian metric $g : TSU(n) \times TSU(n) \to \mathbb{R}$ on $SU(n)$ s.t. $\forall \hat{A} \in \mathcal{A}_F$, $g(\hat{A}, \hat{A}) \leq 1$ satisfies for each smooth, constraint compatible curve on $\hat{U}(t) \in SU(n)$:*

$$L[\hat{U}(t)] \leq \tau \quad (24)$$

Proof.

$$L[\hat{U}(t)] = \int_{t=0}^{t=\tau} \sqrt{g(\hat{A}, \hat{A})}\, dt \leq \int_{t=0}^{t=\tau} 1 \; dt = \tau \quad (25)$$

□

4 Application To Spin Chains

We now apply these results to a specific quantum system of relevance in quantum computing: a spin chain. A controlled Heisenberg spin chain of N spins (with coupling constants J_x, J_y, J_z has Hamiltonian [12]:

$$\left(\sum_{k\in\{x,y,z\}} J_k \left(\sum_{n=0}^{N-2} \hat{I}_2^{\otimes^n} \otimes \sigma^k \otimes \sigma^k \otimes \hat{I}_2^{\otimes^{N-n-2}} \right) \right) \tag{26}$$
$$+ \left(\sum_{n=0}^{N-1} f_n(t) \hat{I}_2^{\otimes^n} \otimes \sigma^z \otimes \hat{I}_2^{\otimes^{N-n-1}} \right)$$

Apply the constraint that the total energy used to produce the control functions is less than κ^2, to obtain:

$$\sum_{k=0}^{N-1} f_n(t)^2 \leq \kappa^2 \tag{27}$$

This implies (after some algebra) the following theorem:

Theorem 2. *The Riemannian metric that is the largest multiple of the Killing form of $su(2^N)$ such that all controlled Heisenberg model Hamiltonians (with N spins) that obey the constraint satisfy $g_{op}(H_t, H_t) \leq 1$ is given by:*

$$\begin{aligned} g_{op}(x, y) &= \frac{-B(x, y)}{2^{N+1}\left((N-1)({J_x}^2 + {J_y}^2 + {J_z}^2) + \kappa^2\right)} \\ &= \frac{\mathrm{Tr}(x^\dagger y)}{(N-1)({J_x}^2 + {J_y}^2 + {J_z}^2) + \kappa^2} \end{aligned} \tag{28}$$

Proof. Omitted: simple but laborious.

This theorem, combined with eqn.(25), yields:

$$\tau \geq \int_{t=0}^{t=\tau} \sqrt{g_{op}(\hat{H}_t, \hat{H}_t)}\, dt \tag{29}$$

$$= \frac{1}{\sqrt{(2^{N+1})\left((N-1)({J_x}^2 + {J_y}^2 + {J_z}^2) + \kappa^2\right)}} \int_{t=0}^{t=\tau} \sqrt{-B(-i\hat{H}_t, -i\hat{H}_t)}\, dt \tag{30}$$

$$= \frac{1}{\sqrt{(N-1)({J_x}^2 + {J_y}^2 + {J_z}^2) + \kappa^2}} \int_{t=0}^{t=\tau} \sqrt{Tr({\hat{H}_t}^2)}\, dt \tag{31}$$

for any values of the control functions. We appeal to the following facts:

1. The one parameter subgroups of unitary groups are the geodesics of the Killing form since it is (strictly $-B$ is the metric) a bi-invarient metric [8].

2. Stone's theorem [13] guarantees that the one parameter subgroups take the form $\exp(-it\hat{H})$ for some hermitian $\hat{H}$, which is nothing other that the form of the time evolution operator for a time independent quantum system with Hamiltonian $\hat{H}$.

Using these, the known length of the appropriate geodesic (which can be readily calculated by finding the length of the one-parameter subgroup connecting the identity to the desired transformation according to the metric given by $-B$), and statements made above, we get:

$$\tau \geq \frac{1}{\sqrt{(2^{N+1})\left((N-1)({J_x}^2+{J_y}^2+{J_z}^2)+\kappa^2\right)}} \int_{t=0}^{t=\tau} \sqrt{-B(-i\hat{H}_t, -i\hat{H}_t)}\, dt \tag{32}$$

$$\geq \frac{\pi}{\sqrt{(N-1)({J_x}^2+{J_y}^2+{J_z}^2)+\kappa^2}} \tag{33}$$

Thus we conclude that a Heisenberg spin chain with N spins, constrained as described, cannot transfer from one state to an orthogonal state in less time than:

$$\tau \geq \pi / \sqrt{(N-1)({J_x}^2+{J_y}^2+{J_z}^2)+\kappa^2} \tag{34}$$

The uniqueness properties of the Killing form guarantee that this is the best (largest lower bound on t) speed limit available from any bi-invariant, unitarily invariant metric, as any such metric is a multiple of the Killing form and g_{opt} is the largest multiple meeting the premises of theorem eqn.(1).

5 Conclusions and Further Work

We have demonstrated the use of geometric formalisms in deriving time bounds on quantum systems. We have used this approach to rederive known results in a more compact and elegant manner. More importantly, the new formulation allows us to extend the approach to time dependent and constrained systems, as relevant to quantum computation. We have demonstrated this for a spin chain system.

Next steps include:

- Determining which metric-like structures on unitary group can be used to derive quantum speed limits, including an investigation into the possible role of Finsler functions as these generalise Riemannian metrics but can still produce speed limits.
- Understanding more clearly the relationship between problems 1 and 2 (as discussed in §3). Understanding this relationship in terms of homogeneous spaces (in the sense that $CP^N \cong SU(N)/U(N-1)$) and Riemannian symmetric spaces. Are metric-like structures on special unitary groups the best method for deriving speed limits for state transfer problems 1 or do metrics on complex projective spaces suffice.

– Extending the analysis of the relationship between constrained control functions and geometric derivations of quantum speed limits, to determine which classes of metric-like structures can yield speed limit theorems for which classes of constraints.
– Extending the approach to other Hamiltonian systems, particularly to investigate how quickly one can transfer from a separable state to a maximally entangled state in the presence of constrained constrol functions.

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