Boolean inverse semigroups

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The classical Stone duality

- A Boolean algebra is a relatively complemented distributive lattice with 0 but in general without 1.
- Distributive lattices have 0, but in general do not have 1.
  - Finite Boolean algebras are precisely powersets of finite sets.
  - There are infinite Boolean algebras which are not powersets.
  - Too many subsets? Topologize!
- A Stone space is a Hausdorff space with a basis of compact-open sets.
- A spectral space is a a sober space that has a basis of compact-open sets which is closed under finite non-empty intersections.
- The classical Stone duality (Stone, 1937; Doctor, 1964):
  - The categories of Boolean algebras (resp. unital Boolean algebras) and Stone spaces (resp. compact Stone spaces) are dually equivalent.
  - The categories of distributive lattices (resp. bounded distributive lattices) and spectral spaces (resp. compact spectral spaces) are dually equivalent.

Boolean and distributive inverse semigroups

- A Boolean inverse semigroup is an inverse semigroup S such that:
  - $E(S)$ admits the structure of a Boolean algebra;
  - If $a \sim b$ ($\sim$ is the compatibility relation) then $a \vee b$ exists in S.
- A distributive inverse semigroup is an inverse semigroup S such that:
  - $E(S)$ admits the structure of a distributive lattice;
  - If $a \sim b$ then $a \vee b$ exists in S.
- Any distributive lattice is a distributive inverse semigroup with $a \cdot b = a \wedge b$; likewise any Boolean algebra is a Boolean inverse semigroup.
- $\mathcal{I}_n$; $\mathcal{I}_X$ (X – any set); $E(\mathcal{I}_X) \simeq \mathcal{P}(X)$.

Étale groupoids

- A groupoid is a small category where every arrow is invertible.
- $\mathcal{G}$ – groupoid, $\mathcal{G}^{(0)} = \{a^{-1}a\colon a \in \mathcal{G}\}$ – the set of units of $\mathcal{G}$.
- $d\colon \mathcal{G} \to \mathcal{G}^{(0)}$, $d(a) = a^{-1}a$ – the domain (or sourse) map;
  $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$, $r(a) = aa^{-1}$ – the range map.
- The set of composable pairs: $\mathcal{G}^{(2)} = \{(a, b) \in \mathcal{G} \times \mathcal{G}\colon r(b) = d(a)\}$.
- A local bisection is a subset $U \subseteq \mathcal{G}$ such that $d|_U$ and $r|_U$ are injective maps.
- $\mathcal{G}$ is a topological groupoid if $\mathcal{G}$ is a topological space and the inversion map $\mathcal{G} \to \mathcal{G}$ and the product map $\mathcal{G}^{(2)} \to \mathcal{G}$ are both continuous.
- $\mathcal{G}$ is étale if d is a local homeomorphism ($\Leftrightarrow$ r is a local homeomorphism $\Leftrightarrow$ m is a local homeomorphism)
  - If $\mathcal{G}$ is étale than $\mathcal{G}^{(0)}$ is an open subspace and $\mathcal{G}$ has a basis of open local bisections; also $\mathcal{G}$ is R-discrete, that is, $d^{-1}(x)$ is a discrete subspace of $\mathcal{G}$ for any $x \in \mathcal{G}^{(0)}$.

Non-commutative Stone dualities

- A spectral groupoid is an étale groupoid $\mathcal{G}$ such that $\mathcal{G}^{(0)}$ is a spectral space.
- A Stone groupoid is an étale groupoid $\mathcal{G}$ such that $\mathcal{G}^{(0)}$ is a Stone space.

Theorem (Lawson, 2010-2013, more morphisms: GK and Lawson, 2017, very relevant work: Resende, 2007, Lawson and Lenz, 2013.)

- The categories of Boolean inverse semigroups and Stone groupoids are dually equivalent.
- The categories of distributive inverse semigroups and spectral groupoids are dually equivalent.

- Local bisections of a Stone groupoid form a Boolean inverse semigroup.
- Germs of elements of a Boolean inverse semigroup S over points of the space of ultracharacters (resp. prime characters) of $E(S)$ give rise to a Boolean (resp. spectral) groupoid.

Morphisms

- A morphism $\varphi\colon S \to T$ between Boolean inverse semigroups is a semigroup homomorphism such that $\varphi|_{E(S)}$ is a non-degenerate morphism of Boolean algebras. (Non-degenerate: for any $e \in E(T)$ there is $f \in E(S)$: $\varphi(f) \geq e$.)
- A continuous relational covering morphism between Boolean (or spectral) groupoids is a map $f\colon \mathcal{G}_1 \to \mathcal{P}(\mathcal{G}_2)$ such that:

(RM1) for any $t \in \mathcal{G}_1^{(0)} : |f(t)| = 1$ and $f|_{\mathcal{G}^{(0)}}$ is a continuous proper map;

(RM2) for all $y \in f(x)$: $d(y) = fd(x)$ and $r(y) = fr(x)$;

(RM3) if $(x, y) \in \mathcal{G}_1^{(2)}$ and $s \in f(x)$, $t \in f(y)$ then $st \in f(xy)$;

(RM4) for any $x \in \mathcal{G}_1 : f(x^{-1}) = (f(x))^{-1}$;

(RM5) if $A \subseteq \mathcal{G}_2$ is compact-open local bisection, then $f^{-1}(A) = \{x \in \mathcal{G}_1 \colon f(x) \cap A \neq \varnothing\}$ is a compact-open local bisection in $\mathcal{G}_1$;

(RM6) if $d(x) = d(y)$ (or $r(x) = r(y)$) and $f(x) \cap f(y) \neq \varnothing$ then $x = y$ (star-injectivity);

(RM7) if $d(t) = y$ (resp. $r(t) = y$) where $y = f(x)$ then there is $s \in \mathcal{G}_1$ such that $d(s) = x$ (resp. $r(s) = x$) and $t \in f(s)$ (star-surjectivity).

- Morphisms between Boolean inverse semigroups are dualized by continuous relational covering morphisms (GK and Lawson, 2017).

Morphisms: variations

	semigroups	groupoids
type 1	morphisms	continuous relational covering morphisms (CRCMs)
type 2	proper moprhisms	at least single valued CRCMs
type 3	weakly meet-preserving moprhisms	at most single valued CRCMs
type 4	proper and weakly meet preserving morphisms	continuous covering functors (= single-valued CRCMs)

- A morphism $\varphi\colon S \to T$ is proper if any $t \in T$ can be written as $t = \bigvee_{i=1}^{n} t_i$ where $n \geq 1$ so that there are $s_1, \ldots, s_n \in S$ satisfying $\varphi(s_i) \geq t_i$ for all $i = 1, \ldots, n$. Briefly, $T = ((\mathrm{im}\varphi)^{\downarrow})^{\vee}$.
- A morphism $\varphi\colon S \to S$ is weakly meet-preserving if $t \leq f(a), f(b)$ implies that there is $c \leq a, b$ such that $t \leq f(c)$.
- In the case where S, T are $\wedge$-semigroups, weakly meet preserving $=$ $\wedge$-preserving.

Character space of a semilattice

- E – a semilattice with 0, B – a Boolean algebra (or a distributive lattice).
- A representation $\varphi\colon E \to B$ is a map such that
  - $\varphi(0) = 0$;
  - $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$ for all $a, b \in E$.
- A character of E is a non-zero representation $E \to \{0, 1\}$.
- $\widehat{E}$ - character set of E, topology is inherited from $\{0, 1\}^E$ (with 0 removed), called the patch topology. Basis of the patch topology:

$$M_{a;b_1,\dots,b_n} = \{\varphi \in \widehat{E}\colon \varphi(a) = 1, \varphi(b_1) = \cdots = \varphi(b_n) = 0\},$$

$n \geq 1$ and $a, b_1, \dots, b_n \in S$ are such that $b_i \leq a$ for all $i = 1, \dots, n$.

- Remark. There is another, spectral, topology on $\widehat{E}$ with the basis:

$$M_a = \{\varphi \in \widehat{E}\colon \varphi(a) = 1\}, a \in S.$$

The groupoid of germs of an inverse semigroup

- S – an inverse semigroup with 0 (assumed throughout the talk!).
- S acts on $\widehat{E(S)}$ by partial maps: if $s \in S$ and $\varphi \in \widehat{E(S)}$ then $s \cdot \varphi$ is defined $\Leftrightarrow \varphi(s^{-1}s) = 1$,

  in which case $(s \cdot \varphi)(e) = \varphi(s^{-1}es)$, $e \in E(S)$.
- Let $s, t \in S$ and $\varphi \in \widehat{E(S)}$ be such that $s \cdot \varphi$ and $t \cdot \varphi$ are both defined.
- s and t define the same germ over φ if there is $e \in E(S)$ such that $\varphi(e) = 1$ and $se = te$.
- Notation: $[s, \varphi]$ – the germ defined by s over φ.
- We look at the germ $[s, \varphi]$ as an arrow from φ to $s \cdot \varphi$.
- This leads to the groupoid of germs $\mathcal{G}(S)$ of the natural action of S on $\widehat{E(S)}$.

The universal groupoid of an inverse semigroup

- The patch topology on $\mathcal{G}(S)$ has a basis consisting of the sets

$$\Theta[s; s_1, \dots s_n] = \{[s, \varphi] \in \mathcal{G}(S) \colon \varphi(s^{-1}s) = 1, \forall i \colon \varphi(s_i^{-1}s_i) = 0\},$$

  where $n \geq 1$, $s \in S$ and $s_1, \dots, s_n \leq s$.
- $\mathcal{G}(S)$ – Paterson's universal groupoid of S. It is a Stone groupoid.
- $B(S)$ – the dual Boolean inverse semigroup of $\mathcal{G}(S)$, the universal Booleanization of S.

Cover-to-join representations

- E – semilattice, B – Boolean algebra (or a distributive lattice)
- $Z \subseteq E$ is a cover of $e \in E$ if $f \leq e$ such that $ef \neq 0$ there is $z \in Z$ satisfying $zf \neq 0$. From now on we consider finite covers.
- $\varphi\colon E \to B$ is cover-to-join, if for $e \in E$ and any finite cover $Z \subseteq E$ of e we have:
$$\varphi(e) = \bigvee_{z \in Z} \varphi(z).$$
- Cover-to-join represenations (Donsig & Milan, 2014) are closely relatedy to tight representations (Exel, 2009) (B is a uinital Boolean algebra).
- A non-degenerate representation $E \to B$ is tight if and only if it is cover-to-join (Exel, 2019, B – Boolean algebra).
- Cover-to-join characters of E = tight characters of E.
- $\widehat{E}_{tight}$ is a closed subset of $\widehat{E}$.

The tight groupoid of an inverse semigroup

- $\widehat{E}_{tight}$ is invariant under the natural action of S on $\widehat{E}$: if $\varphi \in \widehat{E}_{tight}$ and $\varphi(s^{-1}s) = 1$ then $s \cdot \varphi \in E_{tight}$.
- $\widehat{E}_{tight}$ is a closed subset of $\widehat{E}$.
- Let $\mathcal{G}(S)_{tight}$ be the groupoid of germs attached to the induced action of S on $\widehat{E}_{tight}$.
- $\mathcal{G}(S)_{tight}$ – the tight groupoid of S.
- $B_{tight}(S)$ – the dual Boolean inverse semigroup of $\mathcal{G}_{tight}(S)$, the tight Booleanization of S.
- $\iota_{B_{tight}(S)}\colon S \to B_{tight}(S)$, $s \mapsto \Theta[s]$, is a morphism of semigroups (not injective in general!)
- Example. Let S be a Boolean inverse semigroup. Then its dual Stone groupoid is $\mathcal{G}_{tight}(S)$ whence $S \simeq B_{tight}(S)$.

X-to-join representations of semilattices

- E – a semilattice, B a – Boolean algebra, $X \subseteq E \times \mathcal{P}_{fin}(E)$.
- A representation $\varphi\colon E \to B$ will be called an X-to-join representation, if
$$\varphi(e) = \bigvee_{i=1}^{n} \varphi(e_i)$$
for all $(e, \{e_1, \dots, e_n\}) \in X$.
- $\widehat{E}_X$ – the set of all X-to-join characters of E. It is a closed subset of $\widehat{E}$.
- $\widehat{E}_X$ – the space of X-to-join characters with the subspace topology inherited from $\widehat{E}$.
- $\widehat{E}_X$ is a Stone space.
- $B_X(E)$ – the X-to-join Booleanization of E.

Connection with π-tight representations

- Let $\pi\colon E \to B$ be a representation of a semilattice E in a Boolean algebra B.
- Define X_π as the set consisting of all $(e, \{e_1, \ldots, e_n\}) \in E \times \mathcal{P}_{fin}(E)$ such that $\pi(e) = \bigvee_{i=1}^{n} \pi(e_i)$.
- Then X_π-to-join representations of E coincide with π-tight representations considered by Exel and Steinberg in 2018.
- The following is a consequence of a result by Exel and Steinberg:

Theorem

Let $\pi\colon E \to B$ be a representation of a semilattice E in a Boolean algebra B such that $\pi(E)$ generates B as a Boolean algebra. Then B is isomorphic to $\mathrm{B}_{X_\pi}(E)$.

Quotients of $\mathrm{B}(S)$ via X-to-join representations

- The canonical quotient morphism $\mathrm{B}(S) \to \mathrm{B}_X(S)$ corresponds to the inclusion map $\mathcal{G}_X(S) \hookrightarrow \mathcal{G}(S)$. Since this map is single-valued, $\mathrm{B}(S) \to \mathrm{B}_X(S)$ is weakly meet-preserving.
- $\mathcal{X}$ – a closed and invariant subset of $\widehat{E(S)}$
- $I_{\mathcal{X}}$ – the ideal of $E(\mathrm{B}(S))$ consisting of those compact-opens of $\widehat{E(S)}$ which do not intersect with $\mathcal{X}$.
- $a, b \in \mathrm{B}(S)$: define $a \sim_{\mathcal{X}} b$ iff there are $e, f, g \in E(\mathrm{B}(S))$ such that $\mathbf{d}(a) = e \vee f$, $\mathbf{d}(b) = e \vee g$ where $f, g \in I_{\mathcal{X}}$ and $ae = be$.
- Theorem. Let S be an inverse semigroup and $\mathcal{X}$ a closed invariant subset of $\widehat{E(S)}$. Then $\mathrm{B}(S)/\sim_{\mathcal{X}} \simeq \mathrm{B}_{X_{\pi_{\mathcal{X}}}}(S)$ where $\pi_{\mathcal{X}} \colon S \to \mathrm{B}(S)/\sim_{\mathcal{X}}$ is the composition of $\iota_{\mathrm{B}(S)} \colon S \to \mathrm{B}(S)$ and the quotient map $\mathrm{B}(S) \to \mathrm{B}(S)/\sim_{\mathcal{X}}$.
- Corollary. Let $\varphi \colon \mathrm{B}(S) \to T$ be a surjective weakly meet-preserving additive morphism where T is a Boolean inverse semigroup. Then there is an invariant subset $X \subseteq E(S) \times \mathcal{P}_{fin}(E(S))$ such that $T \simeq \mathrm{B}_X(S)$.

The X-to-join Booleanization of an inverse semigroup

- S – an inverse semigroup, $X \subseteq E(S) \times \mathcal{P}_{fin}(E(S))$.
- X is said to be S-invariant if $(e, \{e_1, \dots, e_n\}) \in X$ implies that $(s^{-1}es, \{s^{-1}e_1s, \dots, s^{-1}e_ns\}) \in X$, for all $s \in S$.
- If X is S-invariant then $\widehat{E(S)}_X$ is invariant under the natural action of S on $\widehat{E(S)}$.
- X' – the smallest S-invariant subset that contains X.
- The natural action of S on $\widehat{E(S)}$ restricts to the closed invariant subset $\widehat{E(S)}_{X'}$.
- The groupoid of germs of this restricted action is denoted by $\mathcal{G}_X(S)$.
- Example: $X = \varnothing \Rightarrow$ the universal groupoid, X defines cover-to-join representations $\Rightarrow$ the tight groupoid.
- The dual Boolean inverse semigroup of $\mathcal{G}_X(S)$, denoted $B_X(S)$, will be called the X-to-join Booleanization of S.

The universal property of $B_X(S)$

- The canonical map $\iota_{B_X(S)}\colon S \to B_X(S)$ is given by $\iota_{B_X(S)}(s) = \Theta[s] \cap \mathcal{G}_X(S)$.
- A representation of S in a Boolean inverse semigroup B is a morphism of semigroups $\varphi\colon S \to B$ satisfying $\varphi(0) = 0$. It is called an X-to-join representation if $\varphi|_{E(S)}\colon E(S) \to E(T)$ is an X-to-join representation.
- Example: $\iota_{B_X(S)}$ is a proper X-to-join representation.

Theorem (GK, 2019) Let S be an inverse semigroup and $X \subseteq E(S) \times \mathcal{P}_{fin}(E(S))$. Let further B be a Boolean inverse semigroup and $\varphi\colon S \to B$ an X-to-join representation (resp. a proper X-to-join representation). Then there is a unique morphism (resp. a unique proper morphism) of Boolean inverse semigroups $\psi\colon \mathrm{B}_X(S) \to B$ such that $\varphi = \psi\iota_{\mathrm{B}_X(S)}$.

Prime representations of semilattices

- E – a semilattice, B – a Boolean algebra
- A representation $\varphi\colon E \to D$ will be called prime, if that for any $e \in E$ and any finite cover Y of e the following implication holds:

$$\text{if } y = \bigvee Y \text{ then } \varphi(y) = \bigvee_{y \in Y} \varphi(y) \qquad (1)$$

- Prime = (cover&join)-to-join
- Any tight representation is prime.
- Let B be a Boolean algebra and suppose that the semilattice E admits the structure of a distributive lattice. Then a proper representation $\varphi\colon E \to B$ is prime if and only if it is a lattice morphism.
- Prime representations generalize proper morphisms from distributive lattices to Boolean algebras.

Example

- $n \geq 1$, $E_n = \{0, e_1, \ldots, e_n\}$ with $0 \leq e_1 \leq \cdots \leq e_n$
- Since E_n is a distributive lattice, $\iota_{\mathrm{B}_{prime}(E_n)} \colon E \to \mathrm{B}_{prime}(E_n)$ is an injective lattice morphism and $\mathrm{B}_{prime}(E_n)$ is isomorphic to the Booleanization E_n^- of the distributive lattice E_n
- $\iota_{\mathrm{B}_{prime}(E_n)}$ is prime but not cover-to-join and that $\iota_{\mathrm{B}_{tight}(E_n)}$ maps E_n onto a two-element Boolean algebra

Core and prime represenations of inverse semigroups

- E – a semilattice, $e, f \in E$, $f \leq e$, $f \neq 0$
- f is dense in e (Exel, 2009) if there is no non-zero element $d \leq e$ satisfying $d \wedge f = 0$
- f is dense in e if and only if $\{f\}$ is a cover of e
- A representation $\varphi\colon E \to B$ of E in a Boolean algebra B is called core, provided that for any $e, f \in E$, $f \neq 0$, such that $f \leq e$ and f is dense in e we have: $\varphi(f) = \varphi(e)$
- A representation $\varphi\colon S \to T$ of an inverse semigroup S in a Boolean inverse semigroup T is called a core (resp. prime) if $\varphi|_{E(S)}\colon E(S) \to E(T)$ is core (resp. prime).
- Core and prime represenations are a special case of X-to-join representations
- Any tight representation is core and prime
- Other representations that are different form tight ones have been recently studied by Exel and Steinberg (2019, arXiv)

Boolean inverse semigroups in extended signature

S – Boolean inverse semigroup. The operations $\setminus$ and $\vee$ on $E(S)$ can be extended to S: $s \setminus t = (\mathbf{r}(s) \setminus \mathbf{r}(t))\, s\, (\mathbf{d}(s) \setminus \mathbf{d}(t)), s \triangledown t = (s \setminus t) \vee t.$

Theorem [Wehrung, 2017]. An algebra $(S; 0,^{-1}, \cdot, \setminus, \triangledown)$ is an algebra attached to a Boolean inverse semigroup iff $(S; 0,^{-1}, \cdot)$ is an inverse semigroup with zero 0 and:

(1) $(\mathbf{d}(x) \setminus \mathbf{d}(y))^2 = \mathbf{d}(x) \setminus \mathbf{d}(y)$, $(\mathbf{d}(x) \triangledown \mathbf{d}(y))^2 = \mathbf{d}(x) \triangledown \mathbf{d}(y)$;

(2) all the defining identities (and hence all the identities) of the variety of Boolean algebras with $x, y, \ldots,$ replaced by $\mathbf{d}(x)$, $\mathbf{d}(y)$, $\ldots$, and 0, $\wedge$, $\vee$ and $\setminus$ replaced by 0, $\cdot$, $\triangledown$ and $\setminus$;

(3) $x \triangledown y \geq x \setminus y$, $x \triangledown y \geq y$;

(4) $\mathbf{d}(x \triangledown y) = \mathbf{d}(x \setminus y) \triangledown \mathbf{d}(y)$;

(5) $x \setminus y = (\mathbf{r}(x) \setminus \mathbf{r}(y)) x (\mathbf{d}(x) \setminus \mathbf{d}(y))$;

(6) $z((\mathbf{d}(x) \setminus \mathbf{d}(y) \triangledown \mathbf{d}(y)) = z(\mathbf{d}(x) \setminus \mathbf{d}(y)) \triangledown z\mathbf{d}(y)$.

If $(S; 0,^{-1}, \cdot, \setminus, \triangledown)$ is an algebra where $(S; 0,^{-1}, \cdot)$ is an inverse semigroup with zero 0 and (1)–(6) hold, then $(S; 0,^{-1}, \cdot)$ is a Boolean inverse semigroup and $(S; 0,^{-1}, \cdot, \setminus, \triangledown)$ is the algebra attached to $(S; 0,^{-1}, \cdot)$.

Free Boolean inverse semigroups

A map $\varphi\colon S \to T$ between Boolean inverse semigroups is a morphisms between their attached algebras if and only if:

1. $\varphi(st) = \varphi(s)\varphi(t)$ for all $s, t \in S$;
2. $\varphi(0) = 0$;
3. $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ for all $a, b \in S$ such that $a \sim b$.

They are called additive morphisms.

Proposition

Let X be a set and let $FI(X)$ be the free inverse semigroup on X. Then the free Boolean inverse semigroup (in the extended signature), $FBI(X)$, on X is isomorphic to $\mathrm{B}(FI(X) \cup \{0\})$.

Defining relations

Proposition (GK, 2019)

Let S be an inverse semigroup and $X \subseteq E(S) \times \mathcal{P}_{fin}(E(S))$. Then $\mathrm{B}_X(S)$ is generated by the set $\{[s] \colon s \in S\}$ subject to the relations:

(1) $[0] = 0$;

(2) $[st] = [s][t]$ *for all* $s, t \in S$;

(3) $[e] = \bigtriangledown_{i=1}^{n} [e_n]$ *for all* $(e, \{e_1, \ldots, e_n\}) \in X$.

Corollary

Let S be an inverse semigroup.

1. *The universal Booleanization $\mathrm{B}(S)$ is generated by the set $\{[s] \colon s \in S\}$ subject to the relations (1) and (2) above.*
2. *Let $X \subseteq E(S) \times \mathcal{P}_{fin}(E(S))$. Then $\mathrm{B}_X(S)$ is a quotient of $\mathrm{B}(S)$ obtained by adding relations (3) above $\mathrm{B}(S)$.*

X-to-join representations of inverse semigroups in C^*-algebras

- S – an inverse semigroup, A – a C^*-algebra.
- A map $\sigma\colon S \to A$ is a representation if the following conditions hold:
  1. $\sigma(0) = 0$;
  2. $\sigma(st) = \sigma(s)\sigma(t)$ for all $s, t \in S$;
  3. $\sigma(s^{-1}) = (\sigma(s))^*$ for all $s \in S$.
- D_σ – the C^*-subalgebra of A generated $\sigma(E(S))$,

$$B_\sigma = \{e \in D_\sigma \colon e^2 = e\}.$$

  B_σ is a Boolean algebra with respect to the operations

$$a \wedge b = ab, \; a \vee b = a + b - ab, \; a \setminus b = a - ab.$$

- Let $X \subseteq E(S) \times \mathcal{P}_{fin}(E(S))$. A representation $\sigma\colon S \to A$ is called X-to-join if the restriction of σ to $E(S)$ is an X-to-join representation of $E(S)$ in the Boolean algebra B_σ.

Right LCM semigroups and their C^*-algebras

- A semigroup P is right LCM if it is left cancellative and the intersection of any two principal right ideals is either empty or a principal right ideal.
- We assume that P has the identity element denoted 1_P.
- $\mathcal{J}(P)$ – the set of all principal right ideals of P, plus $\varnothing$.
- The full C^*-algebra $C^*(P)$ of P (Li, 2012, P any left cancellative semigroup, $\mathcal{J}(P)$ the set of constructible right ideals).
- $C^*(P)$ is the universal C^*-algebra generated by a set of isometries $\{v_p\colon p \in P\}$ and a set of projections $\{e_X\colon X \in J(P)\}$ subject to the following relations:

  (L1) $v_p v_q = v_{pq}$ for all $p, q \in P$,

  (L2) $v_p e_X v_p^* = e_{pX}$ for all $p \in P$ and $X \in J(P)$,

  (L3) $e_P = 1$ and $e_\varnothing = 0$,

  (L4) $e_X e_Y = e_{X\cap Y}$ for all $X, Y \in J(P)$.

$C^*(P)$ is a groupoid C^*-algebra

- $U(P)$ – the group of units of P.
- $(p,q) \sim (a,b) \Leftrightarrow$ there is $u \in U(P)$ such that $p = au$ and $q = bu$.
- $[p,q]$ – the $\sim$-class of (p,q), for any $p,q \in P$.
- $\mathcal{S} = \{[p,q] \colon p,q \in P\} \cup \{0\}$

  is an inverse semigroup with the identity $[1_P, 1_P]$,

  $$[a,b][c,d] = \begin{cases} [ab', dc'], & \text{if } cP \cap bP = rP \text{ and } cc' = bb' = r, \\ 0, & \text{if } cP \cap bP = \varnothing, \end{cases}$$

  $s0 = 0s = 0$ and $[p,q]^{-1} = [q,p]$.
- $E(\mathcal{S}) = \{[p,p] \colon p \in P\}$.
- The inverse semigroup $\mathcal{S}$ is called the left inverse hull of P.
- Another consruction of $\mathcal{S}$:
  - $\lambda_p \colon P \to pP$, $p \in P$, is a bijection $\Rightarrow \lambda_p \in \mathcal{I}(P)$.
  - $\mathcal{I}_l(P)$ – the inverse subsemigroup of $\mathcal{I}(P)$ generated by λ_p, $p \in P$.
  - $\mathcal{S} \to \mathcal{I}_l(P)$, $[p,q] \mapsto \lambda_p \lambda_q^{-1}$, is an isomorphism.
- $C^*(P)$ is isomorphic to the universal C^*-algebra $C^*(\mathcal{S})$ of S (Norling, 2014). It is a groupoid C^*-algebra of the groupoid $\mathcal{G}(S)$.

The boundary quotient $\mathcal{Q}(P)$

- A finite subset $F \subset P$ is a foundation set if for all $p \in P$ there exists $f \in F$ such that $fP \cap pP \neq \varnothing$.
- $F \subseteq P$ is a foundation set $\Leftrightarrow \{[f,f] \colon f \in F\}$ is a cover of $[1_P, 1_P]$ in $E(\mathcal{S})$.
- The boundary quotient $\mathcal{Q}(P)$ of $C^*(P)$ is defined as the universal C^*-algebra given by the defining relations of $C^*(P)$ plus the relations

$$\prod_{p \in F}(1 - e_{pP}) = 0 \quad \text{for all foundation sets } F \subseteq P$$

  (Brownlowe, Ramagge, Robertson and Whittaker, 2014).
- $\mathcal{Q}(P)$ is isomorphic to the tight C^*-algebra $C^*_{tight}(S)$ of $\mathcal{S}$ (Starling, 2015). It is a groupoid C^*-algebra of the groupoid $\mathcal{G}_{tight}(\mathcal{S})$ (P – countable).
- Example: if $P = \{a, b\}^*$ then characters of $E(\mathcal{S})$ are in a bijection with path in the binary tree (both finite and infinite), the cover-to-join characters correspond to infinite paths).

An example: the semigroup $\mathbb{N} \rtimes \mathbb{N}^\times$

- $\mathbb{N}^\times = \{n \in \mathbb{N} \colon n \geq 1\}$
- $\mathbb{N}^\times$ acts on $\mathbb{N}$ by multiplication.
- $\mathbb{N} \rtimes \mathbb{N}^\times$ – the semidirect product, i.e.

$$(r, a)(q, b) = (r + aq, ab).$$

- $P = \mathbb{N} \rtimes \mathbb{N}^\times$ is right LCM:

$$(r, a)P \cap (q, b)P = \begin{cases} (l, \mathrm{lcm}(a, b)), & \text{if } (r + a\mathbb{N}) \cap (q + b\mathbb{N}) \neq \varnothing, \\ \varnothing, & \text{otherwise.} \end{cases}$$

- The C^*-albebra $\mathcal{Q}_\mathbb{N}$ (Cuntz, 2008), it is isomorphic to the Crisp-Laca quotient $\mathcal{Q}(\mathbb{N} \rtimes \mathbb{N}^\times)$ of the Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ (Laca and Raeburn, 2010). This is precisely the tight C^*-algebra $C^*_{tight}(\mathcal{S})$ (follows from the result by Starling, 2015).

A construction of a Zappa-Szép product

- P – a semigroup with unit e; $U, A \subseteq P$ – subsemigroups and
  - $U \cap A = \{e\}$;
  - $\forall p \in P \; \exists! \; (u, a) \in U \times A$ such that $p = ua$.
- Then P is an internal Zappa-Szép product $P \simeq U \bowtie A$ of U and A.
- If $u \in U$ and $a \in A$ we write $au = (a \cdot u)a|_u$ where $(a \cdot u) \in U$ and $a|_u \in A$. These define the action and the restriction maps.
- Brin (2005) defined external Zappa-Szép products of U and A and proved the equivalence of the 'external' and the 'internal' definitions.
- For groups: Zappa (1942) and Szép (1950, 1958, 1962).
- Brownlowe, Ramagge, Robertson, Whittaker (2014) considered Zappa-Szép products $P \simeq U \bowtie A$ such that:

  (C1) U, A are right LCM;
  (C2) $\mathcal{J}(A)$ is totally ordered by inclusion;
  (C3) The map $u \mapsto a \cdot u$ is a bijection for each $a \in A$.
- Then the P is right LCM.

Zappa-Szép products: examples

- $\mathbb{N} \rtimes \mathbb{N}^{\times}$
    - $U = \{(r, x) \in \mathbb{N} \times \mathbb{N}^{\times} : 0 \leq r \leq x - 1\}$, $A = \{(m, 1) : m \in \mathbb{N}\}$.
    - Axioms (C1), (C2), (C3) hold.
- Baumslag - Solitar semigroups $B(c, d)^{+}$
    - $c, d \in \mathbb{Z}$, $c, d > 0$, $BS(c, d)$ is a group given by the group presentation $BS(c, d) = \langle a, b \colon ab^{c} = b^{d}a \rangle$.
    - $B(c, d)^{+}$ is the submonoid in $BS(c, d)$, generated by a and b.
    - every element of $B(c, d)^{+}$ can be uniquely written as $\prod_{i=1}^{n}(b^{\alpha_i}a)b^{\beta}$, where $\alpha_i \in \{0, \dots, d-1\}$ and $\beta \geq 0$.
    - $U = \langle a, ba, \dots, b^{d-1}a \rangle$, $A = \langle b \rangle$.
    - Axioms (C1), (C2), (C3) hold.
- Self-similar group actions: X a finite alphabet, G a group acting faithfully on the rooted tree X^{*}. The action is self-similar if $\forall g \in G, x \in X \ \exists! g|_x \in G \colon g \cdot (xw) = (g \cdot x)(g|_x \cdot w)$.

The additive and the multiplicative boundary quotients of $C^*(U \bowtie A)$

- Suppose that axioms (C1), (C2) and (C3) hold. Brownlowe, Ramagge, Robertson, Whittaker (2014) have shown that $\mathcal{Q}(U \bowtie A)$ is a quotient of $C^*(U \bowtie A)$ by the relations:

  (Q1) $e_{aP} = 1$ for all $a \in A$ and
  (Q2) $\prod_{p \in F} (1 - e_{pP}) = 0$ for all foundation sets $F \subseteq U$.

- The additive boundary quotient $\mathcal{Q}_A(U \bowtie A)$ - only relations (Q1)
- The multiplicative boundary quotient $\mathcal{Q}_U(U \bowtie A)$ - only relations (Q2)
- $\mathcal{Q}_U(\mathbb{N} \rtimes \mathbb{N}^\times)$ and $\mathcal{Q}_A(\mathbb{N} \rtimes \mathbb{N}^\times)$ were studied before that by Brownlowe, An Huef, Laca and Raeburn (2012).

- The boundary quotient diagram:

$$\begin{array}{ccc} & C^*(U \bowtie A) & \\ \swarrow & & \searrow \\ \mathcal{Q}_U(U \bowtie A) & & \mathcal{Q}_A(U \bowtie A) \\ \searrow & & \swarrow \\ & \mathcal{Q}(U \bowtie A) & \end{array}$$

The core subsemigroup of a right LCM semigroup

- Stammeier (2015) studies right LCM semigroups P that are decomposable as $P \simeq U \bowtie A$ where

$$A = P_c = \{p \in P \colon pP \cap qP \neq \varnothing \text{ for all } q \in P\}$$

  - the core subsemigroup of P (the term stems from Crisp and Laca, 2007, in the context of quasi-lattice ordered groups).

- Examples:
  - $\mathbb{N} \rtimes \mathbb{N}^{\times}$, $B(c,d)^{+}$, $X^* \bowtie G$ - in their presented decompositions as $P \simeq U \bowtie A$ we have $A = P_c$.

- Stammeier (2015) asked for a realisation of $\mathcal{Q}_U(U \bowtie A)$ and $\mathcal{Q}_A(U \bowtie A)$ as groupoid C^*-algebras.

$\mathcal{Q}_U(U \bowtie A)$ and $\mathcal{Q}_A(U \bowtie A)$ are groupoid C^*-algebras

- $P = U \bowtie A$, $\mathcal{S}$ – the left inverse hull of P.
- The set X_A consists of all $([a,a],\{[b,b]\})$ where $a \in A$ and $b \in aA$;
- The set X_U consists of all $([s,s],\{[s_1,s_1],\dots,[s_n,s_n]\})$ where $s \in U$, $s_i \in sU$ for all $i \in \{1,\dots,n\}$ and for each $t \in sU$ there is $i \in \{1,\dots,n\}$ satisfying $s_iU \cap tU \neq \varnothing$.
- The C^*-algebras $C^*_A(\mathcal{S}_P)$, and $C^*_U(\mathcal{S}_P)$ are defined as the universal C^*-algebras generated by one element for each element of $\mathcal{S}_P$ subject to the following relations:
  1. for $C^*_A(\mathcal{S}_P)$ these are the relations saying that the standard map $\pi_A\colon \mathcal{S}_P \to C^*_A(\mathcal{S}_P)$ is an X_A-to-join representation.
  2. for $C^*_U(\mathcal{S}_P)$ these are the relations saying that the standard map $\pi_U\colon \mathcal{S}_P \to C^*_U(\mathcal{S}_P)$ is an X_U-to-join representation.
- Result (GK, 2019)
  - $\mathcal{Q}_U(U \bowtie A)$ is isomorphic to the C^*-algebra $C^*_U(\mathcal{S})$.
  - $\mathcal{Q}_A(U \bowtie A)$ is isomorphic to the C^*-algebra $C^*_A(\mathcal{S})$.
- Corollary. $\mathcal{Q}_U(U \bowtie A)$ and $\mathcal{Q}_A(U \bowtie A)$ are groupoid C^*-algebras.