Regular semigroups weakly generated by idempotents

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Introduction

S - semigroup.

$E(S)$ - set of idempotents of the semigroup S.

The (von Neumann) inverses of $x \in S$ are the elements $x' \in S$ such that

$$xx'x = x \quad \text{and} \quad x'xx' = x' \, .$$

$V(x)$ - set of inverses of $x \in S$.

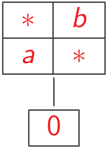
S is regular $\quad \Leftrightarrow \quad V(x) \neq \emptyset$ for all $x \in S$.

Introduction

S - regular semigroup.

S is weakly generated by $A \subseteq S$ if S has no proper regular subsemigroup containing A.

- S may not be generated by A .



$(*)$ - idempotents

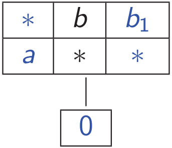
The "red" semigroup is weakly generated by $\{a\}$, but $\langle a\rangle = \{a, 0\}$.

Introduction

S - regular semigroup.

S is weakly generated by $A \subseteq S$ if S has no proper regular subsemigroup containing A.

- S may not be generated by A .
- There may exist more than one subsemigroup weakly generated by A.



$(*)$ - idempotents

The "red" subsemigroup is weakly generated by $\{a\}$.

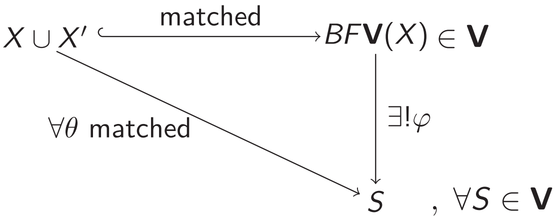
The "blue" subsemigroup is also weakly generated by $\{a\}$.

Motivation

E-variety $\mathbf{V}$ of regular semigroups: class of regular semigroups closed for homomorphic images, direct products and regular subsemigroups.

X': disjoint copy of X. $\theta : X \cup X' \to S$ matched: $x'\theta \in V(x\theta)$.

Bifree object on X:



However, not all e-varieties have bifree objects.

Motivation

$A \subseteq S$ <u>matched</u>: $A \cap V(x) \neq \emptyset$ for all $x \in A$.

Proposition (Yeh'92) The following conditions are equivalent for $|X| \geq 2$:

a) $BF\mathbf{V}(X)$ exists.

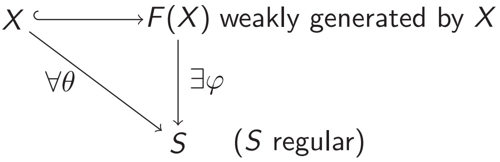
b) For every $S \in \mathbf{V}$ and every $A \subseteq S$ *matched* with $|A| \leq |X|$, there exists a unique subsemigroup of S weakly generated by A.

Theorem (Yeh'92) For $|X| \geq 2$, $BF\mathbf{V}(X)$ exists if and only if $\mathbf{V}$ is an e-variety of locally inverse semigroups or of regular E-solid semigroups.

Motivation

Question: Is there a "free" regular semigroup $F(X)$ weakly generated by X, in the sense that all regular semigroups weakly generated by X are homomorphic images of $F(X)$?

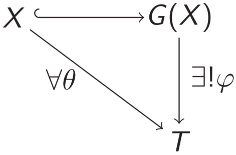
If $F(X)$ exists, it has the following property



Motivation

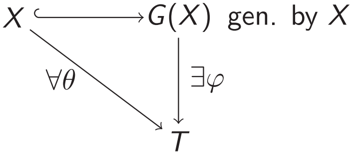
Free objects in varieties of algebras:

usual definition



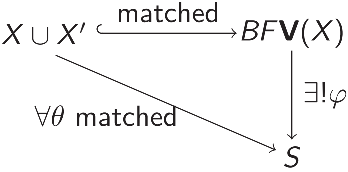
$\Leftrightarrow$

alternative definition

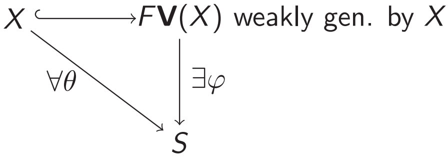


$\cdots$ and for e-varieties:

Bifree object



alternative notion



If $BF\mathbf{V}(X)$ exists, then $BF\mathbf{V}(X) \cong F\mathbf{V}(X)$.

Regular semigroups weakly generated by idempotents

We will focus only on regular semigroups weakly generated by idempotents in this talk.

Trivial observation: Any regular semigroup weakly generated by a set X of idempotents is idempotent generated (not necessarily by X).

Proof: S regular $\implies$ $\langle E(S) \rangle$ is a regular subsemigroup containing X.

Thus $S = \langle E(S) \rangle$ if S is weakly generated by X.

Regular semigroups weakly generated by idempotents

Question: Is there a

free regular semigroup $FI(X)$ weakly generated by $|X|$ idempotents?

(in the sense that all regular semigroups weakly generated by $|X|$ idempotents are homomorphic images of $FI(X)$.)

Answer: Yes, and it is unique up to isomorphism.

In this talk we will:

- Introduce $FI(X)$ by a presentation $\langle G, R\rangle$ with both G and R infinite.
- Present a solution to the word problem for this presentation (there is a "canonical form" despite G and R being infinite sets).
- Give some details about the structure of $FI(X)$.

The presentation $\langle G, R \rangle$

Notation:

- if $g \in L \times C \times R$, then $g = (g^l, g^c, g^r)$.
- $g^{l^2} = (g^l)^l$; $g^{rl} = (g^r)^l$; $g^{lr} = (g^l)^r$.

Recursive definition of G: $G = \cup G_i$ where

- $G_0 = \{1\}$ and $G_1 = X$;
- we identify each $x \in X$ with the triple $(1, x, 1)$;
- if G_{i-1} and G_i are defined, then let $G_{i+1} \subseteq G_i \times G_{i-1} \times G_i$ such that

$$g \in G_{i+1} \iff g^l \neq g^r \text{ and } g^c \in \{g^{l^2}, g^{lr}\} \cap \{g^{r^2}, g^{rl}\}$$

For example:

- $G_2 = \{(x_1, 1, x_2) \mid x_1, x_2 \in X \text{ with } x_1 \neq x_2\}$.

The presentation $\langle G, R \rangle$

The relation R: $R = \rho_e \cup \rho_s$ where

- $\rho_e = \{(1g, g),\ (g1, g),\ (g^2, g) \mid g \in G\}.$
- $\rho_s = \{(g^c g^l g, g),\ (gg^r g^c, g),\ (g^r g^c gg^c g^l, g^r g^c g^l) \mid g \in G_i,\ i \geq 2\}.$

Then $FI^1(X) = G^+/\rho$ where ρ is the congruence generated by R.

- $\rho_e \Rightarrow \begin{cases} 1\rho & \text{is the identity element ;} \\ G\rho & \text{is a set of idempotents of } FI^1(X) \text{ .} \end{cases}$
- $FI^1(X)$ is an idempotent generated monoid with identity element 1ρ.

The presentation $\langle G, R \rangle$

The sandwich set $S(e, f)$ of $e, f \in E(S)$ is the set

$$S(e, f) = \{g \in E(S) \ : \ ge = g\,, \ \ fg = g \ \text{ and } \ egf = ef\}\,.$$

Proposition: An idempotent generated semigroup S is regular if and only if $S(e, f) \neq \emptyset$ for all $e, f \in E(S)$.

- ρ_e and ρ_s $\Rightarrow$ $g^c g^l, g^r g^c \in E(FI^1(X))$.
- Then $\rho_s \Leftrightarrow g \in S(g^r g^c, g^c g^l)$.

The word problem for $\langle G, R\rangle$

The height of $g \in G$ is the index $\upsilon(g) = i$ such that $g \in G_i$.

$$\beta_1(g) \quad = \quad 1 \cdots g^{c^2} g^{cl} g^{c} g^{l} g\, g^{r} g^{c} g^{cr} g^{c^2} \cdots 1$$

height i

height $i-1$

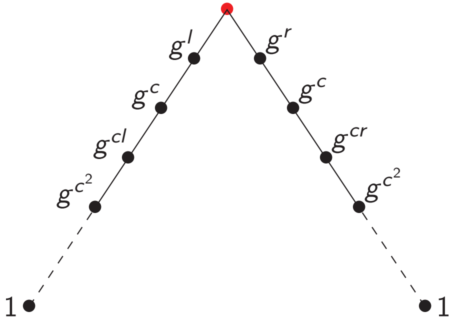
height $i-2$

height $i-3$

height $i-4$

$\vdots$

height 0



For $v = g_1 \cdots g_n \in G^+$, let $\beta_1(v) = \beta_1(g_1) \cdots \beta_1(g_n)$

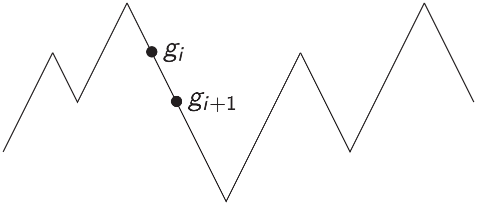
Proposition: $\quad g \;\rho\; \beta_1(g) \quad$ and $\quad v \;\rho\; \beta_1(v)$.

The word problem for $\langle G, R\rangle$

Landscape: word $u = g_1 \cdots g_n \in G^+$ such that

$$g_{i-1} \in \{g_i^l, g_i^r\} \text{ or } g_i \in \{g_{i-1}^l, g_{i-1}^r\} \text{ for all } i\,.$$

We represent the landscapes in drawings that include information about the height of the letters:



$$g_{i+1} \in \{g_i^l, g_i^r\}$$

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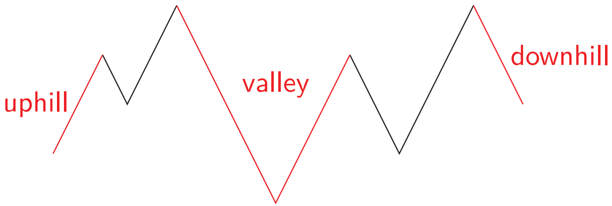

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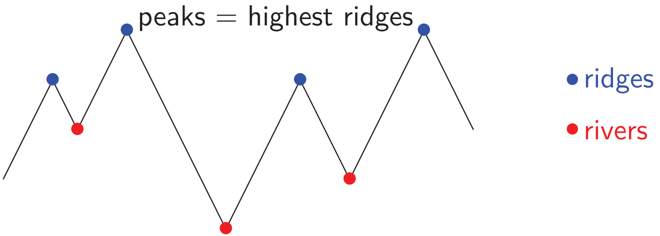

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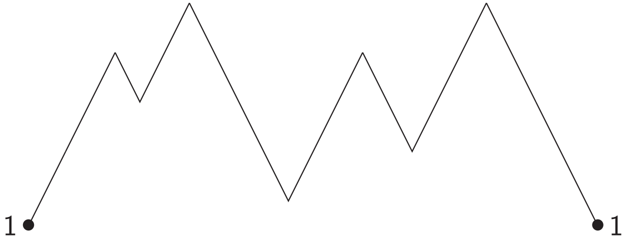
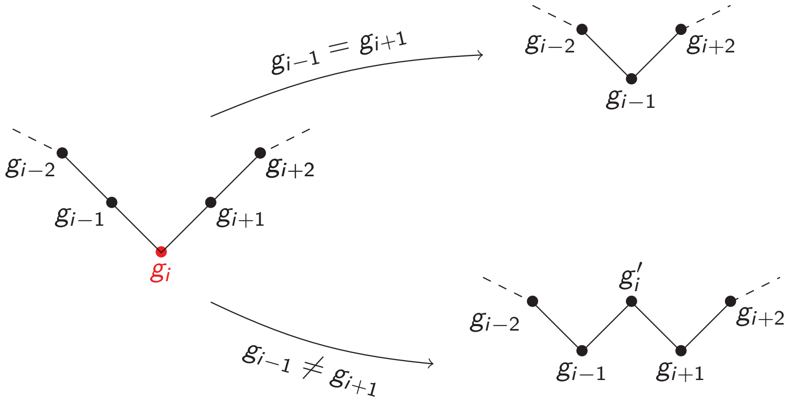


Mountain range: landscape u with $\sigma(u) = \tau(u) = 1$ (examples: $\beta_1(v)$).

Mountain: mountain range with no rivers (examples: $\beta_1(g)$).

Uplifting of rivers



where $g_i' = (g_{i+1}, g_i, g_{i-1}) \in G$

$u \to v$: v is obtained from u by uplifting a river.

$\xrightarrow{*}$: reflexive and transitive closure of $\to$.

The solution to the word problem

Proposition: Applying uplifting of rivers to a mountain range u, we must always stop with the same mountain independently of the order we choose to apply the upliftings.

$\beta_2(u)$: unique mountain such that $u \xrightarrow{*} \beta_2(u)$ (u is a mountain range).

Proposition: $u \; \rho \; \beta_2(u)$ for any mountain range u.

- show that $u \; \rho \; v$ if $u \to v$, and use transitivity.

$\beta(v) = \beta_2(\beta_1(v))$ for any $v \in G^+$.

Proposition: $v_1 \; \rho \; v_2 \quad \Leftrightarrow \quad \beta(v_1) = \beta(v_2)$ for any $v_1, v_2 \in G^+$.

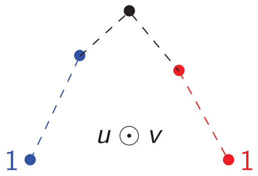
- $v \; \rho \; \beta(v)$ for any $v \in G^+$.
- $\beta(v)$ is the only mountain in $v\rho$.

The semigroup $FI(X)$

$FI(X) = FI^1(X) \setminus \{1_\rho\}$.

$M(X)$: set of all non-trivial mountains of G^+.

Proposition: $FI(X) \cong (M(X), \odot)$, where $u \odot v = \beta_2(u * v)$.



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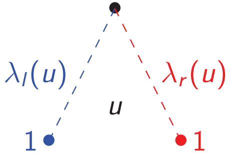
If $u = g_0 g_1 \cdots g_{n-1} g_n \in M(X)$, then

- $\lambda_l(u)$: "left hill" of u;
- $\lambda_r(u)$: "right hill" of u;
- $\overleftarrow{u} = g_n g_{n-1} \cdots g_1 g_0 \in M(X)$.

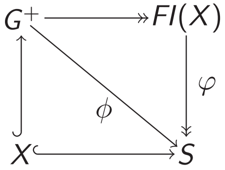


The semigroup $FI(X)$

Proposition: $FI(X)$ is a regular semigroup weakly generated by $X\rho$.

- $\overleftarrow{u} \in V(u)$ for any $u \in M(X)$.
- $S(\beta(g^r g^c), \beta(g^c g^l)) = \{\beta_1(g)\}$ in $M(X)$.

Theorem: Any regular semigroup weakly generated by a set X of idempotents is a homomorphic image of $FI(X)$.



$$\rho_e \cup \rho_s \subseteq \ker \phi$$

However, not all homomorphic images of $FI(X)$ are weakly generated by X.

- Can we (partially) characterize which homomorphic images of $FI(X)$ are weakly generated by X? Is this problem decidable?

Skeletons

Skeleton mapping: $\varphi : G \setminus \{1\} \to E(S)$ "respecting" the structure of G

$$\varphi_{|X} \text{ is one-to-one} \qquad \text{and} \qquad g\varphi \in S((g^r)\varphi\,(g^c)\varphi, (g^c)\varphi\,(g^l)\varphi).$$

Skeleton of S (induced by X): $(G \setminus \{1\})\varphi$.

Proposition: Let S be a regular semigroup.

(*i*) The subsemigroup generated by a skeleton is always regular.

(*ii*) If S is weakly generated by $|X|$ idempotents, then S is generated by any of its skeletons (it can have distinct skeletons).

Open questions:

- When is the semigroup generated by a skeleton weakly generated by X? Is this question decidable?
- When do two skeletons generate the same subsemigroup?

Green's relations

S: regular semigroup

$s \leq_{\mathscr{R}} t \Leftrightarrow sS \subseteq tS$; $s \leq_{\mathscr{L}} t \Leftrightarrow Ss \subseteq St$; $s \leq_{\mathscr{J}} t \Leftrightarrow SsS \subseteq StS$;

$\mathscr{R} = \leq_{\mathscr{R}} \cap \geq_{\mathscr{R}}$; $\mathscr{L} = \leq_{\mathscr{L}} \cap \geq_{\mathscr{L}}$; $\mathscr{J} = \leq_{\mathscr{J}} \cap \geq_{\mathscr{J}}$;

$\leq_{\mathscr{H}} = \leq_{\mathscr{R}} \cap \leq_{\mathscr{L}}$; $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$; $\mathscr{D} = \mathscr{R} \vee \mathscr{L}$.

Proposition: For $u, v \in M(X)$,

(*i*) $u \leq_{\mathscr{R}} v \Leftrightarrow \lambda_l(v)$ prefix of $\lambda_l(u)$;

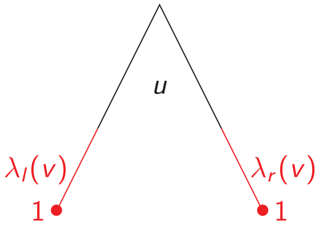
(*ii*) $u \,\mathscr{R}\, v \Leftrightarrow \lambda_l(u) = \lambda_l(v)$;

(*iii*) $u \leq_{\mathscr{L}} v \Leftrightarrow \lambda_r(v)$ suffix of $\lambda_r(u)$;

(*iv*) $u \,\mathscr{L}\, v \Leftrightarrow \lambda_r(u) = \lambda_r(v)$;

(*v*) $u \leq_{\mathscr{H}} v \Leftrightarrow \lambda_l(v)$ prefix of $\lambda_l(u)$ and $\lambda_r(v)$ suffix of $\lambda_r(u)$;

(*vi*) $u \,\mathscr{H}\, v \Leftrightarrow u = v$.



Green's relations

$\kappa(u)$: peak of $u \in M(X)$.

Ground of $g \in G$: defined recursively by $\epsilon(g) = \epsilon(g^l) \cup \{g\} \cup \epsilon(g^r)$.

Ground of $u \in M(X)$: $\epsilon(u) = \epsilon(\kappa(u))$.

Proposition: For $u, v \in M(X)$

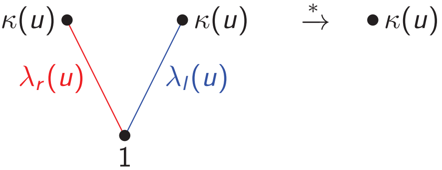
(i) $u \leq_{\mathscr{J}} v \Leftrightarrow \kappa(v) \in \epsilon(u)$;

(ii) $\mathscr{D} = \mathscr{J}$ and $u \mathscr{D} v \Leftrightarrow \kappa(u) = \kappa(v)$.

Corollary: $G \setminus \{1\}$ is a transversal set for the $\mathscr{D}$-classes of $FI(X)$.

Idempotents

Proposition: $u \in E(M(X)) \quad \Leftrightarrow$

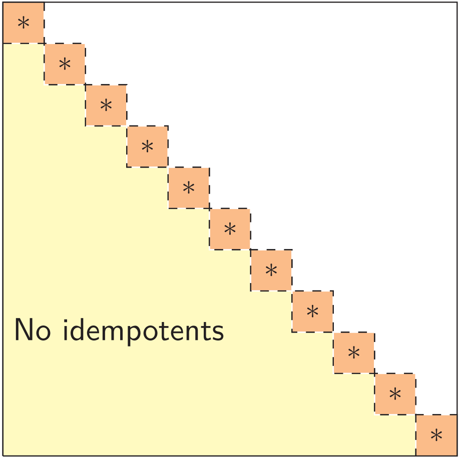


Open questions:

- What is the structure of the biordered set $E(FI(X))$?
- When is a regular biordered set E the biordered set of some regular semigroup weakly generated by $|X|$ idempotents?

The $\mathscr{D}$-class D_g

There is a natural way to order the $\mathscr{L}$ and $\mathscr{R}$-classes of D_g such that



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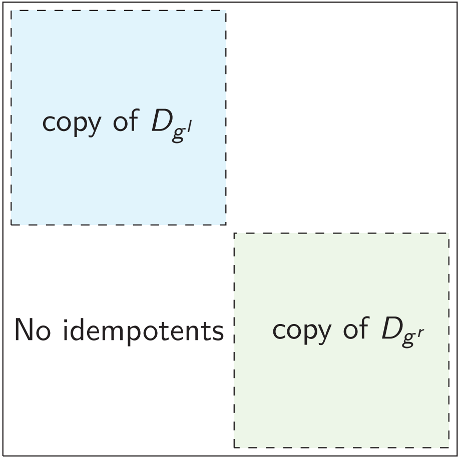

$FI(X)$ for X finite

$FI_n = FI(X)$ for $|X| = n$.

Proposition: FI_2 contains copies of all FI_n as subsemigroups.

Corollary: Every regular semigroup weakly generated by a finite set of idempotents strongly divides FI_2.

The previous corollary applies, in particular, to

(i) regular semigroups generated by a finite set of idempotents.

(ii) finite idempotent generated regular semigroups.

$FI(X)$ for X finite

Theorem [Gray & Ruškuc 2012]: Every group is a maximal subgroup of some free regular idempotent generated semigroup.

Question: What kind of groups can we get as maximal subgroups of free regular idempotent generated semigroups that are also weakly generated by two idempotents?

"Returning to the starting point":

- Is there a free regular semigroup $F(X)$ weakly generated by $|X|$ elements (non-idempotent case)?
- If $F(X)$ exists, what kind of impact can it have in the theory of e-varieties of regular semigroups?

The results presented here can be found in:

L.O., Regular semigroups weakly generated by idempotents, preprint

Thank you for your attention