Malcev expansions of finite semigroups

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Preliminaries

Alphabet – non-empty set (finite or infinite) A
Letter – element of A
Word (non-empty) – finite sequence of letters $a_1 a_2 \dots a_n$ with $n \geqslant 1$

A^+ – free semigroup over A, that is the set of all words with multiplication

$$(a_1 a_2 \dots a_n) \cdot (b_1 b_2 \dots b_p) = a_1 a_2 \dots a_n b_1 b_2 \dots b_p$$

$A^* = A^+ \cup \{1\}$ – free monoid over A

A is a generating set of A^+.

Proposition

Any map $A \to S$, where S is a semigroup, can be extended to a unique morphism $A^+ \to S$.

Identity (in A^+) – formal equality $u = v$, where $u, v \in A^+$

A semigroup S satisfies an identity $u = v$ over A if $u\varphi = v\varphi$ for all morphisms $\varphi\colon A^+ \to S$.

Variety (of semigroups): class of semigroups closed under the formation of homomorphic images, subsemigroups, and direct products (with finitely or infinitely many factors).

For a set Σ of identities,
$[\Sigma]$ – class of the semigroups that satisfy all the identities of Σ

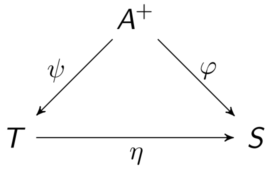
Theorem (Birkhoff, 1935)

A class $\mathcal{V}$ of semigroups is a variety if and only if $\mathcal{V} = [\Sigma]$ for some set Σ of identities.

Examples: $\mathcal{S} = [x = x]$ – (all) semigroups
$\mathcal{I} = [x = y]$ – trivial semigroups
$\mathcal{B} = [x^2 = x]$ – bands (i.e. every element is idempotent)
$\mathcal{SL} = [x^2 = x,\ xy = yx]$ – semilattices
$\mathcal{LZ} = [xy = x]$ – left zero semigroups
$\mathcal{RB} = [xyx = x]$ – rectangular bands

A-generated semigroup: pair (S, φ), where S is a semigroup and $\varphi\colon A^+ \to S$ is a surjective morphism (so S is generated as semigroup by $A\varphi$).

Morphism of A-generated semigroups from (T, ψ) to (S, φ): semigroup morphism $\eta\colon T \to S$ respecting generators, that is $\psi\eta = \varphi$:



Malcev expansion

The Malcev expansion was formally introduced in:
Elston, *Semigroup expansions using the derived category, kernel and Malcev products*, J. Pure Appl. Algebra **136** (1999), 231–265.

$\mathcal{V}$ – variety of semigroups

For a surjective morphism $\eta\colon T \to S$ such that $(e)\eta^{-1} \in \mathcal{V}$ for any $e \in E(S)$, we say that

- the subsemigroups $(e)\eta^{-1}$ of T are the Malcev kernels of η.
- η is a $\mathcal{V}$-morphism.
- T is a Malcev product of a semigroup S by $\mathcal{V}$.

Analogously for A-generated semigroups, in which case η respects generators.

We want the Malcev expansion of an A-generated semigroup (S, φ) over $\mathcal{V}$ to be the “largest” A-generated Malcev product (T, ψ) of (S, φ).

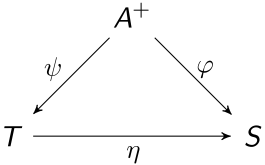
Given an A-generated semigroup (S, φ), that is a surjective morphism

$$\varphi\colon A^+ \to S$$

how to define such a largest (T, ψ)?

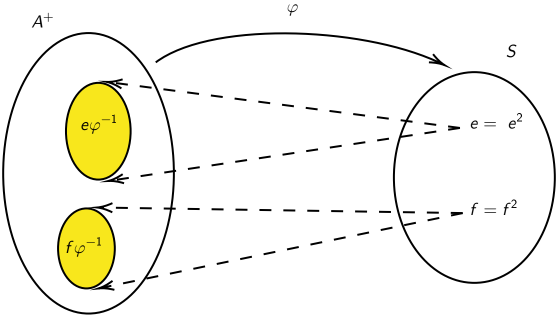
Note that we immediately have

- $T \simeq A^+/\equiv_\psi$, so a quotient of A^+.
- $\equiv_\psi \subseteq \equiv_\varphi$
- η has Malcev kernels in $\mathcal{V}$.



Elston gives the following definition for an A-generated semigroup (S, φ):

The Malcev expansion *of (S, φ) by $\mathcal{V}$ is the semigroup $\mathcal{V}ⓜS = A^+/\mu_\varphi$, where μ_φ is the congruence on A^+ generated by imposing the identities of $\mathcal{V}$ on each $(e)\varphi^{-1}$ for e an idempotent of S.*



$$e\varphi^{-1}, f\varphi^{-1} \in \mathcal{V}$$

Let us give this definition in two more precise ways.

Let $\varphi\colon A^+ \to S$ be a surjective morphism.

1) [McCammond, Steinberg, Rhodes, 2011]
$\mathcal{V} \textcircled{m} S = A^+/\mu_\varphi$, where μ_φ is the intersection of all congruences $\sim$ on A^+ contained in $\equiv_\varphi$ whose natural morphism $A^+/\sim \to S$ has Malcev kernels in $\mathcal{V}$ ($\equiv_\varphi$ is one of such congruences).

2) [Pin, 2006; McCammond, Steinberg, Rhodes, 2011]
Suppose that $\mathcal{V} = [\Sigma]$, with Σ a set of identities over an alphabet B.

$\mathcal{V} \textcircled{m} S = \mathrm{Smg}\langle A \mid R\rangle$, where

$$R = \{u\sigma = v\sigma\colon\ u = v \in \Sigma \text{ and } \sigma\colon B^+ \to A^+ \text{ is a morphism such that } B^+\sigma \subseteq e\varphi^{-1} \text{ for some } e \in E(S)\}$$

Thus $\mathcal{V} \textcircled{m} S = A^+/\mu_\varphi$, where μ_φ is the congruence on A^+ generated by R.

- It can be proved that this definition does not depend on the choosen set Σ.

It follows that, for an A-generated semigroup (A, φ),

- $\mathcal{I}ⓜS = \mathrm{Smg}\langle A \mid \{u = v \colon \; u\varphi = v\varphi = u^2\varphi\}\rangle$
  $(\mathcal{I} = [x = y])$
- $\mathcal{LZ}ⓜS = \mathrm{Smg}\langle A \mid \{uv = u \colon \; u\varphi = v\varphi = u^2\varphi\}\rangle$
  $(\mathcal{LZ} = [xy = x])$
- $\mathcal{RB}ⓜS = \mathrm{Smg}\langle A \mid \{uvu = u \colon \; u\varphi = v\varphi = u^2\varphi\}\rangle$
  $(\mathcal{RB} = [xyx = x])$

Remark: From above, $\mathcal{I}ⓜS = A^+/\mu_\varphi$, where μ_φ is the congruence on A^+ generated by the relation

$$M_\varphi = \left\{(u, v) \in E(S)\varphi^{-1} \times E(S)\varphi^{-1} \colon \; u\varphi = v\varphi\right\}$$

Examples:

- $G = \mathrm{Smg}\langle a \colon a^4 = a \rangle = \{g, g^2, 1\}$ – cyclic group of order 3
  $A = \{a\}$
  $\varphi \colon A^+ \to G$ the natural morphism ($a\varphi = g$)
  - $1\varphi^{-1} = \{a^3, a^6, a^9, \dots\}$
  - $\mathcal{I}ⓜG = \mathrm{Smg}\langle a \colon a^3 = a^6 \rangle$, which has exactly 5 elements: a, a^2, a^3, a^4, a^5
  - $\mathcal{I}ⓜG$ is not a monoid

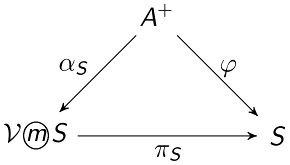
- $G = \mathrm{Smg}\langle a \colon a^4 = a \rangle = \{g, g^2, 1\}$ – cyclic group of order 3
  $A = \{a, b\}$
  $\varphi \colon A^+ \to G$ the morphism such that $a\varphi = b\varphi = g$
  - $1\varphi^{-1} = \{u \in A^+ \colon |u| \text{ is multiple of } 3\}$
  - $\mathcal{I}ⓜG$ has exactly 9 elements: $a, b, a^2, ab, ba, b^2, a^3, a^4, a^5$

Example:

- $G = \mathrm{Smg}\langle a\colon a^4 = a\rangle = \{g, g^2, 1\}$ – cyclic group of order 3
  G^I – G with an extra identity I adjoined
  $A = \{a, b\}$
  $\varphi\colon A^+ \to G$ the morphism such that $a\varphi = g$ and $b\varphi = I$
  - $I\varphi^{-1} = \{b^n\colon n \in \mathbb{N}\} = b^+$
    $1\varphi^{-1} = (b^*ab^*ab^*ab^*)^+$
  - $\mathcal{I}Ⓜ G$ has exactly 16 elements:
    $a, a^2, a^3, a^4, a^5,\ b,$
    $ba, ab, bab \quad (|u|_b \neq 0$ and $|u|_a = 1)$
    $ba^2, aba, a^2b, baba, ba^2b, abab, babab \quad (|u|_b \neq 0$ and $|u|_a = 2)$

Properties:

- For an A-generated semigroup (S, φ),
  - there is a morphism $\pi_S \colon \mathcal{V}\textcircled{m}S \to S$ of A-generated semigroups:



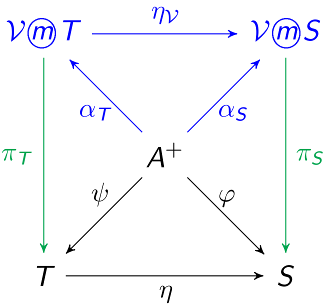
    where α_S is the canonical morphism.
  - π_S is a $\mathcal{V}$-morphism, that is $e\varphi^{-1} \in \mathcal{V}$ for any $e \in E(S)$.

Properties:

- The correspondence

$$S \longmapsto \mathcal{V} \textcircled{m} S$$

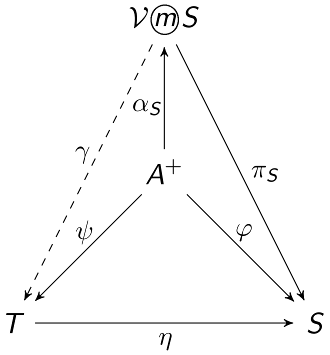
defines an expansion (in sense of Birget-Rhodes) of the category of A-generated semigroups:



Properties:

- $\mathcal{V} ⓜ S$ is the largest Malcev product A-generated semigroup of S:

  If (T, ψ) is an A-generated semigroup and $\eta\colon (T,\psi) \to (S,\varphi)$ is a $\mathcal{V}$-morphism, then there exists a morphism $\gamma\colon (\mathcal{V} ⓜ S, \alpha_S) \to (T,\psi)$ such that the following diagram commutes:



- A semigroup S is locally finite if every finitely generated subsemigroup of S is finite.
- $\mathcal{V}$ is locally finite if all of its semigroups are locally finite.

▶ Every variety of bands and every variety generated by a single semigroup is locally finite.

A – finite alphabet

Theorem (Brown, 1971)

Let S be a locally finite semigroup and $\eta\colon T \to S$ be a surjective morphism such that $e\eta^{-1}$ is a locally finite subsemigroup of T for any $e \in E(S)$. Then T is locally finite.

Corollary

If S is a finite A-generated semigroup and $\mathcal{V}$ is locally finite, then $\mathcal{V}ⓜS$ is finite.

Remark:

- Idempotent-pure morphisms are precisely the $\mathcal{B}$-morphisms.
- Lawson, Margolis, Steinberg (2006) study $\mathcal{B}ⓜS$ when S is an inverse semigroup.

Our focus: $\mathcal{V} = \mathcal{I}$ – trivial variety.

A crucial result:

Proposition

Let S and T be finite semigroups and let $\eta\colon T \to S$ be a surjective $\mathcal{I}$-morphism.
Then η is injective on regular elements.

Corollary

Let S and T be finite semigroups and let $\eta\colon T \to S$ be a surjective $\mathcal{I}$-morphism.
Then, for each regular $\mathcal{J}$-class J of S, there exists a unique regular $\mathcal{J}$-class K of T such that $K\eta = J$. In this situation, K is a subsemigroup of T if and only if J is a subsemigroup of S.

The Malcev expansion by the variety $\mathcal{I}$ is stable:

Proposition

For any A-generated semigroup (S, φ)*, one has* $\mathcal{I}Ⓜ(\mathcal{I}ⓂS) = \mathcal{I}ⓂS$.

$\mathcal{I}ⓜS$, for S locally group

A finite semigroup S is locally group if eSe is a subgroup of S for any $e \in E(S)$.

Proposition

Let S be a finite semigroup and let I be its minimal ideal. The following are equivalent:

1. *S is locally group.*
2. $E(S) \subseteq I$
3. $Reg(S) = I$

- Finite groups, finite unipotent semigroups and finite simple semigroups are locally groups.
- If S is locally group, then $\mathcal{I}ⓜS$ is locally group.

In general, $\mathcal{I}ⓜS \not\simeq S$ when S is locally group.

If (S,φ) is an A-generated semigroup, let

$$\mathrm{Red}(\varphi) = \{u \in A^+ \colon\ \forall v \in \mathrm{Fact}(u),\ u\varphi \notin E(S)\}$$

- Associative multiplication in $\mathrm{Red}(\varphi) \cup \mathrm{Reg}(S)$: for $u, v \in \mathrm{Red}(\varphi)$ and $s, t \in \mathrm{Reg}(S)$,

$$u \cdot v = \begin{cases} uv & \text{if } uv \in \mathrm{Red}(\varphi) \\ (uv)\varphi & \text{otherwise} \end{cases}$$
$$u \cdot s = (u\varphi)s, \qquad s \cdot u = s(u\varphi)$$
$$s \cdot t = st \text{ (product in } S)$$

- Generating morphism $\psi \colon A^+ \to \mathrm{Red}(\varphi) \cup \mathrm{Reg}(S)$ defined by

$$u\psi = \begin{cases} u & \text{if } u \in \mathrm{Red}(\varphi) \\ u\varphi & \text{otherwise} \end{cases}$$

Proposition

For every finite locally group A-generated semigroup (S,φ), one has the following, where $T = Red(\varphi) \cup Reg(S)$:

- $\mathcal{I}Ⓜ S \simeq T$
- *$Reg(S)$ is the minimal ideal of T and $T/Reg(S)$ is nilpotent.*

L-factorization expansion

$L \subseteq A^+$

(S, φ) a finite A-generated semigroup

Let $W_\varphi(L)$ be the smallest subset X of A^+ such that

1. $L \subseteq X$
2. for any $u, w \in A^*$ and $v \in A^+$,

$$uv, vw \in X \implies uvw \in X$$

3. for any $x, y \in X$ such that $x\varphi = y\varphi$ and $u, v \in A^*$,

$$uxv \in X \implies uyv \in X$$

Let $Z(L)$ be the smallest subset X of A^+ satisfying conditions 1 and 2.

A sequence $(u_1, u_2, \ldots, u_n)$ of words of A^* is a factorization of length n. If $u = u_1 u_2 \ldots u_n$, we say that $(u_1, u_2, \ldots, u_n)$ is a factorization of u.

- A condition concerning a factorization (u_0, x, u_1):
  (C) $x \in W_\varphi(L)$ and, for all $y \in \mathrm{Suf}(u_0)$ and $z \in \mathrm{Pref}(u_1)$,
  $$yxz \in W_\varphi(L) \implies y = z = 1$$
  This says that the factor x is "locally maximal" on $u_0 x u_1$ relatively to $W_\varphi(L)$.
- For each $i \in \mathbb{N}$, a condition concerning a factorization $(u_0, x_1, u_1, \ldots, x_n, u_n)$ of odd length:
  (C_i) If $n \geqslant i$, the factorization $(u_0 x_1 u_1 \ldots x_{i-1} u_{i-1}, x_i, u_i x_{i+1} u_{i+1} \ldots x_n u_n)$ satisfies (C).

A factorization $(u_0, x_1, u_1, \ldots, x_n, u_n)$ is an L-factorization if satisfies $(C_1), (C_2), \ldots, (C_n)$ and $u_0, u_1, \ldots, u_n \in A^+ \setminus A^* L A^* \cup \{1\}$.

Proposition

Each word of A^+ has a unique L-factorization.

- Two factorizations

$$\underline{u} = (u_0, x_1, u_1, \ldots, x_n, u_n) \text{ and } \underline{v} = (v_0, y_1, v_1, \ldots, y_m, v_m)$$

are φ-equivalent if $n = m$, $u_0 = v_0$, $u_1 = v_1, \ldots, u_n = v_n$ and $x_1\varphi = y_1\varphi, \ldots, x_n\varphi = y_n\varphi$.

- Binary relation $\theta_\varphi(L)$ on A^+:

$u \; \theta_\varphi(L) \; v$ if and only if the L-factorizations of u and v are φ-equivalent.

Proposition

The relation $\theta_\varphi(L)$ is a congruence on A^+, which is generated by the relation $\{(u, v) \in W_\varphi(L) \times W_\varphi(L) \colon \; u\varphi = v\varphi\}$.

Let $\rho_\varphi(L)$ be the congruence on A^+ generated by

$$\{(u, v) \in Z(L) \times Z(L) \colon \; u\varphi = v\varphi\}$$

Proposition

The correspondences

$$S \longmapsto \tilde{S}_\varphi(L) = A^+/\theta_\varphi(L)$$

and

$$S \longmapsto \bar{S}_\varphi(L) = A^+/\rho_\varphi(L)$$

define expansions on the category of A-generated semigroups, which are both stable.

Recall that $\mathcal{I}ⓜS = A^+/\mu_\varphi$, with μ_φ the congruence on A^+ generated by

$$M_\varphi = \left\{(u, v) \in E(S)\varphi^{-1} \times E(S)\varphi^{-1} : \; u\varphi = v\varphi\right\}$$

E-factorization expansion

(S, φ) an A-generated semigroup

Let

- $L = (E(S))\varphi^{-1} \subseteq A^+$
- $\tilde{S}_E = A^+/\theta_\varphi(L)$, $\quad \tilde{\varphi}_E \colon A^+ \to \tilde{S}_E$ the canonical morphism
- $\bar{S}_E = A^+/\rho_\varphi(L)$, $\quad \bar{\varphi}_E \colon A^+ \to \bar{S}_E$ the canonical morphism

$(E(S))\varphi^{-1} = L \subseteq Z(L) \subseteq W_\varphi(L)$, which gives the following:

Proposition

There exist morphisms $\mathcal{I} \circledm S \to \bar{S}_E \to \tilde{S}_E \to S$ of A-generated semigroups.

Two cases:

1. $E(S)$ is a subsemigroup of S
2. S is locally group

- If $E(S)$ is a subsemigroup of S, then $\mathrm{Reg}(S)$ is a subsemigroup of S (well known).

Proposition

If $Reg(\mathcal{I}ⓜS)$ is a subsemigroup of $\mathcal{I}ⓜS$, then $\mathcal{I}ⓜS = \tilde{S}_E = \bar{S}_E$.

Corollary

If $E(S)$ is a subsemigroup of S or S is locally group, then $\mathcal{I}ⓜS = \tilde{S}_E = \bar{S}_E$.

Thank you!