Endomorphism monoids of countably infinite structures

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Algebraic invariants

Automorphism group

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Automorphism group $<$ Endomorphism monoid

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Automorphism group $<$ Endomorphism monoid $<$ Polymorphism clone

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Purely algebraic?

Algebraic and topological invariants

Δ: countable relational structure with underlying set D

Topology on D^D: base open sets

$$\Omega((a_1, b_1), \ldots, (a_n, b_n)) = \{f \in D^D \mid f(a_i) = b_i\}$$

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Discrete for finite Δ, interesting for countably infinite Δ.

M is closed $\Leftrightarrow$ $M = \mathrm{End}(\Delta)$ for some Δ.

The general problem

Question: Does the abstract algebraic structure of Aut(Δ) / End(Δ) / Pol(Δ) determine its topological structure?

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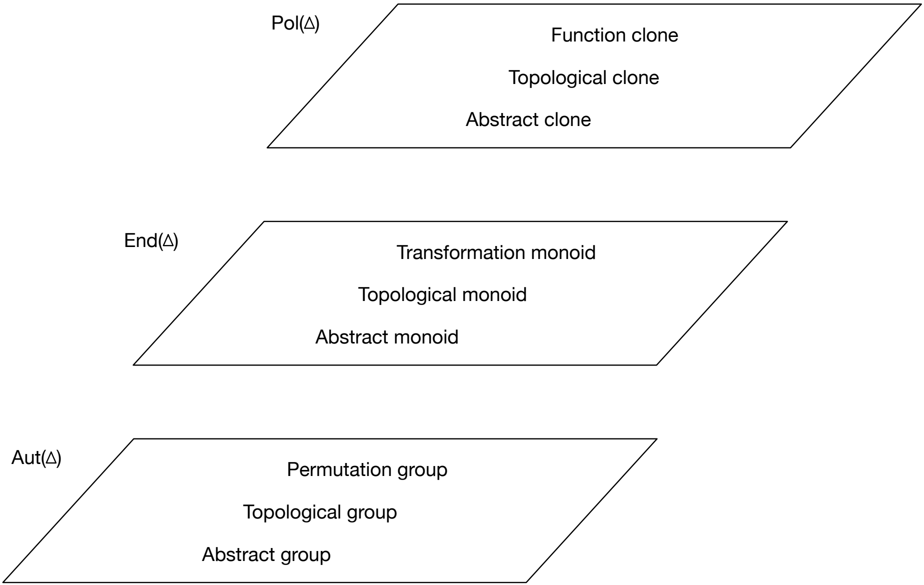
Question: How much information is captured by these invariants?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$\Rightarrow \Delta \cong (\mathbb{Q}; <)$

The automorphism group...

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as a permutation group.

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Let Δ, Γ be ω-categorical. TFAE:

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Topological clone — p. p. biinterpretability

Abstract clone — ?

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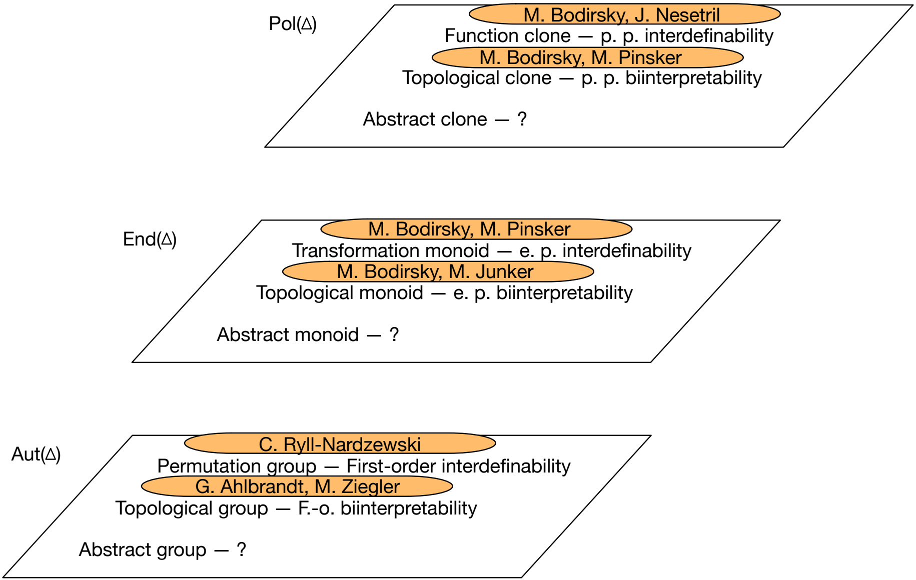
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Reconstruction

For any closed group G, if there exists an isomorphism $\xi : \mathrm{Aut}(\Delta) \to G$, then there exists (possibly another) isomorphism $\xi' : \mathrm{Aut}(\Delta) \to G$ which is a homeomorphism.

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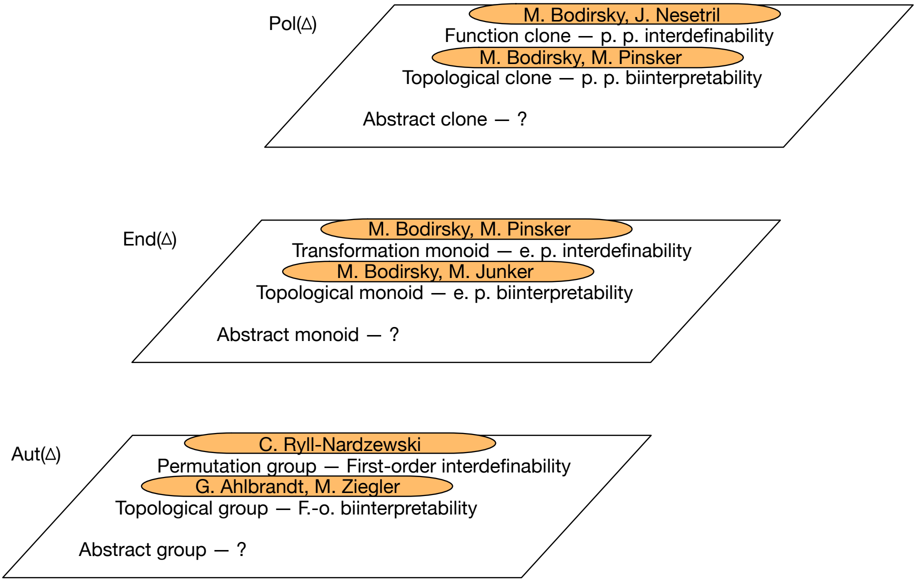
Fact: It is consistent with ZF that for every countable structure Δ the topological group Aut(Δ) has automatic continuity/ automatic homeomorphicity/ reconstruction.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Automatic continuity for monoids

M. Bodirsky, M. Pinsker, AP (2013)

Let $\mathcal{M}$ be a closed submonoid of D^D. Suppose that $\mathcal{M}$ contains a submonoid $\mathcal{N}$ such that $\mathcal{N}$ is not closed in $\mathcal{M}$, and $(\mathcal{M} \setminus \mathcal{N}) \circ \mathcal{M} \subseteq (\mathcal{M} \setminus \mathcal{N})$, $\mathcal{M} \circ (\mathcal{M} \setminus \mathcal{N}) \subseteq (\mathcal{M} \setminus \mathcal{N})$. Then $\mathcal{M}$ does not have automatic continuity.

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Reconstruction?

From groups to monoids

Proposition

Let $\mathcal{M}$ and $\mathcal{M}'$ be closed submonoids of D^D with dense subsets of invertibles $\mathcal{G}$ and $\mathcal{G}'$. Let $\xi : \mathcal{G} \to \mathcal{G}'$ be a continuous isomorphism. Then ξ extends to an isomorphism $\bar{\xi} : \mathcal{M} \to \mathcal{M}'$ which is a homeomorphism.

From groups to monoids

Proposition

Let $\mathcal{M}$ be a closed submonoid of D^D whose group of invertible elements $\mathcal{G}$ is dense in $\mathcal{M}$ and has automatic homeomorphicity. Assume that the only injective endomorphism of $\mathcal{M}$ that fixes every element of $\mathcal{G}$ is the identity function $\mathrm{id}_{\mathcal{M}}$ on $\mathcal{M}$. Then $\mathcal{M}$ has automatic homeomorphicity.

Automatic homeomorphicity of monoids

Theorem

Let Δ be a countable homogeneous relational structure such that $\mathrm{Aut}(\Delta)$ has no algebraicity and with the joint extension property such that $\mathrm{Aut}(\Delta)$ has automatic homeomorphicity. Then the monoid $\overline{\mathrm{Aut}(\Delta)}$ of self-embeddings of Δ has automatic homeomorphicity.

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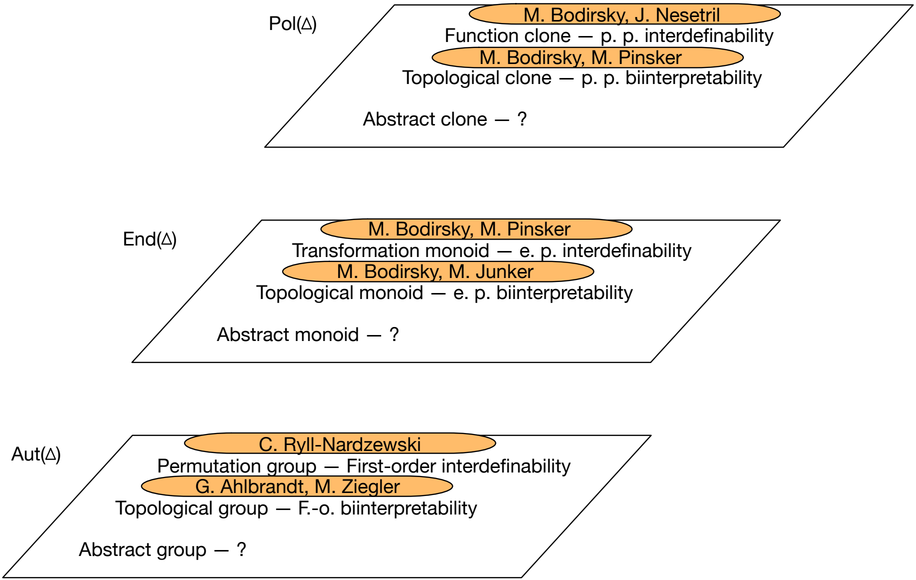
Clone homomorphisms: preserve arities, map π_i^n to π_i^n, compatible with composition, i.e., $\xi(f \circ (g_1, \ldots, g_n)) = \xi(f) \circ (\xi(g_1), \ldots, \xi(g_n)))$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Proposition

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there exist unary $(\alpha^i)_{i\in\mathbb{N}}$ and $(\beta^i_1)_{i\in\mathbb{N}}, \dots, (\beta^i_n)_{i\in\mathbb{N}}$ in $\mathcal{C}$ with

- $g^i(x_1, \dots, x_n) = \alpha^i(f_U(\beta^i(x_1), \dots, \beta^i(x_n)))$ and
- $(\alpha^i)_{i\in\mathbb{N}}$ and $(\beta^i_1)_{i\in\mathbb{N}}, \dots, (\beta^i_n)_{i\in\mathbb{N}}$ converge.

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Then $\mathcal{C}$ has automatic homeomorphicity.

Proposition

$\mathcal{H}$ has automatic homeomorphicity.

Gate coverings II

M. Bodirsky, M. Pinsker, AP (2013)

If $\mathcal{C}$ is a closed subclone of $\mathcal{O}$ such that

- $\mathcal{C}$ acts transitively;
- $\mathcal{C}^{(1)}$ has automatic homeomorphicity;
- $\mathcal{C}$ has a gate covering;

Then $\mathcal{C}$ has automatic homeomorphicity.

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Theorem

$\mathrm{Pol}(V, E)$ has automatic homeomorphicity.

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Every isomorphism ξ : End(V, E) $\to \mathcal{M}$ is continuous.

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Every $f \in \mathrm{Pol}(V, E)$ decomposes as $f = h \circ g$ with $h \in \mathrm{End}(V, E)$, $g \in \mathrm{Pol}(V, E) \cap \mathcal{H}$.

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Moreover, if $(f_i)_{i \in \mathbb{N}}$ is Cauchy, then $\exists$ decompositions $f_i = h_i \circ g_i$ with $h_i \in$ End(V, E), $g_i \in$ Pol(V, E) $\cap \mathcal{H}$ Cauchy sequences.

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All isomorphisms $\xi : \mathrm{Pol}(V, E) \to \mathcal{C}$ are continuous.

Open problems

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Problem 3 Does Aut($P, \leq$) have automatic continuity?