EHRESMANN MONOIDS: ADEQUACY AND EXPANSIONS

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Abstract. It is known that an Ehresmann monoid $\mathcal{P}(T,Y)$ may be constructed from any monoid T acting via order-preserving maps on both sides of a semilattice Y with identity, such that the actions satisfy an appropriate compatibility criterion. Our main result shows that if T is cancellative and equidivisible (as is the case for the free monoid X^*), the monoid $\mathcal{P}(T,Y)$ not only is Ehresmann but also satisfies the stronger property of being adequate.

Fixing T, Y and the actions, we characterise $\mathcal{P}(T,Y)$ as being unique in the sense that it is the initial object in a suitable category of Ehresmann monoids. We also prove that the operator $\mathcal{P}$ defines an expansion of Ehresmann monoids.

Introduction

An inverse monoid M has the property that any of its principal left (right) ideals is projective in the category of left (right) M-acts, and the idempotents of M commute, that is, they form a semilattice $E(M)$. These properties determine the broader class of *adequate monoids*, introduced by Fountain in [5], an article that has given rise to a rich theory. Adequate monoids were first defined via certain equivalence relations $\mathcal{R}^*$ and $\mathcal{L}^*$ that generalise Green's relations $\mathcal{R}$ and $\mathcal{L}$. The more modern approach is to consider them as being monoids equipped with two basic unary operations (usually denoted $^+$ and *), that is, as *bi-unary monoids*.

Adequate monoids form a quasi-variety of bi-unary monoids but not a variety: the variety they generate is that of *Ehresmann monoids*. This follows from the fact that adequate monoids are Ehresmann and that the free Ehresmann monoid on any set is adequate, a result noted in [12], which also follows from [3] and our results here.

In moving away from the arena of regular semigroups, many difficulties arise, and the theory that emerges splits into one and two-sided cases. Statements similar to the ones mentioned above apply in the one-sided case, that is, to the unary monoid quasi-variety of *left adequate monoids* and the variety of *left Ehresmann monoids* [2, 13]. In all cases, the image of the unary operation(s) forms a semilattice, named the *semilattice of projections*, also referred to as the *distinguished semilattice*.

Let σ be the least congruence[1] on a (left) Ehresmann monoid M identifying all the projections, so that if M is inverse, σ is the least group congruence.

For an inverse monoid being *E-unitary* is equivalent to being *proper*, that is, $\mathcal{R} \cap \sigma = \iota$ (and also equivalent to $\mathcal{L} \cap \sigma = \iota$). In [16, 17] McAlister showed that proper inverse monoids are ubiquitous in the sense that any inverse monoid M is closely related to a proper inverse monoid known as a *cover*. Additionally, any proper inverse monoid can be constructed from a group G acting by order automorphisms on a partially ordered set X with a subsemilattice Y, being then isomorphic to a *P-semigroup* $\mathcal{P}(G, X, Y)$. A P-semigroup embeds into a semidirect product of a semilattice by a group [20]. If an inverse monoid M is proper, it may be co-ordinatised by its idempotents (a transversal of its $\mathcal{R}$-classes) and the elements of the group M/σ. The free inverse monoid $\mathcal{FIM}(X)$ on a set X is proper [21, 19].

The crucial role of proper (E-unitary) semigroups within the class of inverse semigroups has inspired the search for analogous results for other classes of semigroups. Fountain [5, 6] made the first steps in extending McAlister's results from inverse to adequate semigroups, by developing an analogous theory for a class of left adequate monoids called *left ample* (formerly, *left type A*). A left adequate monoid is left ample if it satisfies the *ample* identity $xy^+ = (xy)^+x$. The free left ample monoid on X, being embeddable (as a unary monoid) into $\mathcal{FIM}(X)$ [7], is proper now in the sense that $\mathcal{R}^* \cap \sigma = \iota$, where σ becomes the least right cancellative monoid congruence. It was much later that corresponding theories were developed in the two-sided case for ample semigroups [8].

The ample condition is natural in the inverse case, and in some generalisations such as (left) ample and, more generally, (left) restriction semigroups. It is intimately related to the co-ordinatisation results by *two* co-ordinates. However, it does not hold in important examples such as the Ehresmann monoid of binary relations. Without it, neither (left) Ehresmann nor (left) adequate monoids behave like their regular analogues and so new approaches are needed. This is the route forged in this paper.

A (left) Ehresmann monoid M, with semilattice of projections Y and submonoid T, generated (as a semigroup[2]) by $T \cup Y$ is said to be *T-generated*.

In [2], Branco, Gomes and Gould initiated a new approach to the study of left Ehresmann and left adequate monoids. They introduced for a T-generated left Ehresmann monoid M the concept of *T-proper* (the analogue of the aforementioned concept of proper does not hold here); proved that any left adequate monoid M has an X^*-proper cover for some set X, and deduced that the free left Ehresmann monoid on any set X is X^*-proper.

[1]The signature can be that of monoids, or of (bi-)unary monoids; the relation is unaffected.

[2]Again, the signature can also be that of a monoid or (bi-)unary monoid, without the concept being affected.

In a subsequent paper [9], Gomes and Gould constructed a T-generated T-proper left Ehresmann monoid with semilattice of projections Y, which we call here $\mathcal{P}_\ell(T,Y)$[3], from a monoid T acting via order-preserving maps on (the left of) a semilattice Y with identity; elements of $\mathcal{P}_\ell(T,Y)$ may be co-ordinatised by T and Y, but now by tuples of arbitrary length. They proved that if T is right cancellative with trivial group of units, then $\mathcal{P}_\ell(T,Y)$ is left adequate; and that the free left Ehresmann monoid on X is of the form $\mathcal{P}_\ell(X^*,Y)$, coinciding therefore with the free left adequate monoid on X. This monoid was also described in a different and very natural way, using labelled trees, by Kambites [13]. The monoid $\mathcal{P}_\ell(T,Y)$ may be characterised as being the unique T-proper T-generated left Ehresmann monoid having *uniqueness of T-normal forms.* We do not concern ourselves here with T-normal forms [2, 9], however we point out that their presence implies that σ separates the elements of T, a property called *strongly T-proper*, which itself implies that of being T-proper. This terminology is related to the one of Jones [11] in the restriction case; and also to that of Kudryavtseva [14]. Jones's work indicates that, even in the very different case of restriction monoids, the notion of strongly T-proper yields non-trivial insights that are not simple extensions of the approach for inverse monoids.

This paper is the second of a pair (the first being [3]) initiating, developing and implementing a theory for *two-sided* Ehresmann and adequate monoids, corresponding to that in the one-sided case. However, it is very far from true that combining the left and right cases is sufficient. In [3] the first three authors introduced the notions of *T-proper* and *strongly T-proper* for T-generated Ehresmann monoids. The main thrust was to construct a strongly T-proper Ehresmann monoid $\mathcal{P}(T,Y)$ from a semilattice Y with identity acted upon *on both sides* by a monoid T via order-preserving maps satisfying the so called *compatibility conditions* for the actions. We may encode the monoid T, the semilattice Y and the actions via a *$\mathcal{P}$-quadruple,* denoted $\mathcal{T}$ (see Definition 1.5) and for emphasis we also denote $\mathcal{P}(T,Y)$ by $\mathcal{P}(\mathcal{T})$. In addition, it is proved in [3] that any Ehresmann monoid M admits a strongly X^*-proper Ehresmann cover and that the free Ehresmann monoid is of the form $\mathcal{P}(X^*,Y)$.

The first major question we deal with here regards the claim of adequacy for $\mathcal{P}(T,Y)$ in the case T is cancellative, a matter missing from [3]. From [9, Theorem 2.2], provided T is right cancellative with trivial group of units, any monoid of the form $\mathcal{P}_\ell(T,Y)$ is left adequate[4] (with a similar claim holding if we remove the condition on the group of units). The adequacy of $\mathcal{P}(T,Y)$ when T is cancellative does not seem to be easy to determine: we answer it positively when T is equidivisible, but the general case remains open. Recall that a monoid T is *equidivisible* if for any $a,b,c,d \in T$, if $ab = cd$ then for some $u \in T$, $a = cu$

[3]In [9] the notation $\mathcal{P}(T,Y)$ was used for the left-handed case; to distinguish it here from the two-sided case, we now refer to the construction of [9] as $\mathcal{P}_\ell(T,Y)$.

[4]Please see footnote[3] for a remark on notation.

and $ub = d$, or $au = c$ and $b = ud$. Groups and free monoids are clearly the first examples.

Theorem 2.10 *Let T be an equidivisible cancellative monoid acting on both sides upon a semilattice Y with identity, satisfying the compatibility conditions. Then $\mathcal{P}(T, Y)$ is adequate.*

As a consequence of this theorem, $\mathcal{P}(X^*, Y)$ is adequate, thus confirming the free Ehresmann monoid on X is in fact the free adequate monoid, a result essentially shown in [12].

The second question is to find an abstract characterisation of Ehresmann monoids of the form $\mathcal{P}(T, Y)$, built from a $\mathcal{P}$-quadruple $\mathcal{T}$. Note that, unlike the one-sided case, $\mathcal{P}(T, Y)$ does not have uniqueness of T-normal forms and so cannot be distinguished by such a property. By suitably marking the copies of T and Y in $\mathcal{P}(T, Y)$ and the embeddings from T and Y to $\mathcal{P}(T, Y)$, we form $\mathcal{Q}(\mathcal{T})$, which is an object in the category $\mathcal{C}(\mathcal{T})$ of what we call $\mathcal{T}$-marked Ehresmann monoids.

Theorem 3.6 *Let $\mathcal{T} = (T, Y, \cdot, \circ)$ be a $\mathcal{P}$-quadruple. Then*

$$\mathcal{Q}(\mathcal{T}) = \big(\mathcal{P}(T, Y), T\nu_{\mathcal{T},T}, Y\nu_{\mathcal{T},Y}, \nu_{\mathcal{T},T}, \nu_{\mathcal{T},Y}\big)$$

is the initial object in the category $\mathcal{C}(\mathcal{T})$.

For ease of reference in Section 1 we recall some basic facts concerning adequate and Ehresmann monoids, and the construction of $\mathcal{P}(T, Y)$. In Section 2, we show that when T is equidivisible and cancellative the monoid $\mathcal{P}(T, Y)$ is adequate. In Section 3, we go back to considering arbitrary monoids T. Fixing T, Y and the actions of T on Y, we prove that the operator $\mathcal{P}$ determines an expansion of Ehresmann monoids (**Theorem 3.4**), and as a consequence of this we are able to characterise $\mathcal{P}(T, Y)$, suitably augmented to $\mathcal{Q}(\mathcal{T})$, as the initial object in an appropriate category. The final Section 4 ends the article by posing some open questions.

1. Preliminaries

In this section we recall some basic definitions and results concerning adequate and Ehresmann monoids. For further details, we refer the reader to [3] and [10].

Let M be a monoid with set of idempotents $E(M)$. For any $a, b \in M$,

$$a \mathcal{R}^* b \Leftrightarrow \forall x, y \in M \, (xa = ya \Leftrightarrow xb = yb)$$

and

$$a \mathcal{L}^* b \Leftrightarrow \forall x, y \in M \, (ax = ay \Leftrightarrow bx = by).$$

Clearly, $\mathcal{R}^*$ is a left congruence and $\mathcal{L}^*$ a right congruence.

Recall that a monoid M is *adequate* if every $\mathcal{R}^*$-class and every $\mathcal{L}^*$-class contains an idempotent and $E(M)$ forms a semilattice. From the commutativity of idempotents it follows that each $\mathcal{R}^*$-class and $\mathcal{L}^*$-class of an element a contains a unique idempotent, denoted by a^+ and a^*, respectively. Thus we have two unary operations on M given by $a \mapsto a^+$ and $a \mapsto a^*$, whence M becomes an algebra with signature $(2,1,1,0)$; as such we refer to it as a *bi-unary monoid*. The class of adequate monoids forms a quasi-variety of algebras in this signature. The defining quasi-identities are those for monoids together with

$$x^+x = x, \quad (x^+y^+)^+ = x^+y^+ = y^+x^+ \quad \text{and} \quad (xy)^+ = (xy^+)^+,$$
$$x^2 = x \to x = x^+ \quad \text{and} \quad xy = zy \to xy^+ = zy^+,$$

and their left-right duals.

An *Ehresmann monoid* is a bi-unary monoid M, in which we again denote the unary operations by $^+$ and *, satisfying the identities for monoids together with

$$x^+x = x, \quad (x^+y^+)^+ = x^+y^+ = y^+x^+ \quad \text{and} \quad (xy)^+ = (xy^+)^+,$$

their left-right duals, and

$$(x^*)^+ = x^* \quad \text{and} \quad (x^+)^* = x^+.$$

The identities $x^+ = x^+x^+$ and $(x^+)^+ = x^+$, and their left-right duals, are a consequence of those above. Putting $E = \{s^+ : s \in M\} = \{s^* : s \in M\}$ we have that E is a semilattice, the *semilattice of projections*. We have already remarked that the variety generated by the quasi-variety of adequate monoids is the variety of Ehresmann monoids. In particular, an adequate monoid M is Ehresmann, with $E(M) = E$.

Another approach to Ehresmann monoids is based on relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$, which themselves contain $\mathcal{R}^*$ and $\mathcal{L}^*$, respectively. We do not pursue this route here, the interested reader may consult [3].

Let M be an Ehresmann monoid with semilattice of projections E and submonoid T. Recall from the Introduction that M is *T-generated* if M is generated (as a semigroup) by $T \cup E$: we denote this by $M = \langle T \cup E \rangle_{(2)}$.

Lemma 1.1. [3, Lemmas 2.2 and 2.3] *Let M be a T-generated Ehresmann monoid with semilattice of projections E. Then T acts on the left and on the right of E by order-preserving maps defining, for $t \in T$ and $e \in E$,*

$$t \cdot e = (te)^+ \quad \text{and} \quad e \circ t = (et)^*.$$

On the other hand, for any $a \in M$ and $e, f \in E$,

$$(eaf)^+ = e(a(eaf)^*)^+ \quad \text{and} \quad (eaf)^* = ((eaf)^+a)^*f.$$

On an Ehresmann monoid M with semilattice of projections E, the relation σ (σ_E for emphasis) is the *semigroup congruence on S generated by $E \times E$*. It is clear that σ is also the bi-unary monoid congruence on the same generators.

A T-generated Ehresmann monoid M with semilattice of projections E is said to be *T-proper* if for any $s, t \in T$ and $e \in E$,

$$[(se)^+ = (te)^+ \text{ and } se \,\sigma\, te] \Rightarrow se = te$$

and, dually,

$$[(es)^* = (et)^* \text{ and } es \,\sigma\, et] \Rightarrow es = et.$$

Further, M is said to be *strongly T-proper* if, for any $s,t \in T$

$$s\ \sigma\ t \Rightarrow s = t.$$

Note that strongly T-proper implies T-proper; the exact relationship between the two conditions is still under investigation.

We recall from [3] the recipe of the first three authors for constructing T-proper Ehresmann monoids from monoids acting on semilattices.

Let T be a monoid with identity 1_T and let Y be a semilattice with identity 1_Y. To avoid any ambiguity we assume that $T \cap Y = \emptyset$. Let $T * Y$ be the free semigroup product of T and Y. We say that $x \in T * Y$ has a *T-beginning* if x begins with a $t \in T$, that is, $x = tz$ for some $z \in T * Y$. Dually, x has a *T-end* if $x = zt$ for some $t \in T$ and $z \in T * Y$. Correspondingly, we say that x has a *Y-beginning* (*Y-end*) if $x = ez$ ($x = ze$) for some $e \in Y$ and $z \in T * Y$.

If, for example, x has a T-beginning and Y-end, we write x as

$$x = t_0 e_1 t_1 e_2 \ldots t_{n-1} e_n$$

where $t_i \in T$ and $e_j \in Y$, $0 \leq i \leq n-1$ and $1 \leq j \leq n$.

Suppose that T acts on the left of Y via order-preserving maps. We denote the action of $t \in T$ on $y \in Y$ by $t \cdot y$. It follows that there exists a monoid morphism

$$\phi_\ell : T \to \mathcal{O}_Y^*,\ (t\phi_\ell)(y) = t \cdot y,$$

where $\mathcal{O}_Y^*$ is the monoid of order-preserving maps of Y with maps composed from right to left. Now, Y acts on the left of itself by order-preserving maps via multiplication, hence there is a monoid morphism, also denoted ϕ_ℓ, given by

$$\phi_\ell : Y \to \mathcal{O}_Y^*,\ (z\phi_\ell)(y) = zy.$$

By the universal property of free products, we obtain a semigroup morphism

$$\phi_\ell : T * Y \to \mathcal{O}_Y^*$$

defined by

$$(s_1 \ldots s_n)\phi_\ell = (s_1\phi_\ell) \ldots (s_n\phi_\ell),$$

where each $s_i \in T \cup Y$. We thus have a semigroup left action of $T * Y$ on Y, which we may without ambiguity denote by $\cdot$, given by

$$s_1 \ldots s_n \cdot y = s_1 \cdot (s_2 \cdot (\ldots (s_n \cdot y) \ldots)).$$

We now define u^+, for $u \in T * Y$, to be

$$u^+ = u \cdot 1_Y.$$

Therefore $e^+ = e$ for all $e \in Y$. We remark that for any $u \in T * Y$, if v is obtained from u via insertion or deletion of elements 1_Y or 1_T, then $u^+ = v^+$. Notice also that $1_T^+ = 1_Y$. The free product $T * Y$ is now a unary semigroup.

Lemma 1.2. [3, Lemma 4.1] *If $u, v \in T * Y$ and $e \in Y$, then $(uv)^+ = u \cdot v^+ = (uv^+)^+$, $(eu)^+ = eu^+$, $(uv)^+ \leq u^+$ and $(uev)^+ \leq (uv)^+$.*

We also suppose we have a right action of T on Y via order-preserving maps. Denoting the right action of $t \in T$ on $y \in Y$ by $y \circ t$, there exists a monoid morphism

$$\phi_r : T \to \mathcal{O}_Y,\ (y)(t\phi_r) = y \circ t,$$

where $\mathcal{O}_Y$ denotes the dual monoid of $\mathcal{O}_Y^*$ (where maps are composed from left to right). Again Y acts on the right on itself by order-preserving maps via multiplication, and we may consider the monoid morphism, also denoted ϕ_r, given by

$$\phi_r : Y \to \mathcal{O}_Y,\ (y)(z\phi_r) = yz.$$

As before, by the universal property of free products, we have a semigroup morphism

$$\phi_r : T * Y \to \mathcal{O}_Y$$

defined by

$$(s_1 \dots s_n)\phi_r = (s_1\phi_r) \dots (s_n\phi_r),$$

where each $s_i \in T \cup Y$. We thus have a semigroup action of $T * Y$ on Y, which we may without ambiguity denote by $\circ$, where

$$y \circ s_1 \dots s_n = ((\dots(y \circ s_1)\dots) \circ s_{n-1}) \circ s_n.$$

We now define u^* (for $u \in T * Y$) to be

$$u^* = 1_Y \circ u,$$

so that $e^* = e$ for all $e \in Y$. As before, we remark that for any $u \in T * Y$, if v is obtained from u via insertion or deletion of elements 1_Y or 1_T, then $u^* = v^*$. Notice also that $1_T^* = 1_Y$ and we have that the free product $T * Y$ has become a bi-unary semigroup.

Lemma 1.3. [3, Lemma 4.2] *If $u, v \in T * Y$ and $e \in Y$, then $(uv)^* = u^* \circ v = (u^*v)^*$, $(ue)^* = u^*e$, $(uv)^* \leq v^*$ and $(uev)^* \leq (uv)^*$.*

From Lemmas 1.2 and 1.3, we observe that for any $u \in T * Y$ and $e \in Y$, we get

$$u \cdot e = (ue)^+ \text{ and } e \circ u = (eu)^*.$$

However, $T * Y$ is not Ehresmann, since for example it does not satisfy the identity $x^+x = x$.

To proceed, we require the actions to satisfy compatibility conditions that we now define.

Definition 1.4. Let T be a monoid acting on both sides upon a semilattice Y. We say that the *compatibility conditions* are satisfied if, for any $t \in T$ and $e, f \in Y$:

(CC1) $e(t \cdot f) = e(t \cdot ((e \circ t)f))$

and

(CC2) $(e \circ t)f = ((e(t \cdot f)) \circ t)f$.

Definition 1.5. A $\mathcal{P}$*-quadruple* is a quadruple $\mathcal{T} = (T, Y, \cdot, \circ)$ where T is a monoid acting by $\cdot$ on the left and $\circ$ on the right of a semilattice Y with identity via order preserving maps satisying the compatibility conditions.

Observe that by Lemma 1.1, given a T-generated Ehresmann monoid M, the monoid T acts on both sides upon E satisfying the compatibility conditions and with these actions $\mathcal{T} = (T, E, \cdot, \circ)$ is a $\mathcal{P}$-quadruple.

Aiming at constructing the Ehresmann monoid $\mathcal{P}(T, Y)$, now let

$$H_\ell = \{(u^+u, u) : u \in T * Y\} \cup \{(1_T, 1_Y)\}$$

and

$$H_r = \{(uu^*, u) : u \in T * Y\} \cup \{(1_T, 1_Y)\}.$$

We use $\sim$ to denote the semigroup congruence on $T * Y$ generated by $H_\ell \cup H_r$. Thus for any $u, v \in T * Y$, we have that $u \sim v$ if and only if $u = v$ or there is a sequence

$$u = z_0, z_1, \ldots, z_n = v$$

where $n \in \mathbb{N}$ and for $0 \leq i \leq n-1$ we have

$$z_i = c_i\alpha_i d_i, z_{i+1} = c_i\beta_i d_i$$

for some $c_i, d_i \in (T * Y)^1$ and $(\alpha_i, \beta_i) \in (H_\ell \cup H_r) \cup (H_\ell \cup H_r)^{-1}$.

If $n = 1$ and $c_1, d_1 \in T * Y$, we say that $u \sim v$ *via a basic step*. The relation $\sim$ is not just a congruence on $T * Y$ but it is also a bi-unary congruence [3, Lemma 4.9].

An element $z \in T * Y$ can take one of four forms, depending on whether z has T- or Y-beginning and T- or Y-end. For convenience, we introduce a new symbol $\Box$ which we regard as an adjoined identity to the monoid T. By writing an element $z \in T * Y$ as $\Box e_1 z_1 \ldots e_n z_n$, where $e_1, \ldots, e_n \in Y$ and $z_1, \ldots, z_n \in T$, we are indicating that $z = e_1 z_1 \ldots e_n z_n$ has Y-beginning, with similar conventions for Y-ends. The $\Box$ symbol serves as a marker to help us control places in products of elements in $T * Y$.

Lemma 1.6. [3, Lemma 4.12] *The map $\tau : T * Y \to T$ given by: $\tau(t) = t$ if $t \in T$, $\tau(y) = 1_T$ if $y \in Y$, and*

$$\tau(u) = t_0 t_1 \ldots t_n,$$

for $u = t_0 e_1 t_1 \ldots e_n t_n$ with $t_0, t_n \in T \cup \{\Box\}$, $t_1, \ldots, t_{n-1} \in T$ and $e_1, e_2, \ldots, e_n \in Y$, is a well-defined monoid morphism with $\sim \subseteq ker\tau$.

Theorem 1.7. [3, Theorem 4.18] *Let $\mathcal{T} = (T, Y, \cdot, \circ)$ be a $\mathcal{P}$-quadruple. The quotient $\mathcal{P}(T, Y) := (T * Y)/\sim$ is an Ehresmann monoid with semilattice of projections*

$$Y' = \{[e] : e \in Y\}$$

where $[u]^+ = [u^+]$ and $[u]^* = [u^*]$ for any $u \in T * Y$, and $1_{\mathcal{P}(T,Y)} = [1_T] = [1_Y]$.

Further, Y' is isomorphic to Y and the submonoid $T' = \{[t] : t \in T\}$ of $\mathcal{P}(T,Y)$ is isomorphic to T under the natural morphism $\nu_T : T * Y \to \mathcal{P}(T,Y)$. The monoid $\mathcal{P}(T,Y)$ is T'-generated, with $\mathcal{P}(T,Y)/\sigma_{Y'} \simeq T'$ and so strongly T'-proper, hence T'-proper.

2. Sufficient conditions for $\mathcal{P}(T,Y)$ to be adequate

The aim of this section is to show that $\mathcal{P}(T,Y)$ is an adequate monoid when T is an equidivisible cancellative monoid.

The class of equidivisible cancellative monoids includes, as mentioned before, all groups and all free monoids. A classical example of an equidivisible cancellative monoid with trivial group of units that is not free can be found in [18, Example 6.2.4]. In fact as proved in [15], a monoid is free if and only if it is graded and equidivisible.

Throughout this section, we assume that T is a cancellative monoid acting on both sides upon a semilattice Y with identity by order-preserving maps, satisfying the compatibility conditions. We will denote the group of units of T by $U(T)$ and the (group) inverse of an element $t \in U(T)$ by t^{-1}.

Lemma 2.1. *Let $u = t_0e_1t_1 \dots e_nt_n \in T * Y$ be such that $\tau(u) = 1_T$, where $t_i \in T$ $(0 \le i \le n)$ and $e_j \in Y$ $(1 \le j \le n)$. Then*

$$(t_ie_{i+1}t_{i+1} \dots e_nt_nu^+)^+ = (t_ie_{i+1}t_{i+1} \dots e_nt_nu)^+ \le (t_it_{i+1} \dots t_nu)^+ \le e_i$$

for any i with $1 \le i \le n$. In particular, $(t_nu^+) = (t_nu)^+ \le e_n$.

Proof. By Lemma 1.2, we get the equalities as well as

$$(t_ie_{i+1}t_{i+1} \dots e_nt_nu)^+ \le (t_it_{i+1} \dots t_nu)^+,$$

for any i with $1 \le i \le n$. Since $\tau(u) = 1_T$, we have that $t_0 \dots t_n = 1_T$. Then, as T is cancellative, $t_0, \dots, t_n \in U(T)$ and for any $i \in \{1, \dots, n\}$, we obtain $(t_0t_1 \dots t_{i-1})^{-1} = t_it_{i+1} \dots t_n = t_{i-1}^{-1} \dots t_1^{-1}t_0^{-1}$. Thus

$$\begin{aligned}
(t_it_{i+1} \dots t_nu)^+ &= (t_{i-1}^{-1} \dots t_1^{-1}e_1t_1 \dots e_nt_n)^+ \\
&\le (t_{i-1}^{-1} \dots t_1^{-1}t_1e_2t_2 \dots e_nt_n)^+ \quad \big(\text{by Lemma 1.2}\big) \\
&= (t_{i-1}^{-1} \dots t_2^{-1}e_2t_2 \dots e_nt_n)^+ \\
&\ \ \vdots \\
&= (e_it_i \dots e_nt_n)^+ \\
&\le e_i \quad \big(\text{by Lemma 1.2}\big),
\end{aligned}$$

as required. □

Lemma 2.2. *If $u \in T * Y$ is such that $\tau(u) = 1_T$, then $u \sim u^+ \sim u^*$.*

Proof. We begin by supposing that u has a T-beginning and a T-end. Then

$$u = t_0 e_1 t_1 e_1 \dots t_{n-1} e_n t_n$$

where $t_i \in T$ and $e_j \in Y$, $0 \leq i \leq n$ and $1 \leq j \leq n$. Since $\tau(u) = 1_T$, we have that $u^+ = 1_Y u^+ \sim 1_T u^+ = t_0 t_1 \dots t_n u^+$, and so

$$\begin{aligned}
u^+ &\sim t_0 t_1 \dots t_n u^+ \\
&\sim t_0 t_1 \dots t_{n-1} (t_n u^+)^+ t_n u^+ \\
&= t_0 t_1 \dots t_{n-1} e_n (t_n u^+)^+ t_n u^+ \quad \big(\text{by Lemma 2.1}\big) \\
&\sim t_0 t_1 \dots t_{n-1} e_n t_n u^+ \\
&\vdots \\
&\sim t_0 t_1 \dots t_i e_{i+1} t_{i+1} \dots e_n t_n u^+ \\
&\sim (t_0 t_1 \dots t_{i-1})(t_i e_{i+1} t_{i+1} \dots e_n t_n u^+)^+ t_i e_{i+1} t_{i+1} \dots e_n t_n u^+ \\
&= (t_0 t_1 \dots t_{i-1}) e_i (t_i e_{i+1} t_{i+1} \dots e_n t_n u^+)^+ t_i e_{i+1} t_{i+1} \dots e_n t_n u^+ \quad \big(\text{by Lemma 2.1}\big) \\
&\sim (t_0 t_1 \dots t_{i-1}) e_i t_i e_{i+1} t_{i+1} \dots e_n t_n u^+ \\
&\vdots \\
&\sim t_0 e_1 t_1 \dots e_n t_n u^+ \\
&= u u^+.
\end{aligned}$$

Dually, we can show that $u^* \sim u^* u$. As Y is a semilattice, it follows that

$$u^+ u^* \sim u u^+ u^* = u u^* u^+ \sim u u^+ \sim u^+,$$

and similarly, $u^+ u^* \sim u^*$. We now deduce that $u^* \sim u^+$ and finally

$$u^+ \sim u u^+ \sim u u^* \sim u.$$

Now, suppose that u has a Y-beginning. Then $u = 1_Y u$, so that $u \sim 1_T u$ and $1_T u$ has a T-beginning. Similarly, if u has a Y-end, we get $u \sim u 1_T$ and $u 1_T$ has a T-end. Notice that $\tau(1_T u) = \tau(u 1_T) = \tau(1_T u 1_T)$. Thus if u has either a Y-beginning or a Y-end, $u \sim v$ for some v with T-beginning and T-end such that $\tau(v) = \tau(u)$. So $v \sim v^+ \sim v^*$ by the previous case. From $u \sim v$, we get $u^+ \sim v^+$ and $u^* \sim v^*$, since $\sim$ is a bi-unary congruence, and so

$$u \sim v \sim v^+ \sim u^+$$

and

$$u \sim v \sim v^* \sim u^*$$

as required. ∎

We now locate the full set of idempotents of $\mathcal{P}(T, Y)$ in the current case.

Proposition 2.3. *We have* $E(\mathcal{P}(T, Y)) = Y'$.

Proof. We only need to show that $E(\mathcal{P}(T,Y)) \subseteq Y'$. If $[x]^2 = [x]$, then $x^2 \sim x$. It follows from Lemma 1.6 that $\tau(x)^2 = \tau(x)$, which implies that $\tau(x) = 1_T$ since T is cancellative. So by Lemma 2.2 we obtain that $x \sim x^+$. Thus $[x] = [x^+] = [x]^+ \in Y'$. □

Lemma 2.4. *Let $h \in U(T)$. Then for any $u \in T * Y$,*

$$hu^+h^{-1} \sim (hu)^+.$$

If in addition $\tau(u) = h^{-1}$, then

$$hu^+h^{-1} \sim hu.$$

Proof. Clearly $\tau(hu^+h^{-1}) = 1_T$. Applying Lemmas 1.2 and 2.2 we get

$$hu^+h^{-1} \sim (hu^+)^+hu^+h^{-1} = (hu)^+hu^+h^{-1} \sim (hu)^+(hu^+h^{-1})^+ = (hu^+h^{-1})^+(hu)^+$$
$$= ((hu^+h^{-1})^+hu)^+ \sim (hu^+h^{-1}hu)^+ = (hu^+1_Tu)^+ \sim (hu^+u)^+ \sim (hu)^+.$$

If $\tau(u) = h^{-1}$, then $\tau(hu) = 1_T$ and $(hu)^+ \sim hu$ by Lemma 2.2. □

To proceed to our main results, we need to consider factorisations in $T * Y$.

The next lemma, whose proof is clear, tells us that a factorisation wea, we or ea, where $w = w_0h_1w_1 \ldots h_pw_p$ has T-end and fixed "length" p (meaning that consecutive symbols in the expression $w = w_0h_1w_1 \ldots h_pw_p$ do not belong both to T or to Y), a has T-beginning and $e \in Y$, is unique; and dually.

Lemma 2.5. *Let $w, v, a, b \in T * Y$ be such that*

$$w = w_0h_1w_1 \ldots h_pw_p,\ v = v_0g_1v_1 \ldots g_pv_p,$$

and

$$a = a_0e_1a_1 \ldots e_na_n,\ b = b_0f_1b_1 \ldots f_nb_n,$$

where $w_0, v_0 \in T \cup \{\Box\}$, $w_i, v_i \in T$ and $h_i, g_i \in Y$ for $1 \leq i \leq p$, and $a_{k-1}, b_{k-1} \in T$ and $e_k, f_k \in Y$ for $1 \leq k \leq n$, and $a_n, b_n \in T \cup \{\Box\}$. Then for any $e, f \in Y$, if $wea = vfb$, then $w = v, e = f$ and $a = b$. Similar claims follow for equalities of the form $we = vf$ and $ea = fb$.

The analogue of the above is true if w, v have Y-ends, a, b have Y-beginnings and $wsa = vtb$, $ws = vt$ or $sa = tb$, for some $s, t \in T$.

We require a series of technical results on factorisations of the elements of $T*Y$ in case T is equidivisible, starting with a folklore result on the general case.

Lemma 2.6. *Let U and S be semigroups. If $w, a, v, b \in (U * S)^1$ with $wa = vb \neq 1$, then one of the following cases holds:*

(I) $w = v$ and $a = b$;
(II) there exists $u \in U * S$ such that $w = vu$ and $ua = b$;
(III) there exists $u \in U * S$ such that $v = wu$ and $ub = a$;
(IV) there exist $w', a' \in (U * S)^1$ and $e, f, g, h \in U$ or $e, f, g, h \in S$ such that $w = w'e$, $fa' = a$, $v = w'g$, $ha' = b$ and $ef = gh$.

Case (IV) may be refined to our case for T and Y.

Lemma 2.7. *Let T be equidivisible. If $w, a, v, b \in T * Y$ with $wa = vb$, then one of the following cases holds:*

(I) *$w = v$ and $a = b$;*
(II) *there exists $u \in T * Y$ such that $w = vu$ and $ua = b$;*
(III) *there exists $u \in T * Y$ such that $v = wu$ and $ub = a$;*
(IV) *there exist $w', a' \in T * Y$ and $e, f, g, h \in Y$ such that $w = w'e$, $fa' = a$, $v = w'g$, $ha' = b$ and $ef = gh$.*

Proof. In view of the previous lemma, we only need to analyse the case when $w', a' \in (T * Y)^1$ and e, f, g, h lie in T or in Y such that $w = w'e$, $fa' = a$, $v = w'g$, $ha' = b$ and $ef = gh$. Since T and Y are monoids we can assume $w', a' \in T * Y$. If $e, f, g, h \in Y$ we have the new case (IV). If $e, f, g, h \in T$ then as T is equidivisible we have three possibilities to discuss. If $e = g$, then $f = h$ and then we are in case (I). If $e = gu$ and $uf = h$ for some $u \in T$, then $w = w'gu = vu$ and $b = ufa' = ua$ and we are in case (II). Finally, if $g = eu$ and $uh = f$ for some $u \in T$, then $v = w'eu = wu$ and $a = uha' = ub$, the case (III). □

In the proof of the next lemma we will be using very often Lemmas 1.2 and 1.3 as well as the definition of $\sim$ without specific mention. The proof also requires Lemma 2.7 at various instances.

Lemma 2.8. *Let T be equidivisible and let $x, a, z \in T * Y$ such that $xa \sim z$ via a basic step. Then there exist $h \in U(T)$ and $y, b \in T * Y$ such that*

$$z = yb, \quad \tau(x) = \tau(y)h, \quad h^{-1}\tau(b) = \tau(a) \quad \text{and} \quad xa^+ \sim yb^+h.$$

Proof. Since $xa \sim z$ via a basic step, there exist $c, d \in T * Y$ and $(\alpha, \beta) \in (H_\ell \cup H_r) \cup (H_\ell \cup H_r)^{-1}$ such that

$$xa = c\alpha d \quad \text{and} \quad c\beta d = z.$$

Thus $\alpha \sim \beta$ and, by Lemma 1.6, we have $\tau(\alpha) = \tau(\beta)$.

According to Lemma 2.7, the equality $xa = (c\alpha)d$ results in one of the factorisations (I), (II), (III), (IV). We discuss each in turn. Note that we only explicitly mention h in one sub-case, in the others $h = 1_T$.

(I) *$x = c\alpha$ and $a = d$.* Let $y = c\beta$ and $b = a$. Then $z = yb$. Now, $\tau(\alpha) = \tau(\beta)$, $x = c\alpha$ and $y = c\beta$ together imply $\tau(x) = \tau(y)$. Clearly $\tau(a) = \tau(b)$. Notice that $x = c\alpha \sim c\beta = y$ and so $xa^+ \sim ya^+ = yb^+$.

(II) *$x = (c\alpha)s$ and $sa = d$, for some $s \in T * Y$.* Let $y = c\beta s$ and $b = a$. Then $\tau(x) = \tau(y)$, $\tau(a) = \tau(b)$ and also $z = c\beta d = c\beta sa = yb$. As $x = c\alpha s \sim c\beta s = y$, we obtain $xa^+ \sim yb^+$.

(III) *$c\alpha = xs$ and $sd = a$, for some $s \in T * Y$.* We call upon Lemma 2.7 again to discuss the four possibilities for factorising $c\alpha = xs$.

(III.1) *$c = x$ and $\alpha = s$.* Put $y = c$ and $b = \beta d$. Then $z = yb$ and $\tau(x) = \tau(y)$. From $a = sd = \alpha d \sim \beta d = b$, we get $\tau(a) = \tau(b)$ and $a^+ \sim b^+$. We then obtain that $xa^+ \sim yb^+$.

(III.2) *$c = xl$ and $l\alpha = s$ with $l \in T * Y$.* Put $y = x$, $b = l\beta d$. Then $z = c\beta d = xl\beta d = yb$ and $\tau(x) = \tau(y)$. Also $a = sd = l\alpha d \sim l\beta d = b$, whence $\tau(a) = \tau(b)$ and $a^+ \sim b^+$. Thus $xa^+ \sim yb^+$.

(III.3) *$x = ct$ and $ts = \alpha$, for some $t \in T * Y$.* The fact that $(\alpha, \beta) \in (H_\ell \cup H_r) \cup (H_\ell \cup H_r)^{-1}$ leads the discussion to the following six cases:

(i) $\beta = \alpha^+\alpha$. Put $y = c\alpha^+ t$ and $b = a$. Note that $\alpha d = tsd = ta$. Then $z = c\beta d = c\alpha^+\alpha d = c\alpha^+ ta = yb$, $\tau(a) = \tau(b)$ and $\tau(y) = \tau(c\alpha^+ t) = \tau(ct) = \tau(x)$. Moreover

$$\begin{aligned} yb^+ &= c\alpha^+ tb^+ = c\alpha^+ ta^+ \sim c\alpha^+(ta)^+ ta^+ \\ &= c\alpha^+(\alpha d)^+ ta^+ = c(\alpha d)^+ ta^+ = c(ta)^+ ta^+ \sim cta^+ = xa^+. \end{aligned}$$

(ii) $\beta = \alpha\alpha^*$. Put $y = x$ and $b = s\alpha^* d$. Then $z = c\beta d = c\alpha\alpha^* d = cts\alpha^* d = xs\alpha^* d = yb$, $\tau(x) = \tau(y)$ and $\tau(b) = \tau(s\alpha^* d) = \tau(sd) = \tau(a)$. In addition,

$$\begin{aligned} yb^+ &= x(s\alpha^* d)^+ = ct(s\alpha^* d)^+ \sim ctt^*(s\alpha^* d)^+ = ct(t^* s\alpha^* d)^+ = ct(t^* s(ts)^* d)^+ \\ &= ct(t^* s(t^* s)^* d)^+ \sim ct(t^* sd)^+ = ctt^*(sd)^+ \sim ct(sd)^+ = xa^+. \end{aligned}$$

(iii) $\alpha = \beta^+\beta$. Then $ts = \alpha = \beta^+\beta$. In the following, we further use Lemma 2.7 to discuss $ts = \beta^+\beta$ in four cases.

(iii.1) $t = \beta^+$ and $s = \beta$. Put $y = c$ and $b = a$. Then $x = c\beta^+$ and $\beta d = a$. We have $z = c\beta d = yb$, $\tau(y) = \tau(c) = \tau(c\beta^+) = \tau(x)$ and $\tau(a) = \tau(b)$. In addition, we have $yb^+ = ya^+ = c(\beta d)^+ \sim c(\beta^+\beta d)^+ = c\beta^+(\beta d)^+ = xa^+$.

(iii.2) $\beta^+ = tu$ and $u\beta = s$, for some $u \in T * Y$. In this case, we then must have $t, u \in Y$. Put $y = c$ and $b = \beta d$. Then $z = c\beta d = yb$, $\tau(y) = \tau(c) = \tau(ct) = \tau(x)$ and $\tau(b) = \tau(\beta d) = \tau(u\beta d) = \tau(sd) = \tau(a)$. Now

$$yb^+ = c(\beta d)^+ \sim c\beta^+(\beta d)^+ = ctu(\beta d)^+ = ct(u\beta d)^+ = ct(sd)^+ = cta^+ = xa^+.$$

(iii.3) $t = \beta^+ u$ and $us = \beta$, for some $u \in T * Y$. Put $y = cu$ and $b = a$. Then $z = c\beta d = cusd = cua = yb$, $\tau(y) = \tau(cu) = \tau(c\beta^+ u) = \tau(ct) = \tau(x)$ and $\tau(a) = \tau(b)$. Applying Lemma 1.2 again, we obtain

$$yb^+ = cua^+ = cu(sd)^+ = cus^+(sd)^+ \sim c(us^+)^+ us^+(sd)^+$$

$$= c(us)^+ u(sd)^+ = c\beta^+ u(sd)^+ = cta^+ = xa^+.$$

(iii.4) $t = lg$, $pr = s$, $\beta^+ = le$ and $fr = \beta$, where $l, r \in T * Y$, $e, f, g, p \in Y$, and $gp = ef$. Note that $l \in Y$, and thus $t \in Y$ too. Put $y = c$ and $b = \beta d$. Then $z = c\beta d = yb$, $\tau(y) = \tau(c) = \tau(ct) = \tau(x)$ and $\tau(b) = \tau(\beta d) = \tau(frd) = \tau(rd) = \tau(prd) = \tau(sd) = \tau(a)$. We also have that

$$yb^+ = c(\beta d)^+ \sim c(\beta^+\beta d)^+ = c(tsd)^+ = ct(sd)^+ = xa^+.$$

(iv) $\alpha = \beta\beta^*$. Recall that we have $x = ct$, $ts = \alpha = \beta\beta^*$ and $sd = a$.

In the following, once again we use Lemma 2.7 this time to discuss $ts = \beta\beta^*$ in four cases.

(iv.1) $t=\beta$ and $s=\beta^*$. Put $y=c\beta$ and $b=d$. Then $z=c\beta d=yb$. As $y=c\beta=ct=x$, we have $\tau(y)=\tau(x)$. We also have that $\tau(b)=\tau(d)=\tau(\beta^*d)=\tau(sd)=\tau(a)$. In addition, $yb^+=c\beta d^+\sim c\beta\beta^*d^+=c\beta(\beta^*d)^+=ct(sd)^+=xa^+$.

(iv.2) $\beta=tu$ and $u\beta^*=s$, for some $u\in T*Y$. Put $y=x$ and $b=ud$. Then $z=c\beta d=ctud=xb=yb$, $\tau(a)=\tau(sd)=\tau(u\beta^*d)=\tau(ud)=\tau(b)$ and clearly $\tau(x)=\tau(y)$. Now

$$\begin{aligned} yb^+=x(ud)^+=ct(ud)^+ &\sim ctt^*(ud)^+\sim ct(t^*ud)^+\sim ct(t^*u(tu)^*d)^+\\ &=ct(t^*u\beta^*d)^+=ctt^*(u\beta^*d)^+\sim ct(sd)^+=xa^+. \end{aligned}$$

(iv.3) $t=\beta u$ and $us=\beta^*$, for some $u\in T*Y$. In this case we necessarily have $u,s\in Y$. Put $y=c\beta$ and $b=d$. Thus $z=c\beta d=yb$. Also, we have $\tau(y)=\tau(c\beta)=\tau(c\beta u)=\tau(ct)=\tau(x)$ and $\tau(b)=\tau(d)=\tau(sd)=\tau(a)$. Now

$$yb^+=c\beta d^+\sim c\beta\beta^*d^+=c\beta usd^+=ct(sd)^+=xa^+.$$

(iv.4) $t=lg$, $pr=s$, $\beta=le$ and $fr=\beta^*$, where $l,r\in T*Y$ and $e,f,g,p\in Y$ with $gp=ef$. We then must have $r\in Y$ and thus $s\in Y$ too. Put $y=c\beta$ and $b=d$. Then $z=c\beta d=yb$. Also, $\tau(y)=\tau(c\beta)=\tau(cle)=\tau(cl)=\tau(clg)=\tau(ct)=\tau(x)$ and $\tau(b)=\tau(d)=\tau(sd)=\tau(a)$. In addition, we have

$$\begin{aligned} yb^+=c\beta d^+ &\sim c\beta\beta^*d^+=clefrd^+=clgprd^+=clgsd^+\\ &=clgsd^+=ctsd^+=ct(sd)^+=xa^+. \end{aligned}$$

(v) $(\alpha,\beta)=(1_Y,1_T)$. Then $t=s=1_Y$ as $ts=\alpha$. Put $y=c1_T$ and $b=d$. Then $z=c\beta d=c1_Td=yb$, $\tau(y)=\tau(c1_T)=\tau(c)=\tau(c1_Y)=\tau(ct)=\tau(x)$ and $\tau(a)=\tau(sd)=\tau(1_Yd)=\tau(d)=\tau(b)$. Also,

$$yb^+=c1_Td^+\sim c1_Yd^+=c1_Y1_Yd^+=c1_Y(1_Yd)^+=ct(sd)^+=xa^+.$$

(vi) $(\alpha,\beta)=(1_T,1_Y)$. Notice that this is the only situation where we may have $h\neq 1_T$. From $ts=\alpha=1_T$ we must have that t,s in T and are mutually inverse. Put $y=c1_Y$ and $b=d$. Then $z=c\beta d=c1_Yd=yb$. Also, $\tau(x)=\tau(ct)=\tau(c)\tau(t)=\tau(c1_Y)\tau(t)=\tau(y)t$ and $\tau(a)=\tau(sd)=s\tau(d)=s\tau(b)=t^{-1}\tau(b)$, where $ts=1_T$. In addition,

$$\begin{aligned} xa^+=ct(sd)^+ &\sim ct(sd)^+1_T=ct(sd)^+t^t\sim c(tsd)^+t \quad \text{(by Lemma 2.4)}\\ &=c(1_Td)^+t\sim c(1_Yd)^+t\sim c1_Yd^+t=yb^+t. \end{aligned}$$

(III.4) *$c=x'g$, $ps'=\alpha$, $x=x'e$ and $fs'=s$, where $x',s'\in T*Y$, $e,f,g,p\in Y$ and $gp=ef$.* Put $y=c$ and $b=\beta d$. Then $z=c\beta d=yb$, $\tau(y)=\tau(c)=\tau(x'g)=\tau(x')=\tau(x'e)=\tau(x)$ and $\tau(b)=\tau(\beta d)=\tau(\alpha d)=\tau(ps'd)=\tau(s'd)=\tau(fs'd)=\tau(sd)=\tau(a)$. In addition,

$$\begin{aligned} yb^+=c(\beta d)^+ &\sim c(\alpha d)^+=x'g(ps'd)^+=x'gp(s'd)^+\\ &=x'ef(s'd)^+=x'e(fs'd)^+=x(sd)^+=xa^+. \end{aligned}$$

(IV) *$x=x'e$, $fa'=a$, $c\alpha=x'g$, $\ell a'=d$ for some $x',a'\in T*Y$ and $e,f,g,\ell\in Y$ with $ef=g\ell$.* Let $y=c\beta$ and $b=d$. Then $z=c\beta d=yb$, and also we have

$\tau(x) = \tau(x'e) = \tau(x'g) = \tau(c\alpha) = \tau(c\beta) = \tau(y)$ and $\tau(b) = \tau(d) = \tau(\ell a') = \tau(fa') = \tau(a)$. Using Lemma 1.2, we see that $yb^+ = c\beta d^+ \sim c\alpha d^+ = x'g(\ell a')^+ = x'g\ell(a')^+ = x'ef(a')^+ = x'e(fa')^+ = xa^+$. □

Lemma 2.9. *Let T be equidivisible. Let $h, k \in T$ and $x, a, y, b \in T * Y$ with $xa = yb$, $\tau(x) = \tau(y)h$ and $k\tau(b) = \tau(a)$. Then $hk = 1_T$ and $xa^+ \sim yb^+h$.*

Proof. For the first claim, notice that $\tau(y)\tau(b) = \tau(yb) = \tau(xa) = \tau(x)\tau(a) = \tau(y)hk\tau(b)$, so that as T is cancellative $hk = 1_T$.

According to Lemma 2.7 it is necessary to discuss $xa = yb$ in four cases.

(I) When $x = y$ and $a = b$, then $h = k = 1_T$ and the result is obvious.

(II) Assume that $x = yu$ and $ua = b$, for some $u \in T*Y$. Then $\tau(x) = \tau(yu) = \tau(y)\tau(u)$ together with $\tau(x) = \tau(y)h$ give $\tau(u) = h = k^{-1}$, as T is cancellative. Hence $\tau(ku) = \tau(k)\tau(u) = 1_T$. Now we deduce that

$$\begin{aligned}
yb^+h &= y(ua)^+h \sim y1_T(ua)^+h = yhk(ua)^+h \\
&\sim yh(kua)^+ \quad \big(\text{by Lemma 2.4}\big) \\
&\sim yh((ku)^+a)^+ \quad \big(\text{by Lemma 2.2, as } \tau(ku) = 1_T\big) \\
&= yh(ku)^+a^+ \quad \big(\text{by Lemma 1.2}\big) \\
&\sim yhkua^+ \quad \big(\text{by Lemma 2.2 as } \tau(hu) = 1_T\big) \\
&\sim yua^+ \\
&= xa^+.
\end{aligned}$$

(III) This case is dual to (II) and follows in a similar way.

(IV) To conclude, suppose that $x = w'e$, $fs' = a$, $y = w'g$, $ps' = b$, where $w', s' \in T * Y$, $e, f, g, p \in Y$ and $ef = gp$. Then $\tau(x) = \tau(w'e) = \tau(w') = \tau(w'g) = \tau(y)$ and as $\tau(x) = \tau(y)h$ we have $h = 1_T$. Now,

$$\begin{aligned}
xa^+ &= w'ea^+ = w'e(fs')^+ \\
&= w'ef(s')^+ \quad \big(\text{by Lemma 1.2}\big) \\
&= w'gp(s')^+ \\
&= y(ps')^+ \quad \big(\text{by Lemma 1.2}\big) \\
&= yb^+.
\end{aligned}$$

The result follows. □

We now present the main result of this article.

Theorem 2.10. *Let T be an equidivisible cancellative monoid acting on both sides upon a semilattice Y with identity, satisfying the compatibility conditions. Then $\mathcal{P}(T, Y)$ is adequate.*

Proof. The monoid $\mathcal{P}(T, Y)$ is Ehresmann with semilattice of projections Y' and, from Proposition 2.3, we know that $E(\mathcal{P}(T, Y)) = Y'$.

To show that $\mathcal{P}(T,Y)$ is adequate it remains to prove for any $[x],[y],[a] \in \mathcal{P}(T,Y)$, $[x][a]=[y][a]$ implies that $[x][a]^+=[y][a]^+$, so that $[a]\,\mathcal{R}^*\,[a]^+$. If so, the dual argument will give that $[a]\,\mathcal{L}^*\,[a]^*$.

Suppose now that $[x],[y],[a] \in \mathcal{P}(T,Y)$ and $[x][a]=[y][a]$, whence $xa \sim ya$. Thus $xa=ya$ or there exists a sequence

$$xa = z_0' \sim z_1' \sim \ldots \sim z_n' = ya$$

where $(z_i', z_{i+1}') = (c_i'\alpha_i d_i', c_i'\beta_i d_i')$ for some $(\alpha_i,\beta_i) \in (H_\ell \cup H_r) \cup (H_\ell \cup H_r)^{-1}$ and $c_i', d_i' \in (T*Y)^1$.

If $xa=ya$ then as T is cancellative, certainly $\tau(x)=\tau(y)$, and so by Lemma 2.9 (with $k=h=1_T$), we have that $xa^+ \sim ya^+1_T \sim ya^+$, giving $[x][a]^+=[y][a]^+$.

Next, assume that we have a sequence as given above. For convenience, let S and W be either T or Y. We multiply each term of the sequence on the left by 1_S, where x has an S-beginning and on the right by 1_W, where a has a W-end. Let $z_i = 1_S z_i' 1_W, c_i = 1_S c_i'$ and $d_i = d_i' 1_W$ for $1 \leq i \leq n$. Now, $c_i, d_i \in T*Y$. We thus have a sequence

$$xa = z_0 \sim z_1 \sim \ldots \sim z_n = (1_S y)a,$$

every step of which is basic. It follows from Lemma 1.6 that $\tau(x)\tau(a)=\tau(y)\tau(a)$ which implies that $\tau(x)=\tau(y)$. For convenience, put $x=y_0$ and $a=b_0$. By Lemma 2.8, for $1 \leq i \leq n$ there exist $y_i, b_i \in T*Y$ and $h_i \in U(T)$ such that $z_i = y_i b_i$, $\tau(y_{i-1}) = \tau(y_i)h_i$, $h_i^{-1}\tau(b_i) = \tau(b_{i-1})$ and $y_{i-1}b_{i-1}^+ \sim y_i b_i^+ h_i$. Hence

$$xa^+ = y_0 b_0^+ \sim y_1 b_1^+ h_1 \sim y_2 b_2^+ h_2 h_1 \sim \ldots \sim y_n b_n^+ h_n h_{n-1} \ldots h_1$$

and so by Lemma 1.6, we obtain that $\tau(x) = \tau(xa^+) = \tau(y_n)h$ where $h = h_n h_{n-1} \ldots h_1 \in U(T)$.

From $\tau(x)=\tau(1_S y)$, we get $\tau(1_S y) = \tau(y_n)h$. As we have $(1_S y)a = z_n = y_n b_n$ it then follows that $h^{-1}\tau(b_n) = \tau(a)$, and now Lemma 2.9 yields $(1_S y)a^+ \sim y_n b_n^+ h$. Hence $xa^+ \sim y_n b_n^+ h \sim ya^+$ and so $[x][a]^+ = [y][a]^+$ as required. □

From [3, Theorem 6.1] and the remarks preceding it, which together tell us that the free Ehresmann monoid on a set X is of the form $\mathcal{P}(X^*,Y)$, we immediately deduce the following corollary. Note that this result is also mentioned in the Section 6 (Remarks) of [12].

Corollary 2.11. *The free Ehresmann monoid on a set X is adequate, and hence coincides with the free adequate monoid on X.*

Corollary 2.12. *The quasi-variety of adequate monoids generates the variety of Ehresmann monoids.*

3. Characterisation of $\mathcal{P}(T,Y)$

In this section we return to the consideration of $\mathcal{P}$-quadruples $\mathcal{T} = (T,Y,\cdot,\circ)$ for an *arbitrary* monoid T. We show that the Ehresmann monoid $\mathcal{P}(\mathcal{T}) = \mathcal{P}(T,Y)$ is unique, in the sense that it is exactly the initial object in a particular category, and do so after proving that the operator $\mathcal{P}$ defines an expansion of a suitable category of Ehresmann monoids.

We start by recalling the definition of an expansion, here in the case of bi-unary monoids.

Definition 3.1. (*cf*. [1]) An *expansion* of a category $\mathcal{C}$ of bi-unary monoids with $(2,1,1,0)$-morphisms is a "functorial cover", i.e. a functor $\mathcal{E}$ from $\mathcal{C}$ to itself along with a natural transformation π from $\mathcal{E}$ to the identity functor of $\mathcal{C}$ such that, for each object M of $\mathcal{C}$, the morphism π_M is onto. Thus, for any objects M_1 and M_2 of $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}(M_1) & \xrightarrow{\mathcal{E}(\psi)} & \mathcal{E}(M_2) \\ \pi_{M_1}\big\downarrow & \circlearrowleft & \big\downarrow\pi_{M_2} \\ M_1 & \xrightarrow[\psi]{} & M_2 \end{array}$$

Let $\mathcal{T} = (T, Y, \cdot, \circ)$ be a $\mathcal{P}$-quadruple. By Theorem 1.7, we may construct the Ehresmann monoid $\mathcal{P}(\mathcal{T}) := \mathcal{P}(T, Y)$ associated with $\mathcal{T}$. We have that $\mathcal{P}(\mathcal{T})$ is T'-generated with semilattice of projections Y', and $\nu_{\mathcal{T},T} = \nu_{\mathcal{T}}|_T : T \to T'$ and $\nu_{\mathcal{T},Y} = \nu_{\mathcal{T}}|_Y : Y \to Y'$ are isomorphisms, where $T' = \{[t] : t \in T\}$ and $Y' = \{[y] : y \in Y\}$. Further, Lemmas 1.2 and 1.3 give that for any $t \in T$ and $y \in Y$,

$$(t \cdot y)\nu_{\mathcal{T},Y} = t\nu_{\mathcal{T},T} \cdot y\nu_{\mathcal{T},Y} \quad \text{and} \quad (y \circ t)\nu_{\mathcal{T},Y} = y\nu_{\mathcal{T},Y} \circ t\nu_{\mathcal{T},T}.$$

Lemma 3.2. *Let $\mathcal{T}_1 = (T_1, Y_1, \cdot, \circ)$ and $\mathcal{T}_2 = (T_2, Y_2, \cdot, \circ)$ be $\mathcal{P}$-quadruples, and suppose that $\psi_T : T_1 \to T_2$ and $\psi_Y : Y_1 \to Y_2$ are monoid morphisms such that for all $t \in T_1$ and $y \in Y_1$ we have*

$$(t \cdot y)\psi_Y = t\psi_T \cdot y\psi_Y \quad \text{and} \quad (y \circ t)\psi_Y = y\psi_Y \circ t\psi_T.$$

Let $\psi : T_1 * Y_1 \to T_2 * Y_2$ be the (semigroup) morphism that extends ψ_T and ψ_Y. Then ψ is a $(2,1,1)$-morphism, and

$$\mathcal{P}_\psi : \mathcal{P}(\mathcal{T}_1) \to \mathcal{P}(\mathcal{T}_2)$$

given by

$$[u]\mathcal{P}_\psi = [u\psi]$$

is a $(2,1,1,0)$-morphism. If ψ_T and ψ_Y are both onto, then so is $\mathcal{P}_\psi$.

Proof. To show that $\mathcal{P}_\psi$ is well-defined we are required to prove that the congruence $\sim$ giving $\mathcal{P}(\mathcal{T}_1)$ is such that $\sim \subseteq \ker(\psi\nu_{\mathcal{T}_2})$. To do so it is sufficient to see that the generating set $H_\ell \cup H_r$ of $\sim$ lies in $\ker(\psi\nu_{\mathcal{T}_2})$.

First we prove that, for all $u \in T_1 * Y_1$ and $y \in Y_1$,

$$(u \cdot y)\psi_Y = u\psi \cdot y\psi_Y \quad \text{and} \quad (y \circ u)\psi_Y = y\psi_Y \circ u\psi. \tag{3.1}$$

By hypothesis this holds for $u \in T_1$ and if $u \in Y_1$, we have

$$(u \cdot y)\psi_Y = (uy)\psi_Y = (u\psi_Y)(y\psi_Y) = u\psi_Y \cdot y\psi_Y = u\psi \cdot y\psi_Y.$$

Then, by induction on the minimal number of generators from $T_1 \cup Y_1$ of $u \in T_1 * Y_1$, we obtain one half of (3.1) for all $u \in T_1 * Y_1$ and $y \in Y_1$; the other equality may be shown similarly.

Now, for $u \in T_1 * Y_1$, we get

$$u^+\psi = (u \cdot 1_{Y_1})\psi = (u \cdot 1_{Y_1})\psi_Y = u\psi \cdot 1_{Y_1}\psi_Y = u\psi \cdot 1_{Y_2} = (u\psi)^+.$$

Similarly, $u^*\psi = (u\psi)^*$, thus verifying that ψ is a $(2,1,1)$-morphism.

It follows that

$$(u^+u)\psi = (u^+\psi)(u\psi) = (u\psi)^+(u\psi) \sim u\psi.$$

Analogously, $(uu^*)\psi \sim u\psi$. Also $1_{T_1}\psi = 1_{T_1}\psi_T = 1_{T_2} \sim 1_{Y_2} = 1_{Y_1}\psi_Y = 1_{Y_1}\psi$.

Therefore $\sim$ is contained in $\ker(\psi\nu_{\mathcal{T}_2})$, and so $\mathcal{P}_\psi$ is indeed well-defined and clearly is then a semigroup morphism. Given $u \in T_1 * Y_1$,

$$[u]^+\mathcal{P}_\psi = [u^+]\mathcal{P}_\psi = [u^+\psi] = [(u\psi)^+] = [u\psi]^+ = ([u]\mathcal{P}_\psi)^+.$$

In a similar way, $\mathcal{P}_\psi$ respects $*$. As $[1_{T_1}]\mathcal{P}_\psi = [1_{T_1}\psi] = [1_{T_2}]$, we have that $\mathcal{P}_\psi$ is a $(2,1,1,0)$-morphism.

If both ψ_T and ψ_Y are onto, then so is ψ and hence also $\mathcal{P}_\psi$. □

Now let $\mathcal{C}$ be the category whose objects are triples (M,T,Y), where M is a T-generated Ehresmann monoid with semilattice of projections Y (the objects are over-defined as given M we know Y, however it is useful to mention Y explicitly); by a morphism $\varphi : (M_1,T_1,Y_1) \to (M_1,T_1,Y_2)$ of $\mathcal{C}$ we mean that $\varphi : M_1 \to M_2$ is a $(2,1,1,0)$-morphism such that $T_1\varphi = T_2$ and $Y_1\varphi = Y_2$. The composition in $\mathcal{C}$ is the usual composition of maps.

Definition 3.3. The category $\mathcal{C}$ defined above is called the category of *marked* Ehresmann monoids.

Let (M,T,Y) be an object of $\mathcal{C}$. As seen in Lemma 1.1, we then have a $\mathcal{P}$-quadruple $\mathcal{T} = (T,Y,\cdot,\circ)$ where the actions are the standard ones induced by the action of T on Y in M. By Theorem 1.7, $\mathcal{P}(M,T,Y) \coloneqq (\mathcal{P}(\mathcal{T}), T\nu_{\mathcal{T},T}, Y\nu_{\mathcal{T},Y})$ is an object of $\mathcal{C}$. Now let (M_1,T_1,Y_1) and (M_2,T_2,Y_2) be objects of $\mathcal{C}$, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the corresponding $\mathcal{P}$-quadruples. Suppose that $\psi : (M_1,T_1,Y_1) \to (M_2,T_2,Y_2)$ is a morphism of $\mathcal{C}$. Define $\psi_T : T_1 \to T_2$ and $\psi_Y : Y_1 \to Y_2$ as the morphisms induced by ψ. Since $\psi : M_1 \to M_2$ is a $(2,1,1,0)$-morphism, it is clear that, for any $t \in T_1$ and $y \in Y_1$,

$$(t \cdot y)\psi_Y = t\psi_T \cdot y\psi_Y \quad \text{and} \quad (y \circ t)\psi_Y = y\psi_Y \circ t\psi_T.$$

With some abuse of notation, letting ψ also denote the semigroup morphism extension of ψ_T and ψ_Y to $T_1 * Y_1$, Lemma 3.2 gives that

$$\mathcal{P}(\psi) \coloneqq \mathcal{P}_\psi : \mathcal{P}(\mathcal{T}_1) \to \mathcal{P}(\mathcal{T}_2)$$

is a morphism of Ehresmann monoids. Clearly $(T_1\nu_{\mathcal{T}_1,T_1})\mathcal{P}_\psi = T_2\nu_{\mathcal{T}_2,T_2}$ and $(Y_1\nu_{\mathcal{T}_1,Y_1})\mathcal{P}_\psi = Y_2\nu_{\mathcal{T}_2,Y_2}$. Thus $\mathcal{P}_\psi$ is a morphism in $\mathcal{C}$.

It follows easily that $\mathcal{P}$ is a functor from $\mathcal{C}$ to $\mathcal{C}$.

Theorem 3.4. *The functor $\mathcal{P}$ determines an expansion of the category of marked Ehresmann monoids.*

Proof. Let $\mathcal{M} = (M, T, Y)$ be an object of $\mathcal{C}$. From [3, Theorem 5.2], the morphism $\iota : T * Y \to M$ extending the inclusion maps $\iota_T : T \to M$ and $\iota_Y : Y \to M$ factors through $\mathcal{P}(M, T, Y)$ to produce an onto morphism

$$\pi_{\mathcal{M}} : \mathcal{P}(M, T, Y) \to (M, T, Y)$$

given by

$$[u_1 \dots u_n]\pi_{\mathcal{M}} = u_1 \dots u_n,$$

where $u_1, \dots, u_n \in T \cup Y$ for $1 \leq i \leq n$, the product on the left hand side is in $T * Y$, and that on the right hand side is taken in M.

It is easy to check that if $\mathcal{M}_1 = (M_1, T_1, Y_1)$ and $\mathcal{M}_2 = (M_2, T_2, Y_2)$ are objects of $\mathcal{C}$ and $\psi : \mathcal{M}_1 \to \mathcal{M}_2$ is a morphism of $\mathcal{C}$, then $\pi_{\mathcal{M}_1} \psi = \mathcal{P}_\psi \pi_{\mathcal{M}_2}$, so that the diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{M}_1) & \xrightarrow{\mathcal{P}_\psi} & \mathcal{P}(\mathcal{M}_2) \\ {\scriptstyle \pi_{\mathcal{M}_1}} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \pi_{\mathcal{M}_2}} \\ \mathcal{M}_1 & \xrightarrow[\psi]{} & \mathcal{M}_2 \end{array}$$

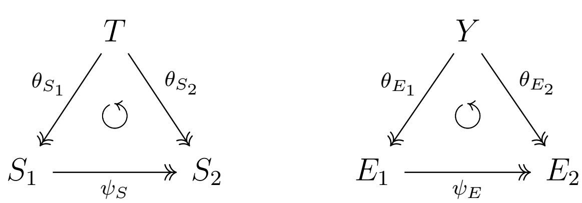
commutes as required. □

We point out that the expansion determined by $\mathcal{P}$ has associated natural transformation π, where, for each object $\mathcal{M}$ of $\mathcal{C}$, the morphism $\pi_{\mathcal{M}} : \mathcal{P}(\mathcal{M}) \to \mathcal{M}$ is defined in the proof of Theorem 3.4.

We now fix a $\mathcal{P}$-quadruple $\mathcal{T} = (T, Y, \cdot, \circ)$, and define a category $\mathcal{C}(\mathcal{T})$ as follows: the objects are quintuples $(M, S, E, \theta_S, \theta_E)$ where (M, S, E) is an object of $\mathcal{C}$, $\theta_S : T \to S$ and $\theta_E : Y \to E$ are onto monoid morphisms such that for all $t \in T$ and $y \in Y$,

$$(t \cdot y)\theta_E = t\theta_S \cdot y\theta_E \quad \text{and} \quad (y \circ t)\theta_E = y\theta_E \circ t\theta_S. \tag{3.2}$$

A morphism $\psi : (M_1, S_1, E_1, \theta_{S_1}, \theta_{E_1}) \to (M_2, S_2, E_2, \theta_{S_2}, \theta_{E_2})$ of $\mathcal{C}(\mathcal{T})$ is simply a morphism $\psi : (M_1, S_1, E_1) \to (M_2, S_2, E_2)$ in $\mathcal{C}$ such that $\theta_{S_1}\psi_S = \theta_{S_2}$ and $\theta_{E_1}\psi_E = \theta_{E_2}$. This can be represented in terms of commutativity of diagrams as follows:



Definition 3.5. The category $\mathcal{C}(\mathcal{T})$ is called the category of $\mathcal{T}$-marked Ehresmann monoids.

Theorem 3.6. *Let $\mathcal{T} = (T, Y, \cdot, \circ)$ be a $\mathcal{P}$-quadruple. Then*

$$\mathcal{Q}(\mathcal{T}) = \big(\mathcal{P}(T, Y), T\nu_{\mathcal{T},T}, Y\nu_{\mathcal{T},Y}, \nu_{\mathcal{T},T}, \nu_{\mathcal{T},Y}\big)$$

is the initial object in the category $\mathcal{C}(\mathcal{T})$.

Proof. Clearly $\mathcal{Q}(\mathcal{T})$ is an object in $\mathcal{C}(\mathcal{T})$. Let $(M, S, E, \theta_S, \theta_E)$ be an object in $\mathcal{C}(\mathcal{T})$. We must show that there is a unique morphism in $\mathcal{C}(\mathcal{T})$ from $\mathcal{Q}(\mathcal{T})$ to $(M, S, E, \theta_S, \theta_E)$.

Let $\mathcal{U} = (S, E, \cdot, \circ)$ be the $\mathcal{P}$-quadruple determined by M and notice that by the very definition of $\mathcal{C}(\mathcal{T})$ we have onto monoid morphisms $\theta_S : T \to S$ and $\theta_E : Y \to E$ satisfying (3.2) for all $t \in T$ and $y \in Y$.

Let $\theta : T * Y \to S * E$ be the natural extension of θ_S, θ_E. By Lemma 3.2 we have an onto $(2, 1, 1, 0)$-morphism

$$\mathcal{P}_\theta : \mathcal{P}(\mathcal{T}) \to \mathcal{P}(\mathcal{U})$$

in $\mathcal{C}$ given by

$$[u]\mathcal{P}_\theta = [u\theta].$$

By Theorem 3.4, there is an onto $(2, 1, 1, 0)$-morphism $\pi : \mathcal{P}(\mathcal{U}) \to M$ that lies in $\mathcal{C}$ and hence $\mathcal{P}_\theta\pi$ lies in $\mathcal{C}$. For any $t \in T$ we have

$$t\nu_{\mathcal{T},T}\mathcal{P}_\theta\pi = [t]\mathcal{P}_\theta\pi = [t\theta]\pi = t\theta = t\theta_S,$$

and similarly $y\nu_{\mathcal{T},Y}\mathcal{P}_\theta\pi = y\theta_E$. Hence $\mathcal{P}_\theta\pi$ lies in $\mathcal{C}(\mathcal{T})$.

Now let $\psi : \mathcal{P}(\mathcal{T}) \to M$ be any morphism in $\mathcal{C}(\mathcal{T})$. By definition, we must have that for any $t \in T$ and $y \in Y$,

$$t\nu_{\mathcal{T},T}\psi = t\theta_S \quad \text{and} \quad y\nu_{\mathcal{T},Y}\psi = y\theta_E.$$

Then $\mathcal{P}_\theta\pi$ and ψ agree on a set of generators of $\mathcal{P}(\mathcal{T})$, so they must be equal, establishing the uniqueness of $\mathcal{P}_\theta\pi$. The result follows. □

It is worth remarking that we have shown that if $(M, S, E, \theta_S, \theta_E)$ is an object in $\mathcal{C}(\mathcal{T})$ for some $\mathcal{P}$-quadruple $\mathcal{T}$, then the expansion $\mathcal{P}(M, S, E)$ of (M, S, E) also lies in $\mathcal{C}(\mathcal{T})$.

4. Open questions

We point to some natural questions that arise from our work.

Let $\mathcal{T} = (T, Y, \cdot, \circ)$ be a $\mathcal{P}$-quadruple.

Open Question 4.1. We have proved that if T is an equidivisible cancellative monoid then the monoid $\mathcal{P}(T, Y)$ is adequate. *Is $\mathcal{P}(T, Y)$ adequate for an arbitrary cancellative monoid T?*

Open Question 4.2. We know that, unlike the one-sided case, $\mathcal{P}(T, Y)$ does not have uniqueness of T-normal forms. *Is there a uniqueness of T-normal forms of minimal length in $\mathcal{P}(T, Y)$, at least in the equidivisible case?*

Open Question 4.3. In the two-sided case, the monoid $\mathcal{P}(T,Y)$ may be thought of as the analogue for Ehresmann monoids of the semidirect product construction known for inverse and for restriction monoids. Similarly for the one-sided case by considering $\mathcal{P}_\ell(T,Y)$, left Ehresmann monoids, and left restriction monoids. *What might be the analogue of the McAlister P-semigroup construction for $\mathcal{P}(T,Y)$ and $\mathcal{P}_\ell(T,Y)$?* Observe that in the two-sided case this may well involve the study of partial actions, in view of the results in [4] for restriction monoids.

Open Question 4.4. That every strongly T-proper Ehresmann monoid M is T-proper is known. *What is the precise connection in the one- and the two-sided cases between the concepts of being strongly T-proper, T-proper, having uniqueness of T-normal forms (the latter in the one-sided case) and indeed other, natural, concepts of T-properness?*

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