Equalizing semigroups, constellations and canonical extension: getting partial transformations from total ones, and other tales of the unexpected.

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We read compositions left to right, so fg denotes "first f, then g".

Throughout, if X is a non-empty set, $PT(X)$ denotes the semigroup of partial functions on X equipped with composition,

It has a unary domain operation D, defined as follows:

$$D(s) = \{(x, x) \mid x \in Dom(s)\}.$$

Here we are mainly interested in ways of building *left restriction semigroups*.

These are ubiquitous unary semigroups that include $PT(X)$ with D.

They have been discovered and re-discovered by a number of authors.

Axioms for left restriction semigroups: semigroups plus

- $D(x)x = x$;
- $D(D(x)) = D(x) = D(x)^2$;
- $D(D(x)y) = D(x)D(y) = D(y)D(x)$;
- $D(xy) = D(xD(y))$;
- $xD(y) = D(xy)x$.

(There is a dual notion of a right restriction semigroup.)

It follows that $D(S) = \{D(s) \mid s \in S\}$ is a semilattice under multiplication,

and that $D(s)$ is the smallest $e \in D(S)$ such that $es = s$.

The main example is $PT(X)$ equipped with domain operation

$$D(s) = \{(x, x) \mid x \in Dom(s)\}.$$

Indeed every left restriction semigroup embeds in such an example.

Any inverse semigroup S also gives an example.

Simply define $D(s) = ss'$ for all $s \in S$.

Interpreting S as 1:1 partial functions (using the Vagner-Preston Theorem)

this is just D as defined earlier for partial functions!

There are other examples involving binary relations, etc.

In a left restriction semigroup S that is a monoid with identity 1,

$1 = D(1)1 = D(1)$, so $1 \in D(S)$,

and the subset $S_1 = \{s \in S \mid D(s) = 1\}$ is a submonoid of S

(since if $s, t \in S_1$ then $D(st) = D(sD(t)) = D(s1) = D(s) = 1$).

We identify $(PT(X))_1$ with $T(X)$, the transformation monoid on X.

Note that $PT(X)$ is also closed under the operation of intersection $\cap$ given by

$$f \cap g = \{(x, y) \mid y = xf = xg\},$$

that is, intersection of the sets of ordered pairs defining the two functions.

So $\cap$ is a semilattice operation on $PT(X)$.

It satisfies a small number of further laws

that can be used to axiomatize subalgebras of $(PT(X), \cdot, \cap)$; see [Garvac'kiĭ, 1971].

Such an abstract semigroup with $\cap$ is called a $\cap$-semigroup.

So $PT(X)$ is both a left restriction semigroup and a $\cap$-semigroup.

Subalgebras of $(PT(X), \cdot, D, \cap)$ have a finite equational axiomatization;

let's call them left restriction $\cap$-semigroups.

In last week's talk, we looked at ways to make certain biunary semigroups correspond to certain types of categories.

Recall that a category C is a class with a partial binary operation $\circ$ and two unary operations D, R such that, for all $x, y \in C$,

1. $D(x) \circ x = x$, $x \circ R(x) = x$

2. $R(D(x)) = D(x)$, $D(R(x)) = R(x)$

3. $x \circ y$ exists if and only if $R(x) = D(y)$

4. if $R(x) = D(y)$ then $D(x \circ y) = D(x)$ and $R(x \circ y) = R(y)$

5. $x \circ (y \circ z) = (x \circ y) \circ z$ whenever the two products are defined.

But what to do for left restriction semigroups where this is domain but no range?

We need to generalise categories somehow.

The result of this generalisation is not a category, but a *constellation*.

These were first defined by [Gould and Hollings, 2009]

(who wanted a “one-sided” version of earlier work on inverse and other bi-unary semigroups)

and then considered further by various authors (eg [Gould and S., 2017]).

Constellations have an (asymmetric) partial binary composition operation and a domain operation but no range.

They model partial functions under partial composition $f \cdot g$,

defined to be the usual composition, but only when $Im(f) \subseteq Dom(g)$

(rather than $Cod(f) = Dom(g)$ as in a category),

with domain defined as for $PT(X)$.

Axioms for a constellation $\mathcal{C}$ can be stated as follows:

1. $D(x) \cdot x = x$
2. if $x \cdot D(y)$ exists then it equals x
3. $x \cdot y$ exists if and only if $x \cdot D(y) = x$
4. if $x \cdot y$ exists then $D(x \cdot y) = D(x)$
5. if $x \cdot (y \cdot z)$ exists then $(x \cdot y) \cdot z$ exists
6. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ whenever the two products are defined.

The constellation laws can be verified for $(PT(X), \cdot, D)$.

A constellation P embeds in such a functional constellation

providing a simple condition on $D(P)$ is satisfied.

Every category is a (special type of) constellation: forget range R.

In a category, $s \circ (t \circ u)$ exists iff $(s \circ t) \circ u$ exists,

but in a constellation only the forward direction holds in general!

[Gould and Hollings, 2009] showed that a left restriction semigroup can be made into a constellation by defining

$s \cdot t := st$ but only when $sD(t) = s$.

For $PT(X)$, this is equivalent to the definition given earlier.

Conversely, a constellation arises from a left restriction semigroup in this way if and only if it is *inductive*.

Note that every constellation has a *natural quasiorder*:

$s \leq t$ if and only if $D(s) \cdot t$ exists and equals s.

The constellation P is inductive iff

- for all $e \in D(P)$ and $s \in P$ there is a unique largest $x \in P$ such that $x \leq s$ and $x \cdot e$ exists, the *co-restriction* of s to e, denoted $s|e$;

- whenever $x \cdot y$ exists and $e \in P$,

$$D((x \cdot y)|e) = D(x|D(y|e)).$$

This forces $\leq$ to be a partial order.

The constellation obtained from a left restriction semigroup is inductive.

Conversely, if P is inductive, we may define the *pseudoproduct*

$s \otimes t := s|D(t) \cdot t$ for all $s, t \in P$,

and then $(P, \otimes, D)$ is a left restriction semigroup.

The constructions are mutually inverse, morphisms correspond etc.

Change of tack!

The idempotent completion (Cauchy completion, Karoubi envelope) of a semigroup S having non-empty idempotent set $E(S)$ is a well-known category.

Generalising slightly, for any $E \subseteq E(S)$, define

$$Cat_E(S) = \{(e, s, f) \in E \times S \times E \mid esf = s\}.$$

Define the partial binary operation $\circ$ on $Cat_E(S)$ by setting

$$(e, s, f) \circ (f, t, g) := (e, s \cdot t, g).$$

It is routine to check that $\circ$ is well-defined.

Indeed $(Cat_E(S), \circ)$ consists of the partial algebra of arrows of a small category.

Define $D((e, s, f)) = (e, e, e)$ and $R((e, s, f)) = (f, f, f)$.

We call this category the *E-completion of S*.

If $E = E(S)$, this is the idempotent completion of the semigroup.

[Tilson, 1987] defined this for an arbitrary semigroup, though not under any of the names now used.

(The concept was already in use by category theorists.)

We are interested in a one-sided version of this construction, which gives a one-sided version of a category.

For any semigroup S and $E \subseteq E(S)$, we define

$$C_E(S) = \{(e, s) \in E \times S \mid es = s\}.$$

Next, define $\cdot$ on $C_E(S)$ by setting

$(e, s) \cdot (f, t) := (e, st)$ whenever $sf = s$,

and define $D((e, s)) = (e, e)$.

PROPOSITION: If S is a semigroup with $E \subseteq E(S)$, then

$$C_E(S) = \{(e, s) \in E \times S \mid es = s\},$$

equipped with composition and domain as just defined

is a constellation we call the *(left) E-completion of S*.

If S is a monoid with $1 \in E$, then there is a copy of S in $C_E(S)$

consisting of the elements with domain 1: i.e. $\{(1, s) \mid s \in S\}$.

One tweaked version of this definition proves useful.

We say the semigroup S is *integral* if it has a zero element but no zero divisors (so $S\backslash\{0\}$ is a subsemigroup).

If S is integral and $0 \in E$, it is fruitful to define $C_E^0(S) \subseteq C_E(S)$ as follows:

$$C_E^0(S) = \{(e, s) \in C_E(S) \mid s = 0 \Rightarrow e = 0\}.$$

The result is a subconstellation of $C_E(S)$ in the natural sense.

NATURAL QUESTION:

if S is a monoid with $1 \in E \subseteq E(S)$, when is $C_E(S)$ inductive?

that is, when is it “really” a left restriction semigroup in a natural way?

Turns out we can get a complete answer to this.

But rather than answer the question in full generality today, we consider the most important special case.

For this, we need one final *change of tack*.

In any semigroup S, we may define, for all $s, t \in S$,

$Eq(s,t) = \{a \in S \mid as = at\}$, a left ideal of S if non-empty.

We say the monoid S *has equalizers* if for all $s, t \in S$, $Eq(s,t)$ is non-empty and generated as a left ideal by an idempotent.

So if S has a zero then it is a left Baer semigroup:

$Eq(s, 0) = \{a \in S \mid as = 0\}$ is generated as a left ideal by an idempotent.

What S having equalizers means is as follows:

for any $s, t \in S$, there is an idempotent e such that $es = et$

(so for any $a \in S$, $(ae)s = (ae)t$),

but e is “universal” in the sense that if $as = at$

then it’s because $a = ae$!

For example, the transformation semigroup $T(X)$ almost has this property:

if $Eq(s,t) \neq \emptyset$, then it is generated by an idempotent.

(Pick any projection onto where s, t agree.)

But $Eq(s,t)$ can be empty!

So introduce a zero to $T(X)$ to give $T(X)^0$, and then 0 equalizes any s, t for which $Eq(s,t) = \emptyset$ in $T(X)$,

and so $T(X)^0$ has equalizers.

How does this connect with left E-completions and left restriction monoids??

Well, let M be a monoid with equalizers.

We need to limit $E(S)$ to $E \subseteq E(S)$, so there is precisely one $e \in E$ that generates each $Eq(s,t)$.

On $E(S)$ define the quasiorder ω_l via $e\omega_l f$ iff $e = ef$;

this is indeed a quasiorder, so consider the associated equivalence relation.

This is nothing but Green's $\mathcal{L}$-relation restricted to E!

Now returning to the construction of E:

it is necessary and sufficient to pick exactly one idempotent from each $\mathcal{L}$-class to give E.

Denote by $s \bowtie t$ the unique generator in E of $Eq(s, t)$.

We can now view $(S, \cdot, \bowtie)$ as an algebra (the image of $\bowtie$ is nothing but E).

We call $(S, \cdot, \bowtie)$ a *generalised EQ-monoid*.

Generalised EQ-monoids can be finitely axiomatized using equations involving multiplication and $\bowtie$.

Surprisingly, E is automatically a meet-semilattice under ω_l,

with $e \wedge f = ((ef \bowtie e)e \bowtie 1)$.

Equational axioms for for generalised EQ-monoids are as follows:

- $(a \bowtie a) = 1$
- $(a \bowtie b)a = (a \bowtie b)b$
- $(a \bowtie 1) \wedge (b \bowtie 1) = (b \bowtie 1) \wedge (a \bowtie 1)\omega_l(a \bowtie 1)$
- $(ca \bowtie cb) = (c(a \bowtie b) \bowtie c)$
- $(a \bowtie b) = (b \bowtie a)$
- $(a \bowtie b)^2 = (a \bowtie b) = ((a \bowtie b) \bowtie 1)$

THEOREM: If $(S, \cdot, \bowtie)$ is a generalised EQ-monoid, then $C_E(S)$ is inductive, hence can be made into a left restriction monoid,

in which in general $(e, s)(f, t) = (e \wedge (sf \bowtie s), (e \wedge (sf \bowtie s))st)$.

(Note: this is NOT a semidirect product in general since the equation $(e \wedge (sf \bowtie s))st = st$ fails in general.)

In fact it is a left restriction $\cap$-monoid, in which

$(e, s) \cap (f, t) = (e \wedge f \wedge (s \bowtie t), (e \wedge f \wedge (s \bowtie t))st)$.

Call this left restriction $\cap$-monoid $Rest(E, S)$.

The structure of $(S, \cdot, \bowtie)$ can depend on the choice of E.

But $Rest(E, S)$ does not depend on the choice of E (up to isomorphism).

So we have a canonical way to extend the monoid with equalizers S to a left restriction $\cap$-monoid into which S embeds as the elements having domain 1.

Which left restriction $\cap$-monoids M arise in this way from monoids with equalizers?

Note that if $(S, \cdot, \bowtie)$ is a generalised EQ-monoid,

then a copy of S sits inside $Rest(E, S)$ as the set of elements having domain 1:

$S \cong \{(1, s) \mid s \in S\} = Rest(E, S)_1$.

Under this isomorphism $e \in E$ corresponds to $(1, e) \in E(Rest(E, S)_1)$,

and in $Rest(E, S)$, $(e, e)(1, e) = (e, e)$ and $(1, e)(e, e) = (1, e)$.

So $(e, e)\mathcal{L}(1, e)$ in $E(Rest(E, S))$.

If M is a left restriction monoid and for all $e \in D(M)$, there is $e' \in E(M_1)$ such that $e\mathcal{L}e'$,

then we will say M *has enough large idempotents*.

So if $(S, \cdot, \bowtie)$ is a generalised EQ-monoid, then $Rest(E, S)$ has enough large idempotents.

THEOREM: Let M be a left restriction $\cap$-monoid having enough large idempotents.

Then $M \cong Rest(E, S)$ for some monoid with equalizers S

(remembering the choice of E does not affect the result).

Specifically, we can let $S = M_1$, and for each $e \in D(M)$, choose any $e' \in E(M_1)$ such that $e\mathcal{L}e'$,

and define $E = \{e' \in E(M_1) \mid e \in D(M)\}$.

Then $(S, \cdot, \bowtie)$ is a generalised EQ-monoid with $s \bowtie t = D(s \cap t)'$, and $M \cong Rest(E, S)$.

This is all well and good...

but note that $Rest(E, S)$ never has a zero element 0 such that $D(0) = 0$,

but many important left restriction monoids do, *e.g.* $PT(X)$.

Luckily, there is a version covering such cases as well.

Recall that if S is integral (has zero and no zero divisors) and $0 \in E$,

then $C^0_E(S) = \{(e, s) \in C_E(S) \mid s = 0 \Rightarrow e = 0\}$,

itself a constellation which might be inductive!

Indeed if $(S, \cdot, \bowtie)$ is a generalised EQ-monoid with S integral then

$C^0_E(S)$ is a subsemigroup of $Rest(E, S)$ closed under D and $\cap$;

so it is itself a left restriction $\cap$-monoid: call it $Rest_0(E, S)$.

Then $(0, 0)$ is a zero of $Rest_0(E, S)$, and $D((0, 0)) = (0, 0)$.

Now, $M = Rest_0(E, S)$ has *almost enough large idempotents*,

meaning that for each *non-zero* domain element, there is an idempotent with domain 1 that is $\mathcal{L}$-related to it.

THEOREM: Let M be a left restriction $\cap$-monoid with zero such that $D(0) = 0$ and having almost enough large idempotents.

Then M is isomorphic to $Rest_0(E, S)$ for some generalised EQ-monoid $(S, \cdot, \bowtie)$ such that S is integral and $0 \in E$.

The details follow the earlier case. Again, the choice of E does not affect the result.

Are there examples of left restriction monoids filling the bill?

Yes, plenty, *e.g.* the canonical example $M = PT(X)$.

This example has a zero with $D(0) = 0$.

Does it have almost enough large idempotents?

Obviously $(PT(X))_1 \cong T(X)$ and indeed we identify the two.

For non-zero $e \in D(PT(X))$, let e' be a projection in $T(X)$ onto $dom(e)$.

Then $ee' = e, e'e = e'$, so $e\mathcal{L}e'$, and so $PT(X)$ has almost enough large idempotents.

But $T(X)^0$ is an integral monoid with equalizers,

so from our second theorem, $PT(X) \cong Rest_0(E \cup \{0\}, T(X)^0)$.

A further (surprising!) example:

$T(X)$ is also a *right* monoid with equalizers!

$REq(s,t) = \{x \in T(X) \mid sx = tx\}$ is always non-empty and is generated by an idempotent:

just pick any projection e whose kernel is the finest equivalence relation equalizing s, t.

Then pick E to contain exactly one idempotent for every possible kernel (= one idempotent from each $\mathcal{R}$-class!), define $\bowtie$ in terms of it,

and $(T(X), \cdot, \bowtie)$ is a right generalised EQ-monoid.

So, form $RRest(T(X), E)$ with underlying set the *right* constellation

$$RC_E(T(X)) = \{(s, e) \mid s \in T(X), e \in E, se = s\}.$$

It will be the opposite of a left restriction $\cap$-monoid.

But what is it??

Recall the partition monoid $P(X)$ on the set X. (Apologies if you're not familiar with this.)

Now consider the submonoid $P^t(X)$ of all "left total" partitions $P^t(X)$ on X (defined in the "obvious" way).

It's not hard to see this is a *right* restriction monoid where $R(s)$ is the finest block partition e such that $se = s$.

And it's got enough large idempotents!

What is $P^t(X)_1$?

Basically a copy of $T(X)$!

And the large idempotents consist of one transformation in E for each partition of X.

So we must have $P^t(X) \cong RRest((T(X), E)$ for E as before!

Because of the additional structure of $(T(X), \cdot, \bowtie_r)$, it follows that

$P^t(X)$ is a right restriction monoid whose opposite is a left restriction $\cap$-monoid:

hence it is the opposite of a function semigroup under $D, \cap$ (and R actually)...

quite unexpected!

This is not easy to verify directly, but easily follows from the properties of the right generalised EQ-monoid $(T(X), \cdot, \bowtie_r)$.

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Other references...

REFERENCES:

V.S. Garvac'kiĭ, ∩-semigroups of transformations, *Teor. Polugrupp Prilozh.* 2:2–13 [Russian] (1971)

V. Gould and C. Hollings, Restriction semigroups and inductive constellations, *Comm. Algebra* 38, 261–287 (2009)

V. Gould and T. Stokes, Constellations and their relationship with categories, *Algebra Universalis* 77, 271–304 (2017)

M. Jackson and T. Stokes, Partial maps with domain and range: extending Schein's representation, *Comm. Alg.* 37, 2845–2870 (2009)

B. Tilson, Categories as algebra: an essential ingredient in the theory of monoids, *J. Pure Appl. Algebra* 48, 83—198 (1987)