Presentations for tensor categories

WESTERN SYDNEY UNIVERSITY

James East

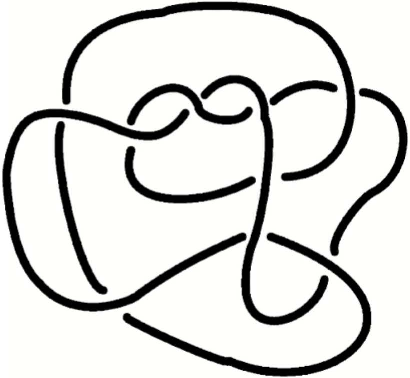
Centre for Research in Mathematics

York/Sydney — 29 April 2020

Analogy: knot theory

Analogy: knot theory

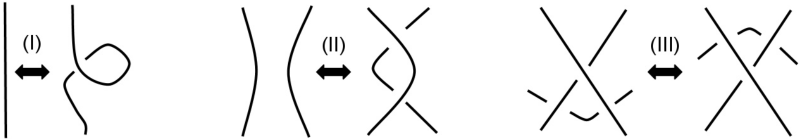
Can this knot be un-knotted?



Analogy: knot theory

Theorem (Reidemeister 1927)

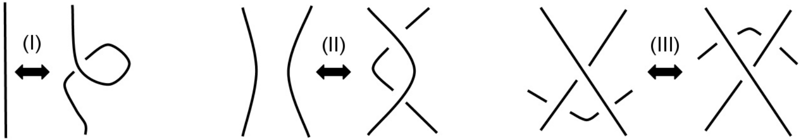
Two knots are equivalent ⇔ they differ by Reidemeister moves.



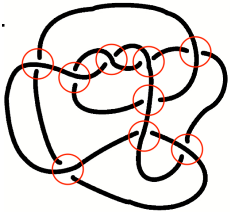
Analogy: knot theory

Theorem (Reidemeister 1927)

Two knots are equivalent $\Leftrightarrow$ they differ by Reidemeister moves.



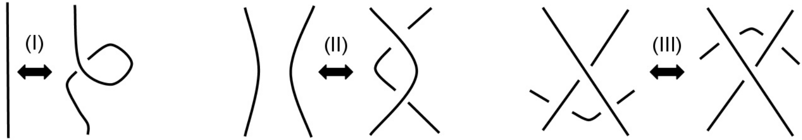
- Knots are built from small pieces (crossings).



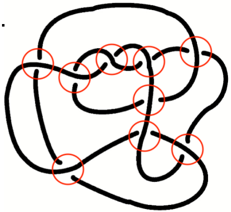
Analogy: knot theory

Theorem (Reidemeister 1927)

Two knots are equivalent ⇔ they differ by Reidemeister moves.



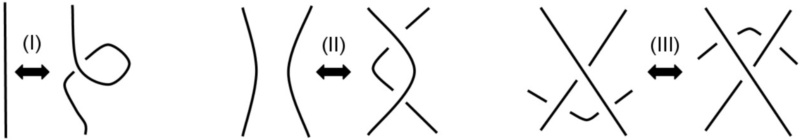
- Knots are built from small pieces (crossings).
- Equivalence is governed by local moves.



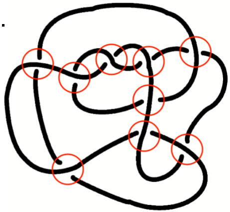
Analogy: knot theory

Theorem (Reidemeister 1927)

Two knots are equivalent $\Leftrightarrow$ they differ by Reidemeister moves.



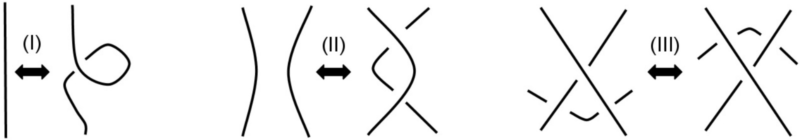
- Knots are built from small pieces (crossings).
- Equivalence is governed by local moves.
- Useful when defining knot invariants.
  - Alexander, Conway, Jones, Kauffman...



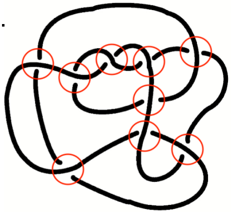
Analogy: knot theory

Theorem (Reidemeister 1927)

Two knots are equivalent ⇔ they differ by Reidemeister moves.



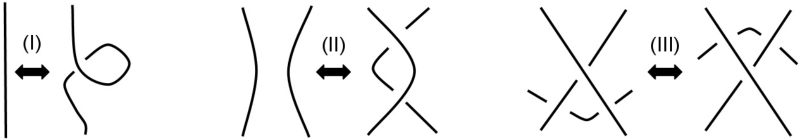
- Knots are built from small pieces (crossings).
- Equivalence is governed by local moves.
- Useful when defining knot invariants.
  - Alexander, Conway, Jones, Kauffman...
  - $f(\text{knot } K)$



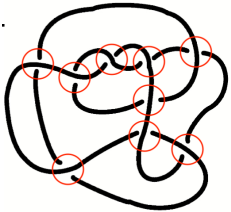
Analogy: knot theory

Theorem (Reidemeister 1927)

Two knots are equivalent $\Leftrightarrow$ they differ by Reidemeister moves.



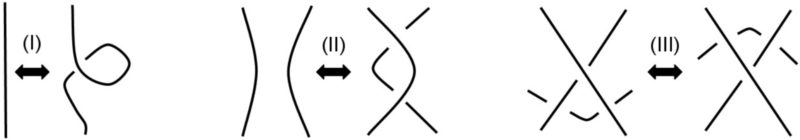
- Knots are built from small pieces (crossings).
- Equivalence is governed by local moves.
- Useful when defining knot invariants.
  - Alexander, Conway, Jones, Kauffman...
  - $f(\text{knot } K) := f(\text{diagram representing } K)$.



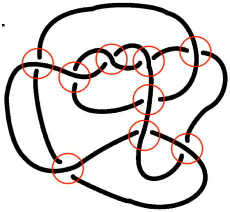
Analogy: knot theory

Theorem (Reidemeister 1927)

Two knots are equivalent ⇔ they differ by Reidemeister moves.



- Knots are built from small pieces (crossings).
- Equivalence is governed by local moves.
- Useful when defining knot invariants.
  - Alexander, Conway, Jones, Kauffman...
  - $f(\text{knot } K) := f(\text{diagram representing } K)$.
  - f is well-defined ⇔ invariant under local moves.



Presentations

Presentations

- Let A be an algebraic structure: e.g., a group, ring, category...

Presentations

- Let A be an algebraic structure: e.g., a group, ring, category...
- Informally, A has presentation $\langle X : R \rangle$ if:

Presentations

- Let A be an algebraic structure: e.g., a group, ring, category...
- Informally, A has presentation $\langle X : R \rangle$ if:
  - X is a generating set for A,

Presentations

- Let A be an algebraic structure: e.g., a group, ring, category...
- Informally, A has presentation $\langle X : R \rangle$ if:
  - X is a generating set for A,
  - R is a set of equations (relations) over X: e.g., $xy = yx$,

Presentations

- Let A be an algebraic structure: e.g., a group, ring, category...
- Informally, A has **presentation** $\langle X : R \rangle$ if:
  - X is a generating set for A,
  - R is a set of equations (relations) over X: e.g., $xy = yx$,
  - R is complete: every equation over X follows from R.

Presentations

- Let A be an algebraic structure: e.g., a group, ring, category...
- Informally, A has presentation $\langle X : R \rangle$ if:
  - X is a generating set for A,
  - R is a set of equations (relations) over X: e.g., $xy = yx$,
  - R is complete: every equation over X follows from R.
- Formally, A has presentation $\langle X : R \rangle$ if $A \cong X^*/R^\sharp$:
  - X is a set and X^* is the free algebra (of same type as A),
  - $R \subseteq X^* \times X^*$ and $R^\sharp$ is the congruence on X^* generated by R.
  - i.e., there is a surmorphism $X^* \to A$ with kernel $R^\sharp$.

Presentations

- Let A be an algebraic structure: e.g., a group, ring, category...
- Informally, A has presentation $\langle X : R \rangle$ if:
  - X is a generating set for A,
  - R is a set of equations (relations) over X: e.g., $xy = yx$,
  - R is complete: every equation over X follows from R.
- Formally, A has presentation $\langle X : R \rangle$ if $A \cong X^*/R^\sharp$:
  - X is a set and X^* is the free algebra (of same type as A),
  - $R \subseteq X^* \times X^*$ and $R^\sharp$ is the congruence on X^* generated by R.
  - i.e., there is a surmorphism $X^* \to A$ with kernel $R^\sharp$.
- We can use presentations to define invariants.

Presentations

- Let A be an algebraic structure: e.g., a group, ring, category...
- Informally, A has presentation $\langle X : R \rangle$ if:
  - X is a generating set for A,
  - R is a set of equations (relations) over X: e.g., $xy = yx$,
  - R is complete: every equation over X follows from R.
- Formally, A has presentation $\langle X : R \rangle$ if $A \cong X^*/R^\sharp$:
  - X is a set and X^* is the free algebra (of same type as A),
  - $R \subseteq X^* \times X^*$ and $R^\sharp$ is the congruence on X^* generated by R.
  - i.e., there is a surmorphism $X^* \to A$ with kernel $R^\sharp$.
- We can use presentations to define homomorphisms $\phi : A \to B$.

Presentations

- Let A be an algebraic structure: e.g., a group, ring, category...
- Informally, A has presentation $\langle X : R \rangle$ if:
  - X is a generating set for A,
  - R is a set of equations (relations) over X: e.g., $xy = yx$,
  - R is complete: every equation over X follows from R.
- Formally, A has presentation $\langle X : R \rangle$ if $A \cong X^*/R^\sharp$:
  - X is a set and X^* is the free algebra (of same type as A),
  - $R \subseteq X^* \times X^*$ and $R^\sharp$ is the congruence on X^* generated by R.
  - i.e., there is a surmorphism $X^* \to A$ with kernel $R^\sharp$.
- We can use presentations to define homomorphisms $\phi : A \to B$.
  - Define $\phi(x)$ for all $x \in X$.

Presentations

- Let A be an algebraic structure: e.g., a group, ring, category...
- Informally, A has presentation $\langle X : R \rangle$ if:
  - X is a generating set for A,
  - R is a set of equations (relations) over X: e.g., $xy = yx$,
  - R is complete: every equation over X follows from R.
- Formally, A has presentation $\langle X : R \rangle$ if $A \cong X^*/R^\sharp$:
  - X is a set and X^* is the free algebra (of same type as A),
  - $R \subseteq X^* \times X^*$ and $R^\sharp$ is the congruence on X^* generated by R.
  - i.e., there is a surmorphism $X^* \to A$ with kernel $R^\sharp$.
- We can use presentations to define homomorphisms $\phi : A \to B$.
  - Define $\phi(x)$ for all $x \in X$.
  - For $a = x_1 \cdots x_k$, define $\phi(a) = \phi(x_1) \cdots \phi(x_k)$.

Presentations

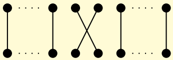
- Let A be an algebraic structure: e.g., a group, ring, category...
- Informally, A has presentation $\langle X : R \rangle$ if:
  - X is a generating set for A,
  - R is a set of equations (relations) over X: e.g., $xy = yx$,
  - R is complete: every equation over X follows from R.
- Formally, A has presentation $\langle X : R \rangle$ if $A \cong X^*/R^\sharp$:
  - X is a set and X^* is the free algebra (of same type as A),
  - $R \subseteq X^* \times X^*$ and $R^\sharp$ is the congruence on X^* generated by R.
  - i.e., there is a surmorphism $X^* \to A$ with kernel $R^\sharp$.
- We can use presentations to define homomorphisms $\phi : A \to B$.
  - Define $\phi(x)$ for all $x \in X$.
  - For $a = x_1 \cdots x_k$, define $\phi(a) = \phi(x_1) \cdots \phi(x_k)$.
  - Well-defined $\Leftrightarrow$ $\phi(u) = \phi(v)$ for every relation $u = v$.

Presentations (examples)

Theorem (Moore, 1897)

The symmetric group $\mathcal{S}_n \cong \langle s_1, \ldots, s_{n-1} : R \rangle$.

$$
\begin{aligned}
s_i^2 &= \iota && \text{for all } i && \text{(R1)} \\
s_i s_j &= s_j s_i && \text{if } |i-j| > 1 && \text{(R2)} \\
s_i s_j s_i &= s_j s_i s_j && \text{if } |i-j| = 1. && \text{(R3)}
\end{aligned}
$$

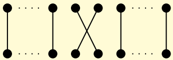
$s_i \equiv$  $= (i, i+1)$

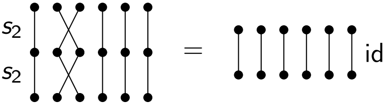
Presentations (examples)

Theorem (Moore, 1897)

The symmetric group $\mathcal{S}_n \cong \langle s_1, \ldots, s_{n-1} : R \rangle$.

$$s_i^2 = \iota \quad \text{for all } i \quad \text{(R1)}$$
$$s_i s_j = s_j s_i \quad \text{if } |i-j| > 1 \quad \text{(R2)}$$
$$s_i s_j s_i = s_j s_i s_j \quad \text{if } |i-j| = 1. \quad \text{(R3)}$$

$s_i \equiv$  $= (i, i+1)$



Presentations (examples)

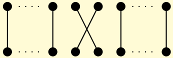
Theorem (Moore, 1897)

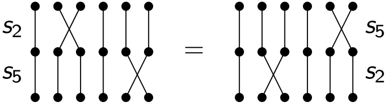
The symmetric group $\mathcal{S}_n \cong \langle s_1, \ldots, s_{n-1} : R \rangle$.

$$s_i^2 = \iota \qquad \text{for all } i \qquad \text{(R1)}$$

$$s_i s_j = s_j s_i \qquad \text{if } |i-j| > 1 \qquad \text{(R2)}$$

$$s_i s_j s_i = s_j s_i s_j \qquad \text{if } |i-j| = 1. \qquad \text{(R3)}$$

$s_i \equiv$  $= (i, i+1)$

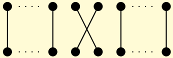


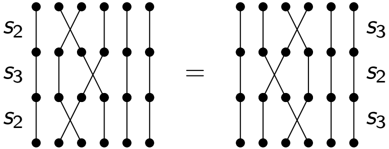
Presentations (examples)

Theorem (Moore, 1897)

The symmetric group $\mathcal{S}_n \cong \langle s_1, \ldots, s_{n-1} : R \rangle$.

$s_i^2 = \iota$	for all i	(R1)
$s_i s_j = s_j s_i$	if $\lvert i - j \rvert > 1$	(R2)
$s_i s_j s_i = s_j s_i s_j$	if $\lvert i - j \rvert = 1.$	(R3)

$s_i \equiv$  $= (i, i+1)$



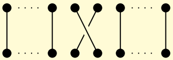
Presentations (examples)

Theorem (Artin, 1925)

The braid group $\mathcal{B}_n \cong \langle \sigma_1, \ldots, \sigma_{n-1} : R \rangle$.

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i-j| > 1 \tag{R1}$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \qquad \text{if } |i-j| = 1. \tag{R2}$$

$\sigma_i \equiv$ 

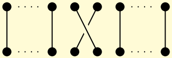
Presentations (examples)

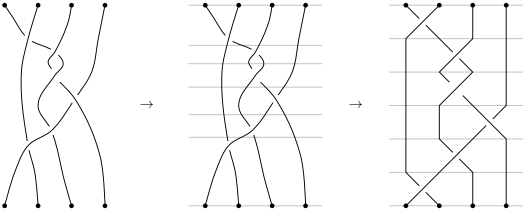
Theorem (Artin, 1925)

The braid group $\mathcal{B}_n \cong \langle \sigma_1, \ldots, \sigma_{n-1} : R \rangle$.

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i - j| > 1 \qquad \text{(R1)}$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \qquad \text{if } |i - j| = 1. \qquad \text{(R2)}$$

$\sigma_i \equiv$ 



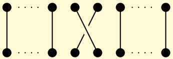
Presentations (examples)

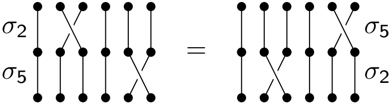
Theorem (Artin, 1925)

The braid group $\mathcal{B}_n \cong \langle \sigma_1, \ldots, \sigma_{n-1} : R \rangle$.

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i-j| > 1 \qquad \text{(R1)}$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \qquad \text{if } |i-j| = 1. \qquad \text{(R2)}$$

$\sigma_i \equiv$ 



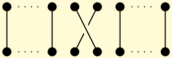
Presentations (examples)

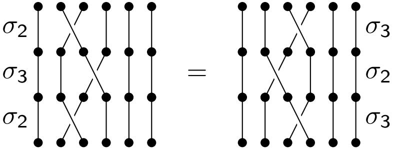
Theorem (Artin, 1925)

The braid group $\mathcal{B}_n \cong \langle \sigma_1, \ldots, \sigma_{n-1} : R \rangle$.

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i-j| > 1 \qquad \text{(R1)}$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \qquad \text{if } |i-j| = 1. \qquad \text{(R2)}$$

$\sigma_i \equiv$ 



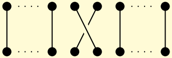
Presentations (examples)

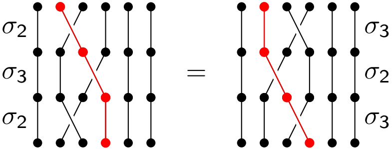
Theorem (Artin, 1925)

The braid group $\mathcal{B}_n \cong \langle \sigma_1, \ldots, \sigma_{n-1} : R \rangle$.

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i-j| > 1 \qquad \text{(R1)}$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \qquad \text{if } |i-j| = 1. \qquad \text{(R2)}$$

$\sigma_i \equiv$ 



A Reidemeister move!

Presentations (examples)

- Those examples were groups.

Presentations (examples)

- Those examples were groups.
- Presentations for many semigroups are known.

Presentations (examples)

- Those examples were groups.
- Presentations for many semigroups are known.
- For an integer $n \geq 0$, let $\mathbf{n} = \{1, \ldots, n\}$.

Presentations (examples)

- Those examples were groups.
- Presentations for many semigroups are known.
- For an integer $n \geq 0$, let $\mathbf{n} = \{1, \ldots, n\}$.
- The full transformation semigroup:

$$\mathcal{T}_n = \{\text{functions } \mathbf{n} \to \mathbf{n}\}.$$

Presentations (examples)

- Those examples were groups.
- Presentations for many semigroups are known.
- For an integer $n \geq 0$, let $\mathbf{n} = \{1, \ldots, n\}$.
- The full transformation semigroup:

$$\mathcal{T}_n = \{\text{functions } \mathbf{n} \to \mathbf{n}\}.$$

- The partial transformation semigroup:

$$\mathcal{PT}_n = \{\text{partial functions } \mathbf{n} \to \mathbf{n}\}.$$

Presentations (examples)

- Those examples were groups.
- Presentations for many semigroups are known.
- For an integer $n \geq 0$, let $\mathbf{n} = \{1, \dots, n\}$.
- The full transformation semigroup:

$$\mathcal{T}_n = \{\text{functions } \mathbf{n} \to \mathbf{n}\}.$$

- The partial transformation semigroup:

$$\mathcal{PT}_n = \{\text{partial functions } \mathbf{n} \to \mathbf{n}\}.$$

- The symmetric inverse semigroup:

$$\mathcal{I}_n = \{\text{partial bijections } \mathbf{n} \to \mathbf{n}\}.$$

Presentations (examples)

- Those examples were groups.
- Presentations for many semigroups are known.
- For an integer $n \geq 0$, let $\mathbf{n} = \{1, \ldots, n\}$.
- The full transformation semigroup:

$$\mathcal{T}_n = \{\text{functions } \mathbf{n} \to \mathbf{n}\}.$$

- The partial transformation semigroup:

$$\mathcal{PT}_n = \{\text{partial functions } \mathbf{n} \to \mathbf{n}\}.$$

- The symmetric inverse semigroup:

$$\mathcal{I}_n = \{\text{partial bijections } \mathbf{n} \to \mathbf{n}\}.$$

- Each contains the symmetric group:

$$\mathcal{S}_n = \{\text{bijections } \mathbf{n} \to \mathbf{n}\}.$$

Presentations (examples)

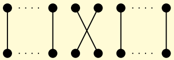
Theorem (Moore, 1897)

The symmetric group $\mathcal{S}_n \cong \langle s_1, \ldots, s_{n-1} : R \rangle$.

$$s_i^2 = \iota \qquad \text{for all } i \qquad \text{(R1)}$$

$$s_i s_j = s_j s_i \qquad \text{if } |i-j| > 1 \qquad \text{(R2)}$$

$$s_i s_j s_i = s_j s_i s_j \qquad \text{if } |i-j| = 1. \qquad \text{(R3)}$$

$s_i \equiv$ 

Presentations (examples)

Theorem (Aĭzenštat, 1958)

The full transformation semigroup $\mathcal{T}_n \cong \langle s_1, \ldots, s_{n-1}, e : R \rangle$.

$$s_i^2 = \iota \qquad \text{for all } i \tag{R1}$$

$$s_i s_j = s_j s_i \qquad \text{if } |i-j| > 1 \tag{R2}$$

$$s_i s_j s_i = s_j s_i s_j \qquad \text{if } |i-j| = 1 \tag{R3}$$

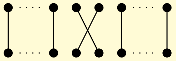
$$s_i e = e s_i \qquad \text{if } i \geq 3 \tag{R4}$$

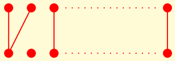
$$e^2 = e = s_1 e \tag{R5}$$

$$e s_2 e s_2 = s_2 e s_2 e = e s_2 e \tag{R6}$$

$$e s_1 s_2 s_1 = e s_1 s_2 e \tag{R7}$$

$$eueu = ueue \qquad \text{where } u = s_2 s_1 s_3 s_2. \tag{R8}$$

$s_i \equiv$ 

$e \equiv$ 

Presentations (examples)

Theorem (Popova, 1961)

The symmetric inverse semigroup $\mathcal{I}_n \cong \langle s_1, \ldots, s_{n-1}, e : R \rangle$.

$$s_i^2 = \iota \qquad \text{for all } i \qquad \text{(R1)}$$

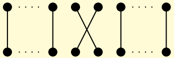
$$s_i s_j = s_j s_i \qquad \text{if } |i-j| > 1 \qquad \text{(R2)}$$

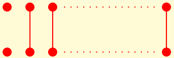
$$s_i s_j s_i = s_j s_i s_j \qquad \text{if } |i-j| = 1 \qquad \text{(R3)}$$

$$s_i e = e s_i \qquad \text{if } i \geq 2 \qquad \text{(R4)}$$

$$e^2 = e \qquad \text{(R5)}$$

$$e s_1 e s_1 = s_1 e s_1 e = e s_1 e. \qquad \text{(R6)}$$

$s_i \equiv$ 

$e \equiv$ 

Categories

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \dots\}$.
- Think of $\mathcal{C}$ as a partial monoid.

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- Think of $\mathcal{C}$ as a partial monoid...... A set with:
  - a partial product $xy = x \circ y$ $(x, y \in \mathcal{C})$,

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- Think of $\mathcal{C}$ as a partial monoid...... A set with:
  - a partial product $xy = x \circ y$ $(x, y \in \mathcal{C})$,
  - domain/range functions $\mathbf{d}, \mathbf{r} : \mathcal{C} \to \mathbb{N}$,

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- Think of $\mathcal{C}$ as a partial monoid...... A set with:
  - a partial product $xy = x \circ y$ $(x, y \in \mathcal{C})$,
  - domain/range functions $\mathbf{d}, \mathbf{r} : \mathcal{C} \to \mathbb{N}$,
  - many identity elements ι_n $(n \in \mathbb{N})$ with $\mathbf{d}(\iota_n) = \mathbf{r}(\iota_n) = n$.

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- Think of $\mathcal{C}$ as a partial monoid...... A set with:
  - a partial product $xy = x \circ y$ $(x, y \in \mathcal{C})$,
  - domain/range functions $\mathbf{d}, \mathbf{r} : \mathcal{C} \to \mathbb{N}$,
  - many identity elements ι_n $(n \in \mathbb{N})$ with $\mathbf{d}(\iota_n) = \mathbf{r}(\iota_n) = n$.
- Properties/axioms:
  - xy defined $\Leftrightarrow \mathbf{r}(x) = \mathbf{d}(y)$,

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- Think of $\mathcal{C}$ as a partial monoid...... A set with:
  - a partial product $xy = x \circ y$ $(x, y \in \mathcal{C})$,
  - domain/range functions $\mathbf{d}, \mathbf{r} : \mathcal{C} \to \mathbb{N}$,
  - many identity elements ι_n $(n \in \mathbb{N})$ with $\mathbf{d}(\iota_n) = \mathbf{r}(\iota_n) = n$.
- Properties/axioms:
  - xy defined $\Leftrightarrow \mathbf{r}(x) = \mathbf{d}(y)$,
    - $\mathbf{d}(xy) = \mathbf{d}(x)$ and $\mathbf{r}(xy) = \mathbf{r}(y)$,

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- Think of $\mathcal{C}$ as a partial monoid...... A set with:
  - a partial product $xy = x \circ y$ $(x, y \in \mathcal{C})$,
  - domain/range functions $\mathbf{d}, \mathbf{r} : \mathcal{C} \to \mathbb{N}$,
  - many identity elements ι_n $(n \in \mathbb{N})$ with $\mathbf{d}(\iota_n) = \mathbf{r}(\iota_n) = n$.
- Properties/axioms:
  - xy defined $\Leftrightarrow$ $\mathbf{r}(x) = \mathbf{d}(y)$,
    - $\mathbf{d}(xy) = \mathbf{d}(x)$ and $\mathbf{r}(xy) = \mathbf{r}(y)$,
  - $(xy)z = x(yz)$ when defined,

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- Think of $\mathcal{C}$ as a partial monoid...... A set with:
  - a partial product $xy = x \circ y$ $(x, y \in \mathcal{C})$,
  - domain/range functions $\mathbf{d}, \mathbf{r} : \mathcal{C} \to \mathbb{N}$,
  - many identity elements ι_n $(n \in \mathbb{N})$ with $\mathbf{d}(\iota_n) = \mathbf{r}(\iota_n) = n$.
- Properties/axioms:
  - xy defined $\Leftrightarrow \mathbf{r}(x) = \mathbf{d}(y)$,
    - $\mathbf{d}(xy) = \mathbf{d}(x)$ and $\mathbf{r}(xy) = \mathbf{r}(y)$,
  - $(xy)z = x(yz)$ when defined,
  - $\iota_m \circ x = x = x \circ \iota_n$ when defined.

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \dots\}$.
- Think of $\mathcal{C}$ as a partial monoid...... A set with:
  - a partial product $xy = x \circ y$ $(x, y \in \mathcal{C})$,
  - domain/range functions $\mathbf{d}, \mathbf{r} : \mathcal{C} \to \mathbb{N}$,
  - many identity elements ι_n $(n \in \mathbb{N})$ with $\mathbf{d}(\iota_n) = \mathbf{r}(\iota_n) = n$.
- Properties/axioms:
  - xy defined $\Leftrightarrow$ $\mathbf{r}(x) = \mathbf{d}(y)$,
    - $\mathbf{d}(xy) = \mathbf{d}(x)$ and $\mathbf{r}(xy) = \mathbf{r}(y)$,
  - $(xy)z = x(yz)$ when defined,
  - $\iota_m \circ x = x = x \circ \iota_n$ when defined.
- Morphism sets: $\mathcal{C}_{m,n} = \{x \in \mathcal{C} : \mathbf{d}(x) = m,\ \mathbf{r}(x) = n\}$.

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- Think of $\mathcal{C}$ as a partial monoid...... A set with:
  - a partial product $xy = x \circ y$ $(x, y \in \mathcal{C})$,
  - domain/range functions $\mathbf{d}, \mathbf{r} : \mathcal{C} \to \mathbb{N}$,
  - many identity elements ι_n $(n \in \mathbb{N})$ with $\mathbf{d}(\iota_n) = \mathbf{r}(\iota_n) = n$.
- Properties/axioms:
  - xy defined $\Leftrightarrow \mathbf{r}(x) = \mathbf{d}(y)$,
    - $\mathbf{d}(xy) = \mathbf{d}(x)$ and $\mathbf{r}(xy) = \mathbf{r}(y)$,
  - $(xy)z = x(yz)$ when defined,
  - $\iota_m \circ x = x = x \circ \iota_n$ when defined.
- Morphism sets: $\mathcal{C}_{m,n} = \{x \in \mathcal{C} : \mathbf{d}(x) = m,\ \mathbf{r}(x) = n\}$.
- Endomorphism monoids: $\mathcal{C}_n = \mathcal{C}_{n,n}$.

Categories

- For us: each category $\mathcal{C}$ has object set $\mathbb{N} = \{0, 1, 2, \dots\}$.
- Think of $\mathcal{C}$ as a partial monoid...... A set with:
  - a partial product $xy = x \circ y$ $(x, y \in \mathcal{C})$,
  - domain/range functions $\mathbf{d}, \mathbf{r} : \mathcal{C} \to \mathbb{N}$,
  - many identity elements ι_n $(n \in \mathbb{N})$ with $\mathbf{d}(\iota_n) = \mathbf{r}(\iota_n) = n$.
- Properties/axioms:
  - xy defined $\Leftrightarrow \mathbf{r}(x) = \mathbf{d}(y)$,
    - $\mathbf{d}(xy) = \mathbf{d}(x)$ and $\mathbf{r}(xy) = \mathbf{r}(y)$,
  - $(xy)z = x(yz)$ when defined,
  - $\iota_m \circ x = x = x \circ \iota_n$ when defined.
- Morphism sets: $\mathcal{C}_{m,n} = \{x \in \mathcal{C} : \mathbf{d}(x) = m, \ \mathbf{r}(x) = n\}$.
- Endomorphism monoids: $\mathcal{C}_n = \mathcal{C}_{n,n}$.
- Familiar example: $\mathcal{M} = \{$all (finite) matrices over $\mathbb{R}\}$.

Transformation categories

Transformation categories

- Recall: $\mathbf{n} = \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

Transformation categories

- Recall: $\mathbf{n} = \{1, \ldots, n\}$ for $n \in \mathbb{N}$.
- The full transformation category:

$$\mathcal{T} = \{\text{functions } \mathbf{m} \to \mathbf{n} : m, n \in \mathbb{N}\}.$$

Transformation categories

- Recall: $\mathbf{n} = \{1, \ldots, n\}$ for $n \in \mathbb{N}$.
- The full transformation category:

$$\mathcal{T} = \{\text{functions } \mathbf{m} \to \mathbf{n} : m, n \in \mathbb{N}\}.$$

  - Morphism sets: $\mathcal{T}_{m,n} = \{\text{functions } \mathbf{m} \to \mathbf{n}\}$.

Transformation categories

- Recall: $\mathbf{n} = \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

- The full transformation category:

$$\mathcal{T} = \{\text{functions } \mathbf{m} \to \mathbf{n} : m, n \in \mathbb{N}\}.$$

  - Morphism sets: $\mathcal{T}_{m,n} = \{\text{functions } \mathbf{m} \to \mathbf{n}\}$.
  - Endomorphism monoids: $\mathcal{T}_n = \mathcal{T}_{n,n} = \{\text{functions } \mathbf{n} \to \mathbf{n}\}$.

Transformation categories

- Recall: $\mathbf{n} = \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

- The full transformation category:

$$\mathcal{T} = \{\text{functions } \mathbf{m} \to \mathbf{n} : m, n \in \mathbb{N}\}.$$

  - Morphism sets: $\mathcal{T}_{m,n} = \{\text{functions } \mathbf{m} \to \mathbf{n}\}$.
  - Endomorphism monoids: $\mathcal{T}_n = \mathcal{T}_{n,n} = \{\text{functions } \mathbf{n} \to \mathbf{n}\}$.

- The partial transformation category:

$$\mathcal{PT} = \{\text{partial functions } \mathbf{m} \to \mathbf{n} : m, n \in \mathbb{N}\}.$$

  - Endomorphism monoids: $\mathcal{PT}_n$.

Transformation categories

- Recall: $\mathbf{n} = \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

- The full transformation category:

$$\mathcal{T} = \{\text{functions } \mathbf{m} \to \mathbf{n} : m, n \in \mathbb{N}\}.$$

  - Morphism sets: $\mathcal{T}_{m,n} = \{\text{functions } \mathbf{m} \to \mathbf{n}\}$.
  - Endomorphism monoids: $\mathcal{T}_n = \mathcal{T}_{n,n} = \{\text{functions } \mathbf{n} \to \mathbf{n}\}$.

- The partial transformation category:

$$\mathcal{PT} = \{\text{partial functions } \mathbf{m} \to \mathbf{n} : m, n \in \mathbb{N}\}.$$

  - Endomorphism monoids: $\mathcal{PT}_n$.

- The symmetric inverse category:

$$\mathcal{I} = \{\text{partial bijections } \mathbf{m} \to \mathbf{n} : m, n \in \mathbb{N}\}.$$

  - Endomorphism monoids: $\mathcal{I}_n$.

Diagram categories — $\mathcal{P}$

Diagram categories — $\mathcal{P}$

- My original motivation.

Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.

Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$

Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$
- Eg: $\alpha = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 5'\}, \{2'\}, \{3'\}\Big\} \in \mathcal{P}_{6,5}$

Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$

$$\equiv \{\text{graphs on } \mathbf{m} \cup \mathbf{n}'\}.$$

- Eg: $\alpha = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 5'\}, \{2'\}, \{3'\}\Big\} \in \mathcal{P}_{6,5}$

Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$

$$\equiv \{\text{graphs on } \mathbf{m} \cup \mathbf{n}'\}.$$

- Eg: $\alpha = \Big\{\{1,3,4'\},\{2,4\},\{5,6,1',5'\},\{2'\},\{3'\}\Big\} \in \mathcal{P}_{6,5}$

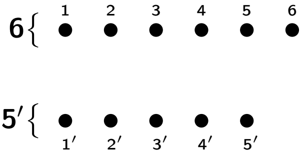


Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \dots, n\}$ and $\mathbf{n}' = \{1', \dots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$

$$\equiv \{\text{graphs on } \mathbf{m} \cup \mathbf{n}'\}.$$

- Eg: $\alpha = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 5'\}, \{2'\}, \{3'\}\Big\} \in \mathcal{P}_{6,5}$

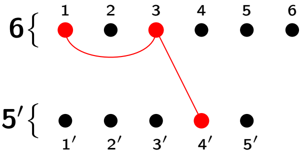


Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$

$$\equiv \{\text{graphs on } \mathbf{m} \cup \mathbf{n}'\}.$$

- Eg: $\alpha = \Big\{ \{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 5'\}, \{2'\}, \{3'\} \Big\} \in \mathcal{P}_{6,5}$

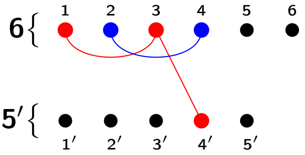


Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$

$$\equiv \{\text{graphs on } \mathbf{m} \cup \mathbf{n}'\}.$$

- Eg: $\alpha = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 5'\}, \{2'\}, \{3'\}\Big\} \in \mathcal{P}_{6,5}$

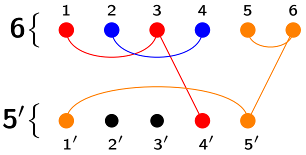


Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$

$$\equiv \{\text{graphs on } \mathbf{m} \cup \mathbf{n}'\}.$$

- Eg: $\alpha = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 5'\}, \{2'\}, \{3'\}\Big\} \in \mathcal{P}_{6,5}$

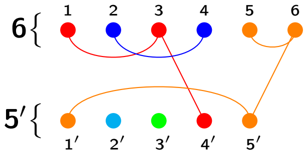


Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$

  $\equiv \{\text{equiv-classes of graphs on } \mathbf{m} \cup \mathbf{n}'\}$.
- Eg: $\alpha = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 5'\}, \{2'\}, \{3'\}\Big\} \in \mathcal{P}_{6,5}$

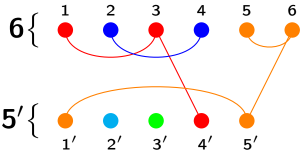


Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$

$$\equiv \{\text{equiv-classes of graphs on } \mathbf{m} \cup \mathbf{n}'\}.$$

- Eg: $\alpha = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 5'\}, \{2'\}, \{3'\}\Big\} \in \mathcal{P}_{6,5}$

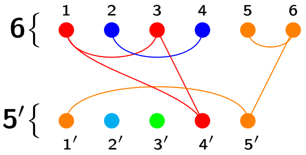


Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$

  $\equiv \{\text{equiv-classes of graphs on } \mathbf{m} \cup \mathbf{n}'\}$.
- Eg: $\alpha = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 5'\}, \{2'\}, \{3'\}\Big\} \in \mathcal{P}_{6,5}$

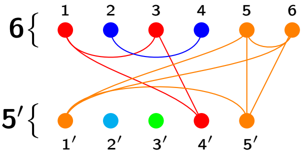
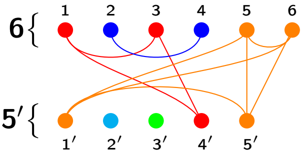


Diagram categories — $\mathcal{P}$

- My original motivation.
- For $n \in \mathbb{N}$, let $\mathbf{n} = \{1, \ldots, n\}$ and $\mathbf{n}' = \{1', \ldots, n'\}$.
- For $m, n \in \mathbb{N}$, let $\mathcal{P}_{m,n} = \{\text{set partitions of } \mathbf{m} \cup \mathbf{n}'\}$

  $\equiv \{\text{equiv-classes of graphs on } \mathbf{m} \cup \mathbf{n}'\}$.
- Eg: $\alpha = \Big\{\{1, 3, 4'\}, \{2, 4\}, \{5, 6, 1', 5'\}, \{2'\}, \{3'\}\Big\} \in \mathcal{P}_{6,5}$



- The partition category is $\mathcal{P} = \bigcup_{m,n \in \mathbb{N}} \mathcal{P}_{m,n}$.

Diagram categories — composition in $\mathcal{P}$

Diagram categories — composition in $\mathcal{P}$

Let $\alpha \in \mathcal{P}_{m,n}$ and $\beta \in \mathcal{P}_{n,q}$.

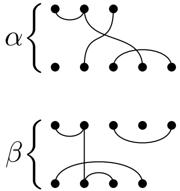


Diagram categories — composition in $\mathcal{P}$

Let $\alpha \in \mathcal{P}_{m,n}$ and $\beta \in \mathcal{P}_{n,q}$. To calculate $\alpha\beta = \alpha \circ \beta \in \mathcal{P}_{m,q}$:

(1) connect bottom of α to top of β,

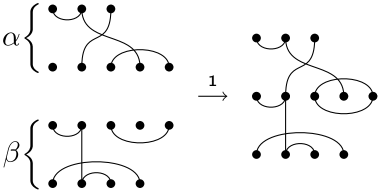


Diagram categories — composition in $\mathcal{P}$

Let $\alpha \in \mathcal{P}_{m,n}$ and $\beta \in \mathcal{P}_{n,q}$. To calculate $\alpha\beta = \alpha \circ \beta \in \mathcal{P}_{m,q}$:

(1) connect bottom of α to top of β,

(2) remove middle vertices and floating components,

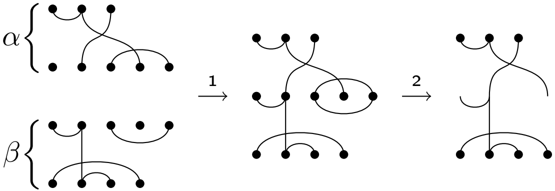


Diagram categories — composition in $\mathcal{P}$

Let $\alpha \in \mathcal{P}_{m,n}$ and $\beta \in \mathcal{P}_{n,q}$. To calculate $\alpha\beta = \alpha \circ \beta \in \mathcal{P}_{m,q}$:

(1) connect bottom of α to top of β,

(2) remove middle vertices and floating components,

(3) smooth out resulting graph to obtain $\alpha\beta$.

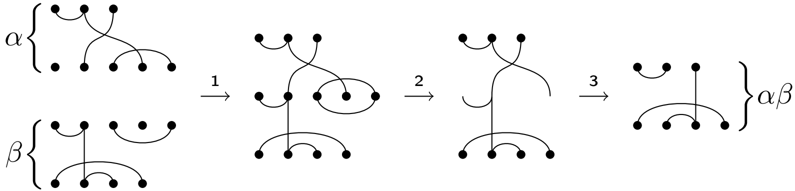


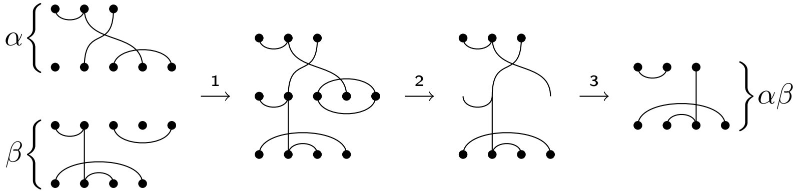
Diagram categories — composition in $\mathcal{P}$

Let $\alpha \in \mathcal{P}_{m,n}$ and $\beta \in \mathcal{P}_{n,q}$. To calculate $\alpha\beta = \alpha \circ \beta \in \mathcal{P}_{m,q}$:

(1) connect bottom of α to top of β,

(2) remove middle vertices and floating components,

(3) smooth out resulting graph to obtain $\alpha\beta$.



- Endomorphism monoids are partition monoids $\mathcal{P}_n = \mathcal{P}_{n,n}$.

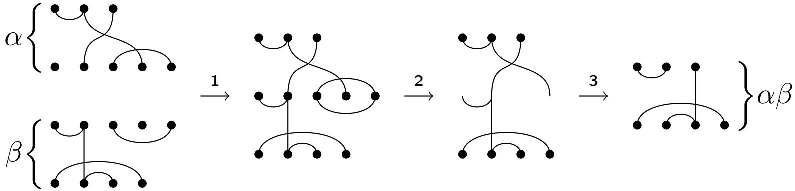
Diagram categories — composition in $\mathcal{P}$

Let $\alpha \in \mathcal{P}_{m,n}$ and $\beta \in \mathcal{P}_{n,q}$. To calculate $\alpha\beta = \alpha \circ \beta \in \mathcal{P}_{m,q}$:

(1) connect bottom of α to top of β,

(2) remove middle vertices and floating components,

(3) smooth out resulting graph to obtain $\alpha\beta$.



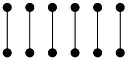
- Endomorphism monoids are partition monoids $\mathcal{P}_n = \mathcal{P}_{n,n}$.
- Identities: $\iota_6 =$  .

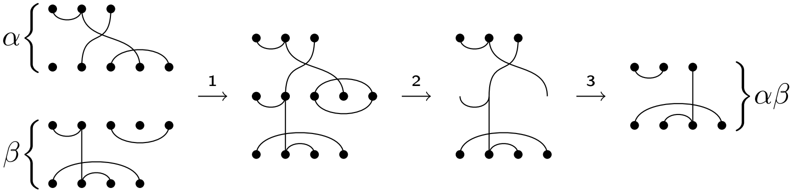
Diagram categories — composition in $\mathcal{P}$

Let $\alpha \in \mathcal{P}_{m,n}$ and $\beta \in \mathcal{P}_{n,q}$. To calculate $\alpha\beta = \alpha \circ \beta \in \mathcal{P}_{m,q}$:

(1) connect bottom of α to top of β,

(2) remove middle vertices and floating components,

(3) smooth out resulting graph to obtain $\alpha\beta$.



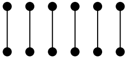
- Endomorphism monoids are partition monoids $\mathcal{P}_n = \mathcal{P}_{n,n}$.
- Identities: $\iota_6 =$  .
- There are linear/twisted versions as well...

Diagram categories — $\mathcal{B}$ and $\mathcal{TL}$

Diagram categories — $\mathcal{B}$ and $\mathcal{TL}$

- The Brauer category $\mathcal{B} = \{$partitions with blocks of size 2$\}$:

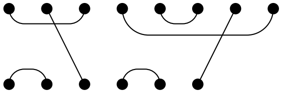
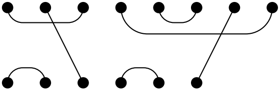


Diagram categories — $\mathcal{B}$ and $\mathcal{TL}$

- The Brauer category $\mathcal{B} = \{$partitions with blocks of size 2$\}$:



- The Temperley-Lieb category $\mathcal{TL} = \{$planar Brauer partitions$\}$:

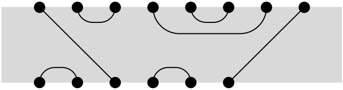
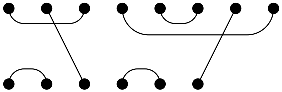
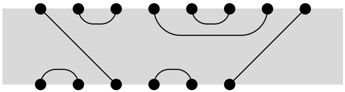


Diagram categories — $\mathcal{B}$ and $\mathcal{TL}$

- The Brauer category $\mathcal{B} = \{$partitions with blocks of size 2$\}$:



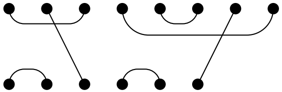
- The Temperley-Lieb category $\mathcal{TL} = \{$planar Brauer partitions$\}$:



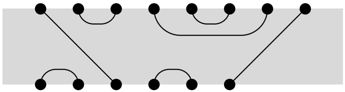
- These are subcategories of $\mathcal{P}$.

Diagram categories — $\mathcal{B}$ and $\mathcal{TL}$

- The Brauer category $\mathcal{B} = \{$partitions with blocks of size 2$\}$:



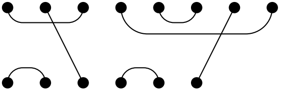
- The Temperley-Lieb category $\mathcal{TL} = \{$planar Brauer partitions$\}$:



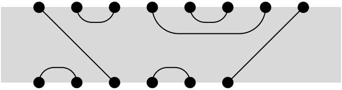
- These are subcategories of $\mathcal{P}$.
- Brauer and Temperley-Lieb monoids: $\mathcal{B}_n$ and $\mathcal{TL}_n$.

Diagram categories — $\mathcal{B}$ and $\mathcal{TL}$

- The Brauer category $\mathcal{B} = \{$partitions with blocks of size 2$\}$:



- The Temperley-Lieb category $\mathcal{TL} = \{$planar Brauer partitions$\}$:



- These are subcategories of $\mathcal{P}$.
- Brauer and Temperley-Lieb monoids: $\mathcal{B}_n$ and $\mathcal{TL}_n$.
- $\mathcal{B}_{m,n} = \mathcal{TL}_{m,n} = \varnothing$ if m and n have opposite parities!

Diagram categories — what are they for?

Diagram categories — what are they for?

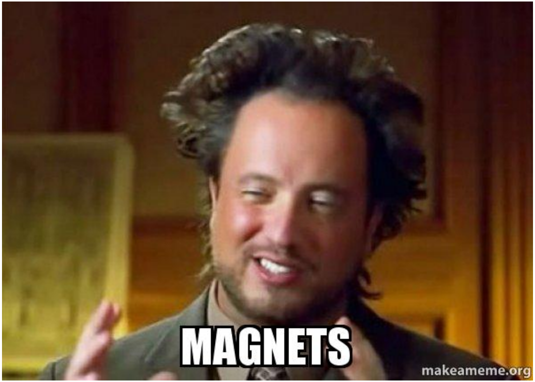
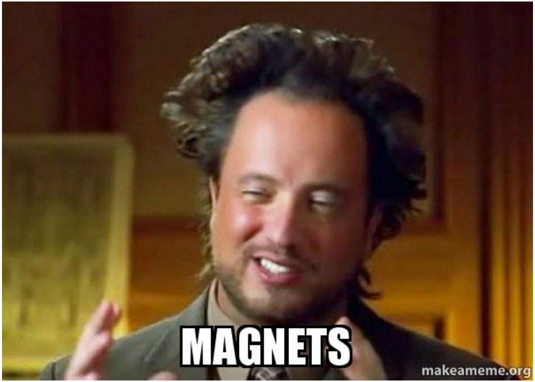
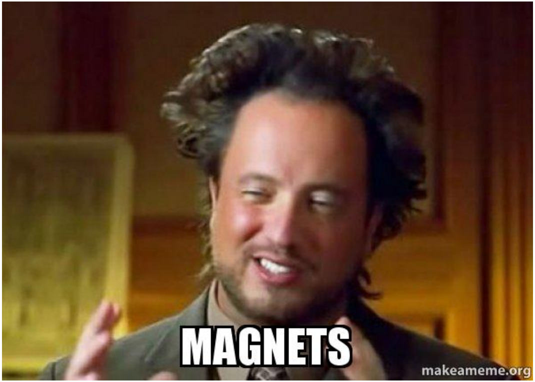


Diagram categories — what are they for?



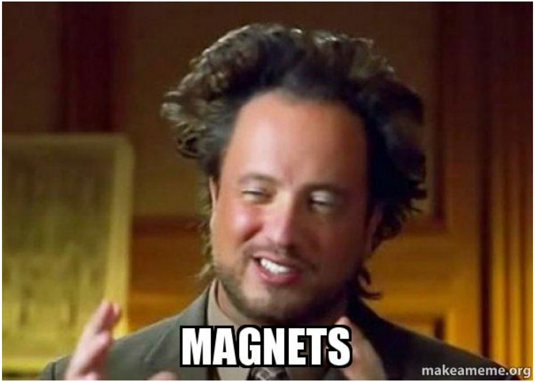
- knot theory, representation theory, category theory, combinatorics...
- computer science, theoretical physics, biology...

Diagram categories — what are they for?



- knot theory, representation theory, category theory, combinatorics...
- computer science, theoretical physics, biology...
- semigroup theory...

Diagram categories — what are they for?



- knot theory, representation theory, category theory, combinatorics...
- computer science, theoretical physics, biology...
- semigroup theory...
- fun!

Diagram categories — presentations

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$  , $\quad \cap \equiv$  , $\quad I = \iota_1 \equiv$  .

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$ , $\cap \equiv$ , $I = \iota_1 \equiv$ .

What?!

Diagram categories — tensor operation

Diagram categories — tensor operation

- The categories $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ have another operation, $\oplus$:

Diagram categories — tensor operation

- The categories $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ have another operation, $\oplus$:

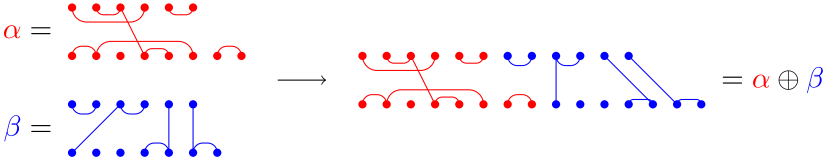
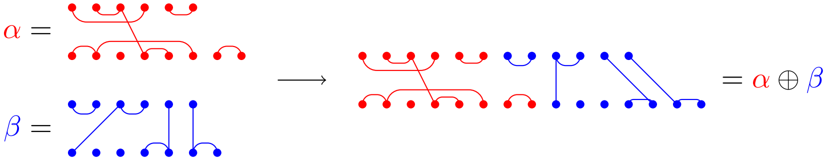


Diagram categories — tensor operation

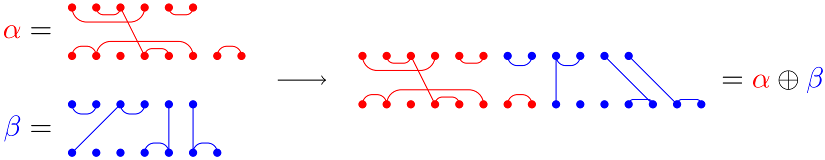
- The categories $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ have another operation, $\oplus$:



- $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ are all (strict) tensor categories.

Diagram categories — tensor operation

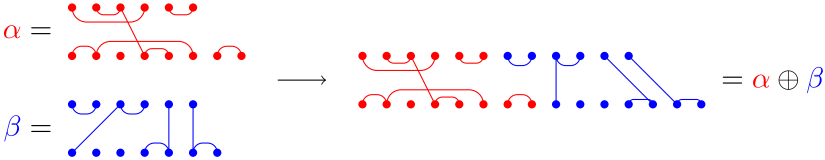
- The categories $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ have another operation, $\oplus$:



- $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ are all (strict) tensor categories.
  - $\mathbf{d}(\alpha \oplus \beta) = \mathbf{d}(\alpha) + \mathbf{d}(\beta)$ and $\mathbf{r}(\alpha \oplus \beta) = \mathbf{r}(\alpha) + \mathbf{r}(\beta)$.
  - $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$,
  - $\alpha \oplus \iota_0 = \alpha = \iota_0 \oplus \alpha$,
  - $\iota_m \oplus \iota_n = \iota_{m+n}$,
  - $(\alpha \circ \beta) \oplus (\gamma \circ \delta) = (\alpha \oplus \gamma) \circ (\beta \oplus \delta)$.

Diagram categories — tensor operation

- The categories $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ have another operation, $\oplus$:



- $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ are all (strict) tensor categories.
  - $\mathbf{d}(\alpha \oplus \beta) = \mathbf{d}(\alpha) + \mathbf{d}(\beta)$ and $\mathbf{r}(\alpha \oplus \beta) = \mathbf{r}(\alpha) + \mathbf{r}(\beta)$.
  - $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$,
  - $\alpha \oplus \iota_0 = \alpha = \iota_0 \oplus \alpha$,
  - $\iota_m \oplus \iota_n = \iota_{m+n}$,
  - $(\alpha \circ \beta) \oplus (\gamma \circ \delta) = (\alpha \oplus \gamma) \circ (\beta \oplus \delta)$.

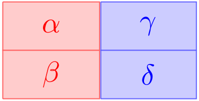
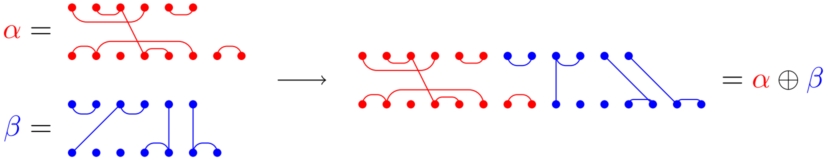


Diagram categories — tensor operation

- The categories $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ have another operation, $\oplus$:



- $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ are all (strict) tensor categories.
  - $\mathbf{d}(\alpha \oplus \beta) = \mathbf{d}(\alpha) + \mathbf{d}(\beta)$ and $\mathbf{r}(\alpha \oplus \beta) = \mathbf{r}(\alpha) + \mathbf{r}(\beta)$.
  - $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$,
  - $\alpha \oplus \iota_0 = \alpha = \iota_0 \oplus \alpha$,
  - $\iota_m \oplus \iota_n = \iota_{m+n}$,
  - $(\alpha \circ \beta) \oplus (\gamma \circ \delta) = (\alpha \oplus \gamma) \circ (\beta \oplus \delta)$.

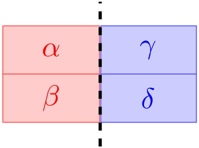
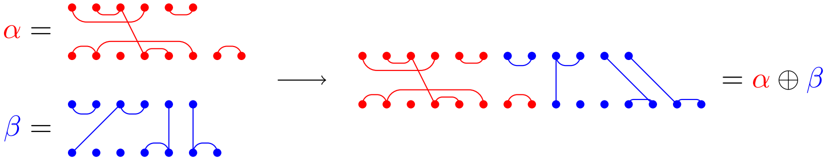


Diagram categories — tensor operation

- The categories $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ have another operation, $\oplus$:



- $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ are all (strict) tensor categories.
  - $\mathbf{d}(\alpha \oplus \beta) = \mathbf{d}(\alpha) + \mathbf{d}(\beta)$ and $\mathbf{r}(\alpha \oplus \beta) = \mathbf{r}(\alpha) + \mathbf{r}(\beta)$.
  - $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$,
  - $\alpha \oplus \iota_0 = \alpha = \iota_0 \oplus \alpha$,
  - $\iota_m \oplus \iota_n = \iota_{m+n}$,
  - $(\alpha \circ \beta) \oplus (\gamma \circ \delta) = (\alpha \oplus \gamma) \circ (\beta \oplus \delta)$.

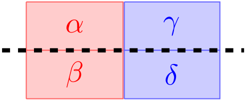
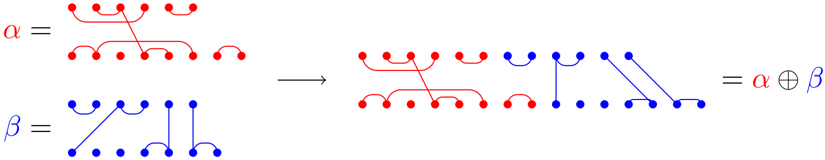
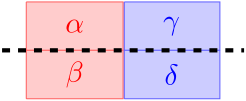


Diagram categories — tensor operation

- The categories $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ have another operation, $\oplus$:



- $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{TL}$ are all (strict) tensor categories.
  - $\mathbf{d}(\alpha \oplus \beta) = \mathbf{d}(\alpha) + \mathbf{d}(\beta)$ and $\mathbf{r}(\alpha \oplus \beta) = \mathbf{r}(\alpha) + \mathbf{r}(\beta)$.
  - $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$,
  - $\alpha \oplus \iota_0 = \alpha = \iota_0 \oplus \alpha$,
  - $\iota_m \oplus \iota_n = \iota_{m+n}$,
  - $(\alpha \circ \beta) \oplus (\gamma \circ \delta) = (\alpha \oplus \gamma) \circ (\beta \oplus \delta)$.



- The categories $\mathcal{T}$, $\mathcal{PT}$ and $\mathcal{I}$ are also tensor categories.

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$  , $\quad \cap \equiv$  , $\quad I = \iota_1 \equiv$  .

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$ , $\cap \equiv$ , $I = \iota_1 \equiv$ .

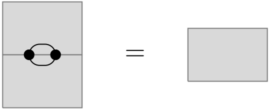


Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$ , $\cap \equiv$ , $I = \iota_1 \equiv$ .

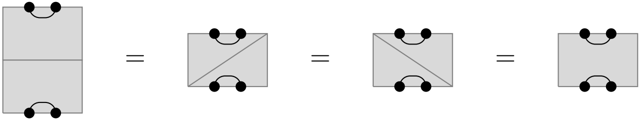


Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$ , $\quad \cap \equiv$ , $\quad I = \iota_1 \equiv$ .

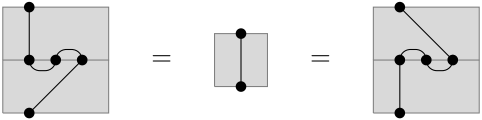


Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$  , $\quad \cap \equiv$  , $\quad I = \iota_1 \equiv$  .

- (R2) is my favourite relation.

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$ , $\cap \equiv$ , $I = \iota_1 \equiv$ .

- (R2) is my favourite relation.
- But it's actually unnecessary :-(

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$ , $\quad \cap \equiv$ , $\quad I = \iota_1 \equiv$ .

- (R2) is my favourite relation.
- But it's actually unnecessary :-(
- e.g., $U \oplus \cap$

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$ , $\quad \cap \equiv$ , $\quad I = \iota_1 \equiv$ .

- (R2) is my favourite relation.
- But it's actually unnecessary :-(
- e.g., $U \oplus \cap = (U \circ \iota_0) \oplus (\iota_0 \circ \cap)$

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$ , $\quad \cap \equiv$ , $\quad I = \iota_1 \equiv$ .

- (R2) is my favourite relation.
- But it's actually unnecessary :-(
- e.g., $U \oplus \cap = (U \circ \iota_0) \oplus (\iota_0 \circ \cap) = (U \oplus \iota_0) \circ (\iota_0 \oplus \cap)$

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$U \circ \cap = U \oplus \cap = \cap \oplus U, \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R3}$$

$U \equiv$ , $\quad \cap \equiv$ , $\quad I = \iota_1 \equiv$ .

- (R2) is my favourite relation.
- But it's actually unnecessary :-(
- e.g., $U \oplus \cap = (U \circ \iota_0) \oplus (\iota_0 \circ \cap) = (U \oplus \iota_0) \circ (\iota_0 \oplus \cap) = U \circ \cap$.

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R2}$$

$U \equiv$  , $\cap \equiv$  , $I = \iota_1 \equiv$  .

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R2}$$

$U \equiv$  , $\cap \equiv$  , $I = \iota_1 \equiv$  .

- Can you show that U and $\cap$ (and I) generate $\mathcal{TL}$?

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$$\cap \circ U = \iota_0, \tag{R1}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U). \tag{R2}$$

$U \equiv$  , $\quad \cap \equiv$  , $\quad I = \iota_1 \equiv$  .

- Can you show that U and $\cap$ (and I) generate $\mathcal{TL}$?

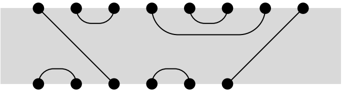
 $= \cap \oplus I \oplus U \oplus \cap \oplus ((I \oplus U \oplus I) \circ U) \oplus I$!!

Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$$X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap, \tag{R1}$$

$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U), \tag{R3}$$

$$(I \oplus X) \circ (U \oplus I) = (X \oplus I) \circ (I \oplus U), \tag{R4}$$

$$(\cap \oplus I) \circ (I \oplus X) = (I \oplus \cap) \circ (X \oplus I). \tag{R5}$$

$X \equiv$ , $\quad U \equiv$ , $\quad \cap \equiv$ , $\quad I = \iota_1 \equiv$ .

Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$$X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap, \qquad \text{(R1)}$$

$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \qquad \text{(R2)}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U), \qquad \text{(R3)}$$

$$(I \oplus X) \circ (U \oplus I) = (X \oplus I) \circ (I \oplus U), \qquad \text{(R4)}$$

$$(\cap \oplus I) \circ (I \oplus X) = (I \oplus \cap) \circ (X \oplus I). \qquad \text{(R5)}$$

$X \equiv$ , $\quad U \equiv$ , $\quad \cap \equiv$ , $\quad I = \iota_1 \equiv$ .

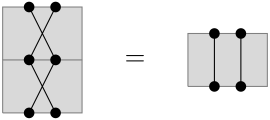


Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$$X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap, \tag{R1}$$

$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U), \tag{R3}$$

$$(I \oplus X) \circ (U \oplus I) = (X \oplus I) \circ (I \oplus U), \tag{R4}$$

$$(\cap \oplus I) \circ (I \oplus X) = (I \oplus \cap) \circ (X \oplus I). \tag{R5}$$

$X \equiv$ , $\quad U \equiv$ , $\quad \cap \equiv$ , $\quad I = \iota_1 \equiv$ .

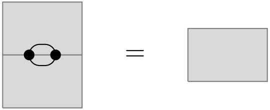


Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$$X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap, \qquad \text{(R1)}$$

$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \qquad \text{(R2)}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U), \qquad \text{(R3)}$$

$$(I \oplus X) \circ (U \oplus I) = (X \oplus I) \circ (I \oplus U), \qquad \text{(R4)}$$

$$(\cap \oplus I) \circ (I \oplus X) = (I \oplus \cap) \circ (X \oplus I). \qquad \text{(R5)}$$

$X \equiv$ , $\quad U \equiv$ , $\quad \cap \equiv$ , $\quad I = \iota_1 \equiv$ .

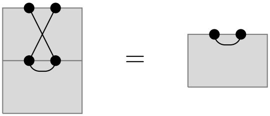


Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$$X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap, \qquad \text{(R1)}$$

$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \qquad \text{(R2)}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U), \qquad \text{(R3)}$$

$$(I \oplus X) \circ (U \oplus I) = (X \oplus I) \circ (I \oplus U), \qquad \text{(R4)}$$

$$(\cap \oplus I) \circ (I \oplus X) = (I \oplus \cap) \circ (X \oplus I). \qquad \text{(R5)}$$

$X \equiv$ , $\quad U \equiv$ , $\quad \cap \equiv$ , $\quad I = \iota_1 \equiv$ .

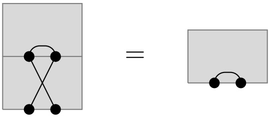


Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$$X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap, \quad \text{(R1)}$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \quad \text{(R2)}$$
$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U), \quad \text{(R3)}$$
$$(I \oplus X) \circ (U \oplus I) = (X \oplus I) \circ (I \oplus U), \quad \text{(R4)}$$
$$(\cap \oplus I) \circ (I \oplus X) = (I \oplus \cap) \circ (X \oplus I). \quad \text{(R5)}$$

$X \equiv$ , $U \equiv$ , $\cap \equiv$ , $I = \iota_1 \equiv$ .

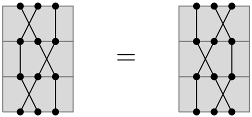


Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$$X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap, \qquad \text{(R1)}$$

$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \qquad \text{(R2)}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U), \qquad \text{(R3)}$$

$$(I \oplus X) \circ (U \oplus I) = (X \oplus I) \circ (I \oplus U), \qquad \text{(R4)}$$

$$(\cap \oplus I) \circ (I \oplus X) = (I \oplus \cap) \circ (X \oplus I). \qquad \text{(R5)}$$

$X \equiv$ , $U \equiv$ , $\cap \equiv$ , $I = \iota_1 \equiv$ .

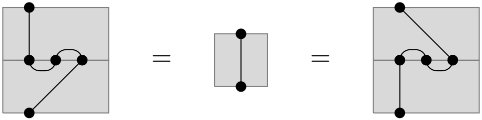


Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$$X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap, \tag{R1}$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \tag{R2}$$
$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U), \tag{R3}$$
$$(I \oplus X) \circ (U \oplus I) = (X \oplus I) \circ (I \oplus U), \tag{R4}$$
$$(\cap \oplus I) \circ (I \oplus X) = (I \oplus \cap) \circ (X \oplus I). \tag{R5}$$

$X \equiv$  , $U \equiv$  , $\cap \equiv$  , $I = \iota_1 \equiv$  .

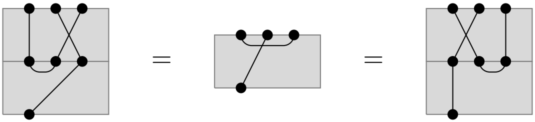


Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$$X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap, \tag{R1}$$

$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \tag{R2}$$

$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U), \tag{R3}$$

$$(I \oplus X) \circ (U \oplus I) = (X \oplus I) \circ (I \oplus U), \tag{R4}$$

$$(\cap \oplus I) \circ (I \oplus X) = (I \oplus \cap) \circ (X \oplus I). \tag{R5}$$

$X \equiv$ , $U \equiv$ , $\cap \equiv$ , $I = \iota_1 \equiv$ .

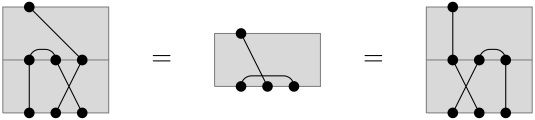


Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$$X \circ X = II, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap, \tag{R1}$$

$$XI \circ IX \circ XI = IX \circ XI \circ IX, \tag{R2}$$

$$I\cap \circ UI = I = \cap I \circ IU, \tag{R3}$$

$$IX \circ UI = XI \circ IU, \tag{R4}$$

$$\cap I \circ IX = I\cap \circ XI. \tag{R5}$$

$X \equiv$ , $\quad U \equiv$ , $\quad \cap \equiv$ , $\quad I \equiv$ .

Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$$X \circ X = II, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap, \qquad \text{(R1)}$$

$$XI \circ IX \circ XI = IX \circ XI \circ IX, \qquad \text{(R2)}$$

$$I\cap \circ UI = I = \cap I \circ IU, \qquad \text{(R3)}$$

$$IX \circ UI = XI \circ IU, \qquad \text{(R4)}$$

$$\cap I \circ IX = I\cap \circ XI. \qquad \text{(R5)}$$

$X \equiv$  , $\quad U \equiv$  , $\quad \cap \equiv$  , $\quad I \equiv$  .

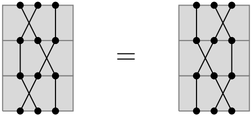


Diagram categories — presentations

Theorem

The partition category $\mathcal{P} \cong \langle X, D, U, \cap : R \rangle$.

$$X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \tag{R1}$$

$$D \circ D = D = D \circ X = X \circ D, \tag{R2}$$

$$(D \oplus I) \circ (I \oplus D) = (I \oplus D) \circ (D \oplus I), \tag{R3}$$

$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \tag{R4}$$

$$(X \oplus I) \circ (I \oplus D) \circ (X \oplus I) = (I \oplus X) \circ (D \oplus I) \circ (I \oplus X), \tag{R5}$$

$$X \circ (I \oplus U) = U \oplus I, \quad (I \oplus \cap) \circ X = \cap \oplus I, \tag{R6}$$

$$(I \oplus \cap) \circ D \circ (I \oplus U) = I, \quad D \circ (I \oplus U \oplus \cap) \circ D = D. \tag{R7}$$

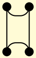
$X \equiv$ , $D \equiv$ , $U \equiv$ , $\cap \equiv$ , $I = \iota_1 \equiv$ .

Diagram categories — presentations

Theorem (Comes, 2017)

The partition category $\mathcal{P} \cong \langle X, U, \cap, V, \Lambda : R \rangle$.

$$X \circ X = I \oplus I, \quad \Lambda \circ V = I, \quad \cap \circ U = \iota_0, \tag{R1}$$
$$X \circ V = V, \quad \Lambda \circ X = \Lambda, \tag{R2}$$
$$X \circ (I \oplus U) = U \oplus I, \quad (I \oplus \cap) \circ X = \cap \oplus I, \tag{R3}$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \tag{R4}$$
$$(I \oplus V) \circ X = (X \oplus I) \circ (I \oplus X) \circ (V \oplus I), \tag{R5}$$
$$X \circ (I \oplus \Lambda) = (\Lambda \oplus I) \circ (I \oplus X) \circ (X \oplus I), \tag{R6}$$
$$\Lambda \circ (I \oplus U) = I = (I \oplus \cap) \circ V, \tag{R7}$$
$$(\Lambda \oplus I) \circ (I \oplus V) = V \circ \Lambda = (I \oplus \Lambda) \circ (V \oplus I). \tag{R8}$$

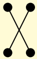
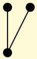
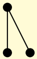
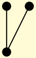
$X \equiv$ , $U \equiv$ , $\cap \equiv$ , $V \equiv$ , $\Lambda \equiv$ , $I \equiv$ .

Diagram categories — presentations

Theorem (Comes, 2017)

The partition category $\mathcal{P} \cong \langle X, U, \cap, V, \Lambda : R \rangle$.

$X \equiv$ , $U \equiv$ , $\cap \equiv$ , $V \equiv$ , $\Lambda \equiv$ .

- Jellyfish partition categories
  - Jonathan Comes
  - *Algebras and representation theory*, to appear.

Diagram categories — presentations

Theorem (Comes, 2017)

The partition category $\mathcal{P} \cong \langle X, U, \cap, V, \Lambda : R \rangle$.

- Jellyfish partition categories
  - Jonathan Comes
  - *Algebras and representation theory*, to appear.
- The proof relies on some heavy machinery:
  - Frobenius algebras and cobordism categories (Abrams, Kock).

Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$X \equiv$  , $U \equiv$  , $\cap \equiv$  .

- The Brauer category and invariant theory
  - Gus Lehrer and Ruibin Zhang
  - *J. European Mathematical Society*, 2015.

Diagram categories — presentations

Theorem (cf. Lehrer and Zhang, 2015)

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : R \rangle$.

$X \equiv$ , $\quad U \equiv$ , $\quad \cap \equiv$ .

- The Brauer category and invariant theory
  - Gus Lehrer and Ruibin Zhang
  - *J. European Mathematical Society*, 2015.
- Quite detailed proof from scratch.

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$U \equiv$  , $\quad \cap \equiv$  .

- Many proofs have been given.

Diagram categories — presentations

Theorem (folklore?)

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : R \rangle$.

$U \equiv$  , $\quad \cap \equiv$  .

- Many proofs have been given.
- The level of rigour varies...

Diagram categories — presentations

Original goal

Find a general framework for presentations of diagram categories.

Diagram categories — presentations

Original goal

Find a general framework for presentations of diagram categories.

- Completely rigorous.

Diagram categories — presentations

Original goal

Find a general framework for presentations of diagram categories.

- Completely rigorous.
- Use known presentations for diagram monoids.

Diagram categories — presentations

Original goal

Find a general framework for presentations of diagram categories.

- Completely rigorous.
- Use known presentations for diagram monoids.

Eventually

- A method that works.

Diagram categories — presentations

Original goal

Find a general framework for presentations of diagram categories.

- Completely rigorous.
- Use known presentations for diagram monoids.

Eventually

- A method that works.
- Key properties axiomatised.

Diagram categories — presentations

Original goal

Find a general framework for presentations of diagram categories.

- Completely rigorous.
- Use known presentations for diagram monoids.

Eventually

- A method that works.
- Key properties axiomatised.
- General results, applicable to many categories.

Diagram categories — presentations

Original goal

Find a general framework for presentations of diagram categories.

- Completely rigorous.
- Use known presentations for diagram monoids.

Eventually

- A method that works.
- Key properties axiomatised.
- General results, applicable to many categories.
  - $\mathcal{P}$, $\mathcal{B}$, $\mathcal{TL}$

Diagram categories — presentations

Original goal

Find a general framework for presentations of diagram categories.

- Completely rigorous.
- Use known presentations for diagram monoids.

Eventually

- A method that works.
- Key properties axiomatised.
- General results, applicable to many categories.
  - $\mathcal{P}$, $\mathcal{B}$, $\mathcal{TL}$, $\mathcal{PT}$, $\mathcal{T}$, $\mathcal{I}$

Diagram categories — presentations

Original goal

Find a general framework for presentations of diagram categories.

- Completely rigorous.
- Use known presentations for diagram monoids.

Eventually

- A method that works.
- Key properties axiomatised.
- General results, applicable to many categories.
  - $\mathcal{P}$, $\mathcal{B}$, $\mathcal{TL}$, $\mathcal{PT}$, $\mathcal{T}$, $\mathcal{I}$, $\mathcal{PO}$, $\mathcal{O}$, $\mathcal{OI}$

Diagram categories — presentations

Original goal

Find a general framework for presentations of diagram categories.

- Completely rigorous.
- Use known presentations for diagram monoids.

Eventually

- A method that works.
- Key properties axiomatised.
- General results, applicable to many categories.
  - $\mathcal{P}$, $\mathcal{B}$, $\mathcal{TL}$, $\mathcal{PT}$, $\mathcal{T}$, $\mathcal{I}$, $\mathcal{PO}$, $\mathcal{O}$, $\mathcal{OI}$, $\mathcal{PV}$, $\mathcal{V}$, $\mathcal{IB}$......

Categories — presentations

Categories — presentations

Basic pattern

- Given a tensor category $\mathcal{C}$ (over $\mathbb{N}$).

Categories — presentations

Basic pattern

- Given a tensor category $\mathcal{C}$ (over $\mathbb{N}$).
- Check it satisfies some basic structural properties.

Categories — presentations

Basic pattern

- Given a tensor category $\mathcal{C}$ (over $\mathbb{N}$).
- Check it satisfies some basic structural properties.
- Find presentations for endomorphism monoids $\mathcal{C}_n$.

Categories — presentations

Basic pattern

- Given a tensor category $\mathcal{C}$ (over $\mathbb{N}$).
- Check it satisfies some basic structural properties.
- Find presentations for endomorphism monoids $\mathcal{C}_n$.
- Theorem A: a (big) category presentation $\langle \Gamma : \Omega \rangle$ for $\mathcal{C}$.

Categories — presentations

Basic pattern

- Given a tensor category $\mathcal{C}$ (over $\mathbb{N}$).
- Check it satisfies some basic structural properties.
- Find presentations for endomorphism monoids $\mathcal{C}_n$.
- Theorem A: a (big) category presentation $\langle \Gamma : \Omega \rangle$ for $\mathcal{C}$.
- Theorem B: a (small?) tensor category presentation $\langle \Delta : \Xi \rangle$.

Categories — presentations

Basic pattern

- Given a tensor category $\mathcal{C}$ (over $\mathbb{N}$).
- Check it satisfies some basic structural properties.
- Find presentations for endomorphism monoids $\mathcal{C}_n$.
- Theorem A: a (big) category presentation $\langle \Gamma : \Omega \rangle$ for $\mathcal{C}$.
- Theorem B: a (small?) tensor category presentation $\langle \Delta : \Xi \rangle$.

- $\langle \Delta : \Xi \rangle$ is what we really want.

Categories — presentations

Basic pattern

- Given a tensor category $\mathcal{C}$ (over $\mathbb{N}$).
- Check it satisfies some basic structural properties.
- Find presentations for endomorphism monoids $\mathcal{C}_n$.
- Theorem A: a (big) category presentation $\langle \Gamma : \Omega \rangle$ for $\mathcal{C}$.
- Theorem B: a (small?) tensor category presentation $\langle \Delta : \Xi \rangle$.

- $\langle \Delta : \Xi \rangle$ is what we really want.
- $\langle \Gamma : \Omega \rangle$ is a means to an end.

Categories — presentations

Basic pattern

- Given a tensor category $\mathcal{C}$ (over $\mathbb{N}$).
- Check it satisfies some basic structural properties.
- Find presentations for endomorphism monoids $\mathcal{C}_n$.
- Theorem A: a (big) category presentation $\langle \Gamma : \Omega \rangle$ for $\mathcal{C}$.
- Theorem B: a (small?) tensor category presentation $\langle \Delta : \Xi \rangle$.

- $\langle \Delta : \Xi \rangle$ is what we really want.
- $\langle \Gamma : \Omega \rangle$ is a means to an end.
- The Micky-Ricky-Vicky Trick!

Theorem A — Key assumptions

Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.

Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$

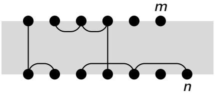
Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$

- $d = 1$ for $\mathcal{C} = \mathcal{P}$.



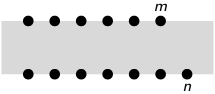
Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$

- $d = 1$ for $\mathcal{C} = \mathcal{P}$.



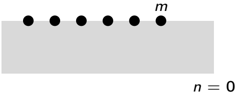
Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$

- $d = 1$ for $\mathcal{C} = \mathcal{P}$.



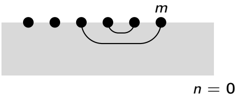
Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$

- $d = 1$ for $\mathcal{C} = \mathcal{P}$.



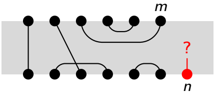
Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$

- $d = 1$ for $\mathcal{C} = \mathcal{P}$.
- $d = 2$ for $\mathcal{C} = \mathcal{B}$.



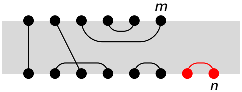
Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$

- $d = 1$ for $\mathcal{C} = \mathcal{P}$.
- $d = 2$ for $\mathcal{C} = \mathcal{B}$.



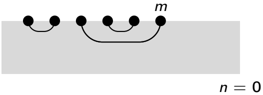
Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$

- $d = 1$ for $\mathcal{C} = \mathcal{P}$.
- $d = 2$ for $\mathcal{C} = \mathcal{B}$.



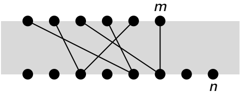
Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$

- $d = 1$ for $\mathcal{C} = \mathcal{P}$.
- $d = 2$ for $\mathcal{C} = \mathcal{B}$.
- $d = 1$ for $\mathcal{C} = \mathcal{T}$?



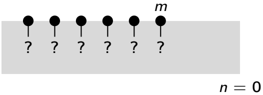
Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$

- $d = 1$ for $\mathcal{C} = \mathcal{P}$.
- $d = 2$ for $\mathcal{C} = \mathcal{B}$.
- $d = 1$ for $\mathcal{C} = \mathcal{T}$?



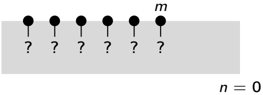
Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$

- $d = 1$ for $\mathcal{C} = \mathcal{P}$.
- $d = 2$ for $\mathcal{C} = \mathcal{B}$.
- $d = 1$ for $\mathcal{C} = \mathcal{T}$?
  - $\mathcal{T}_{m,n} = \varnothing \Leftrightarrow m > 0 = n$.

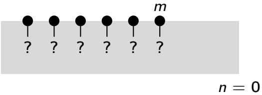


Theorem A — Key assumptions

Assumption 1

- $\mathcal{C}$ is a category over $\mathbb{N}$.
- There is an integer $d \geq 1$ such that

$$\mathcal{C}_{m,n} \neq \varnothing \Leftrightarrow m \equiv n \pmod{d}.$$



- $d = 1$ for $\mathcal{C} = \mathcal{P}$.
- $d = 2$ for $\mathcal{C} = \mathcal{B}$.
- $d = 1$ for $\mathcal{C} = \mathcal{T}$?
  - $\mathcal{T}_{m,n} = \varnothing \Leftrightarrow m > 0 = n$.
  - Things are a little more complicated for $\mathcal{T}$...

Theorem A — Key assumptions

Assumption 2

For each $n \in \mathbb{N}$ there exist $\lambda_n \in \mathcal{C}_{n,n+d}$ and $\rho_n \in \mathcal{C}_{n+d,n}$ such that

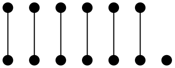
$$\lambda_n \circ \rho_n = \iota_n.$$

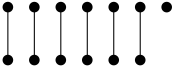
Theorem A — Key assumptions

Assumption 2

For each $n \in \mathbb{N}$ there exist $\lambda_n \in \mathcal{C}_{n,n+d}$ and $\rho_n \in \mathcal{C}_{n+d,n}$ such that

$$\lambda_n \circ \rho_n = \iota_n.$$

$\lambda_n =$  $\in \mathcal{P}_{n,n+1}$

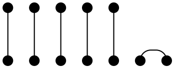
$\rho_n =$  $\in \mathcal{P}_{n+1,n}$

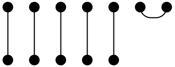
Theorem A — Key assumptions

Assumption 2

For each $n \in \mathbb{N}$ there exist $\lambda_n \in \mathcal{C}_{n,n+d}$ and $\rho_n \in \mathcal{C}_{n+d,n}$ such that

$$\lambda_n \circ \rho_n = \iota_n.$$

$\lambda_n =$  $\in \mathcal{B}_{n,n+2}$

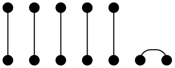
$\rho_n =$  $\in \mathcal{B}_{n+2,n}$

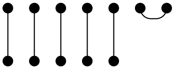
Theorem A — Key assumptions

Assumption 2

For each $n \in \mathbb{N}$ there exist $\lambda_n \in \mathcal{C}_{n,n+d}$ and $\rho_n \in \mathcal{C}_{n+d,n}$ such that

$$\lambda_n \circ \rho_n = \iota_n.$$

$\lambda_n =$  $\in \mathcal{TL}_{n,n+2}$

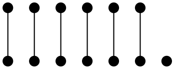
$\rho_n =$  $\in \mathcal{TL}_{n+2,n}$

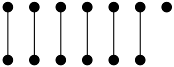
Theorem A — Key assumptions

Assumption 2

For each $n \in \mathbb{N}$ there exist $\lambda_n \in \mathcal{C}_{n,n+d}$ and $\rho_n \in \mathcal{C}_{n+d,n}$ such that

$$\lambda_n \circ \rho_n = \iota_n.$$

$\lambda_n =$  $\in \mathcal{PT}_{n,n+1}$

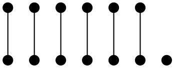
$\rho_n =$  $\in \mathcal{PT}_{n+1,n}$

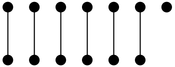
Theorem A — Key assumptions

Assumption 2

For each $n \in \mathbb{N}$ there exist $\lambda_n \in \mathcal{C}_{n,n+d}$ and $\rho_n \in \mathcal{C}_{n+d,n}$ such that

$$\lambda_n \circ \rho_n = \iota_n.$$

$\lambda_n =$  $\in \mathcal{I}_{n,n+1}$

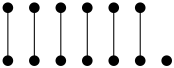
$\rho_n =$  $\in \mathcal{I}_{n+1,n}$

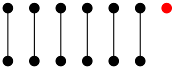
Theorem A — Key assumptions

Assumption 2

For each $n \in \mathbb{N}$ there exist $\lambda_n \in \mathcal{C}_{n,n+d}$ and $\rho_n \in \mathcal{C}_{n+d,n}$ such that

$$\lambda_n \circ \rho_n = \iota_n.$$

$\lambda_n =$  $\in \mathcal{T}_{n,n+1}$

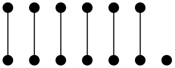
$\rho_n =$  $\in \mathcal{T}_{n+1,n}$

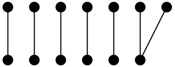
Theorem A — Key assumptions

Assumption 2

For each $n \in \mathbb{N}$ there exist $\lambda_n \in \mathcal{C}_{n,n+d}$ and $\rho_n \in \mathcal{C}_{n+d,n}$ such that

$$\lambda_n \circ \rho_n = \iota_n.$$

$\lambda_n =$  $\in \mathcal{T}_{n,n+1}$

$\rho_n =$  $\in \mathcal{T}_{n+1,n}$

Theorem A — Key assumptions

Assumption 3

We have presentations $\mathcal{C}_n \cong \langle X_n : R_n \rangle$ for each n.

Theorem A — Key assumptions

Assumption 3

We have presentations $\mathcal{C}_n \cong \langle X_n : R_n \rangle$ for each n.

- $\mathcal{T}_n$ (Aĭzenštat), $\mathcal{PT}_n$ and $\mathcal{I}_n$ (Popova),
- $\mathcal{P}_n$ (Halverson and Ram; E),
- $\mathcal{B}_n$ (Kudryatseva and Mazorchuk),
- $\mathcal{TL}_n$ (Jones; Kauffman; Borisavljević, Došen and Petrić),
- $\mathcal{V}_n$ (Lavers), $\mathcal{IB}_n$ (Easdown and Lavers), $\mathcal{PV}_n$ (E).

Theorem A — Key assumptions

Assumption 3

We have presentations $\mathcal{C}_n \cong \langle X_n : R_n \rangle$ for each n.

- $\mathcal{T}_n$ (Aĭzenštat), $\mathcal{PT}_n$ and $\mathcal{I}_n$ (Popova),
- $\mathcal{P}_n$ (Halverson and Ram; E),
- $\mathcal{B}_n$ (Kudryatseva and Mazorchuk),
- $\mathcal{TL}_n$ (Jones; Kauffman; Borisavljević, Došen and Petrić),
- $\mathcal{V}_n$ (Lavers), $\mathcal{IB}_n$ (Easdown and Lavers), $\mathcal{PV}_n$ (E).

Lemma

We have $\mathcal{C} = \langle \Gamma \rangle$, where $\Gamma = \{\lambda_n, \rho_n : n \in \mathbb{N}\} \cup \bigcup_{n \in \mathbb{N}} X_n$.

Theorem A — Key assumptions

Assumption 4

We assume that Ω is a set of relations over Γ such that:

Theorem A — Key assumptions

Assumption 4

We assume that Ω is a set of relations over Γ such that:

- Each relation holds in $\mathcal{C}$.

Theorem A — Key assumptions

Assumption 4

We assume that Ω is a set of relations over Γ such that:

- Each relation holds in $\mathcal{C}$.
- Ω contains:
  - each R_n,

Theorem A — Key assumptions

Assumption 4

We assume that Ω is a set of relations over Γ such that:

- Each relation holds in $\mathcal{C}$.
- Ω contains:
  - each R_n,
  - $\lambda_n \rho_n = \iota_n$,

Theorem A — Key assumptions

Assumption 4

We assume that Ω is a set of relations over Γ such that:

- Each relation holds in $\mathcal{C}$.
- Ω contains:
  - each R_n,
  - $\lambda_n \rho_n = \iota_n$,
  - $\rho_n \lambda_n = w_n$ for some $w_n \in X_{n+d}^*$.

Theorem A — Key assumptions

Assumption 4

We assume that Ω is a set of relations over Γ such that:

- Each relation holds in $\mathcal{C}$.
- Ω contains:
  - each R_n,
  - $\lambda_n \rho_n = \iota_n$,
  - $\rho_n \lambda_n = w_n$ for some $w_n \in X_{n+d}^*$.
- For all $w \in X_{n+d}^*$, $\lambda_n w \rho_n \sim w'$ for some $w' \in X_n^*$.

Theorem A

Theorem A

If Assumptions 1–4 hold, then $\mathcal{C}$ has presentation $\langle \Gamma : \Omega \rangle$.

Theorem A

Theorem A

If Assumptions 1–4 hold, then $\mathcal{C}$ has presentation $\langle \Gamma : \Omega \rangle$.

- Theorem A applies to *many* categories.

Theorem A

Theorem A

If Assumptions 1–4 hold, then $\mathcal{C}$ has presentation $\langle \Gamma : \Omega \rangle$.

- Theorem A applies to *many* categories.
  - $\mathcal{P}$, $\mathcal{B}$, $\mathcal{TL}$, $\mathcal{PT}$, $\mathcal{T}^+$, $\mathcal{I}$, $\mathcal{PO}$, $\mathcal{O}^+$, $\mathcal{OI}$, $\mathcal{PV}$, $\mathcal{V}^+$, $\mathcal{IB}$......

Theorem A

Theorem A

If Assumptions 1–4 hold, then $\mathcal{C}$ has presentation $\langle \Gamma : \Omega \rangle$.

- Theorem A applies to *many* categories.
  - $\mathcal{P}$, $\mathcal{B}$, $\mathcal{TL}$, $\mathcal{PT}$, $\mathcal{T}^+$, $\mathcal{I}$, $\mathcal{PO}$, $\mathcal{O}^+$, $\mathcal{OI}$, $\mathcal{PV}$, $\mathcal{V}^+$, $\mathcal{IB}$......
- Most parts of the assumptions are easy to check.

Theorem A

Theorem A

If Assumptions 1–4 hold, then $\mathcal{C}$ has presentation $\langle \Gamma : \Omega \rangle$.

- Theorem A applies to *many* categories.
  - $\mathcal{P}$, $\mathcal{B}$, $\mathcal{TL}$, $\mathcal{PT}$, $\mathcal{T}^+$, $\mathcal{I}$, $\mathcal{PO}$, $\mathcal{O}^+$, $\mathcal{OI}$, $\mathcal{PV}$, $\mathcal{V}^+$, $\mathcal{IB}$......
- Most parts of the assumptions are easy to check.
- Exceptions:
  - Presentations for endomorphism monoids $\mathcal{C}_n$.

Theorem A

Theorem A

If Assumptions 1–4 hold, then $\mathcal{C}$ has presentation $\langle \Gamma : \Omega \rangle$.

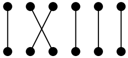
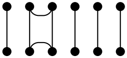
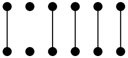
- Theorem A applies to *many* categories.
  - $\mathcal{P}$, $\mathcal{B}$, $\mathcal{TL}$, $\mathcal{PT}$, $\mathcal{T}^+$, $\mathcal{I}$, $\mathcal{PO}$, $\mathcal{O}^+$, $\mathcal{OI}$, $\mathcal{PV}$, $\mathcal{V}^+$, $\mathcal{IB}$......
- Most parts of the assumptions are easy to check.
- Exceptions:
  - Presentations for endomorphism monoids $\mathcal{C}_n$.
  - For all $w \in X_{n+d}^*$, $\quad \lambda_n w \rho_n \sim w' \quad$ for some $w' \in X_n^*$.

Theorem A — applications

Theorem

The partition category $\mathcal{P} \cong \langle \Gamma : \Omega \rangle$:

$$
\begin{array}{rll}
\sigma_{i;n}^2 = \iota_n, \quad \varepsilon_{i;n}^2 = \varepsilon_{i;n}, & \tau_{i;n}^2 = \tau_{i;n} = \tau_{i;n}\sigma_{i;n} = \sigma_{i;n}\tau_{i;n}, & \\
\sigma_{i;n}\varepsilon_{i;n} = \varepsilon_{i+1;n}\sigma_{i;n}, & \varepsilon_{i;n}\varepsilon_{i+1;n}\sigma_{i;n} = \varepsilon_{i;n}\varepsilon_{i+1;n}, & \\
\varepsilon_{i;n}\varepsilon_{j;n} = \varepsilon_{j;n}\varepsilon_{i;n}, & \tau_{i;n}\tau_{j;n} = \tau_{j;n}\tau_{i;n}, & \\
\sigma_{i;n}\sigma_{j;n} = \sigma_{j;n}\sigma_{i;n}, & \sigma_{i;n}\tau_{j;n} = \tau_{j;n}\sigma_{i;n}, & \text{if } |i-j| > 1, \\
\sigma_{i;n}\sigma_{j;n}\sigma_{i;n} = \sigma_{j;n}\sigma_{i;n}\sigma_{j;n}, & \sigma_{i;n}\tau_{j;n}\sigma_{i;n} = \sigma_{j;n}\tau_{i;n}\sigma_{j;n}, & \text{if } |i-j| = 1, \\
\sigma_{i;n}\varepsilon_{j;n} = \varepsilon_{j;n}\sigma_{i;n}, & \tau_{i;n}\varepsilon_{j;n} = \varepsilon_{j;n}\tau_{i;n}, & \text{if } j \neq i, i+1, \\
\tau_{i;n}\varepsilon_{j;n}\tau_{i;n} = \tau_{i;n}, & \varepsilon_{j;n}\tau_{i;n}\varepsilon_{j;n} = \varepsilon_{j;n}, & \text{if } j = i, i+1, \\
\lambda_n\rho_n = \iota_n, & \rho_n\lambda_n = \varepsilon_{n+1;n+1}, & \\
\theta_{i;n}\lambda_n = \lambda_n\theta_{i;n+1}, & \rho_n\theta_{i;n} = \theta_{i;n+1}\rho_n, & \text{for } \theta \in \{\sigma, \varepsilon, \tau\}.
\end{array}
$$

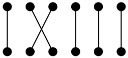
$\sigma_{2;6} =$  $\tau_{2;6} =$  $\varepsilon_{2;6} =$ 

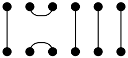
Theorem A — applications

Theorem

The Brauer category $\mathcal{B} \cong \langle \Gamma : \Omega \rangle$:

$$\begin{aligned}
\sigma_{i;n}^2 &= \iota_n, & \tau_{i;n}^2 &= \tau_{i;n} = \tau_{i;n}\sigma_{i;n} = \sigma_{i;n}\tau_{i;n}, & & \\
\sigma_{i;n}\sigma_{j;n} &= \sigma_{j;n}\sigma_{i;n}, & \tau_{i;n}\tau_{j;n} &= \tau_{j;n}\tau_{i;n}, \quad \sigma_{i;n}\tau_{j;n} = \tau_{j;n}\sigma_{i;n}, & & \text{if } |i-j| > 1, \\
\sigma_{i;n}\sigma_{j;n}\sigma_{i;n} &= \sigma_{j;n}\sigma_{i;n}\sigma_{j;n}, & \sigma_{i;n}\tau_{j;n}\sigma_{i;n} &= \sigma_{j;n}\tau_{i;n}\sigma_{j;n}, \quad \tau_{i;n}\sigma_{j;n}\tau_{i;n} = \tau_{i;n}, & & \text{if } |i-j| = 1, \\
\lambda_n\rho_n &= \iota_n, & \rho_n\lambda_n &= \tau_{n+1;n+2}, & & \\
\theta_{i;n}\lambda_n &= \lambda_n\theta_{i;n+2}, & \rho_n\theta_{i;n} &= \theta_{i;n+2}\rho_n, & & \text{for } \theta \in \{\sigma, \tau\}.
\end{aligned}$$

$\sigma_{2;6} =$ 

$\tau_{2;6} =$ 

Theorem A — applications

Theorem

The Brauer category $\mathcal{B} \cong \langle \Gamma : \Omega \rangle$:

$$\begin{array}{rlll}
\sigma_{i;n}^2 = \iota_n, & \tau_{i;n}^2 = \tau_{i;n} = \tau_{i;n}\sigma_{i;n} = \sigma_{i;n}\tau_{i;n}, & & \\
\sigma_{i;n}\sigma_{j;n} = \sigma_{j;n}\sigma_{i;n}, & \tau_{i;n}\tau_{j;n} = \tau_{j;n}\tau_{i;n}, & \sigma_{i;n}\tau_{j;n} = \tau_{j;n}\sigma_{i;n}, & \text{if } |i-j| > 1, \\
\sigma_{i;n}\sigma_{j;n}\sigma_{i;n} = \sigma_{j;n}\sigma_{i;n}\sigma_{j;n}, & \sigma_{i;n}\tau_{j;n}\sigma_{i;n} = \sigma_{j;n}\tau_{i;n}\sigma_{j;n}, & \tau_{i;n}\sigma_{j;n}\tau_{i;n} = \tau_{i;n}, & \text{if } |i-j| = 1, \\
\lambda_n\rho_n = \iota_n, & \rho_n\lambda_n = \tau_{n+1;n+2}, & & \\
\theta_{i;n}\lambda_n = \lambda_n\theta_{i;n+2}, & \rho_n\theta_{i;n} = \theta_{i;n+2}\rho_n, & & \text{for } \theta \in \{\sigma, \tau\}.
\end{array}$$

Theorem

The Temperley-Lieb category $\mathcal{TL} \cong \langle \Gamma : \Omega \rangle$:

$$\tau_{i;n}^2 = \tau_{i;n}, \quad \tau_{i;n}\tau_{j;n} = \tau_{j;n}\tau_{i;n} \text{ if } |i-j| > 1, \quad \tau_{i;n}\tau_{j;n}\tau_{i;n} = \tau_{i;n} \text{ if } |i-j| = 1,$$
$$\lambda_n\rho_n = \iota_n, \quad \rho_n\lambda_n = \tau_{n+1;n+2}, \quad \tau_{i;n}\lambda_n = \lambda_n\tau_{i;n+2}, \quad \rho_n\tau_{i;n} = \tau_{i;n+2}\rho_n.$$

Theorem A

Theorem A

If Assumptions 1–4 hold, then $\mathcal{C}$ has presentation $\langle \Gamma : \Omega \rangle$.

Theorem A

Theorem A

If Assumptions 1–4 hold, then $\mathcal{C}$ has presentation $\langle \Gamma : \Omega \rangle$.

- But we really want a tensor presentation.

Theorem A

Theorem A

If Assumptions 1–4 hold, then $\mathcal{C}$ has presentation $\langle \Gamma : \Omega \rangle$.

- But we really want a tensor presentation.
- We make two further assumptions...

Theorem B — Key assumptions

Theorem B — Key assumptions

Assumption 5

We assume that $\mathcal{C}$ is a (strict) tensor category over $\mathbb{N}$.

Theorem B — Key assumptions

Assumption 5

We assume that $\mathcal{C}$ is a (strict) tensor category over $\mathbb{N}$.

Assumption 6

We assume that $\Delta \subseteq \mathcal{C}$, and Ξ is a set of relations:

Theorem B — Key assumptions

Assumption 5

We assume that $\mathcal{C}$ is a (strict) tensor category over $\mathbb{N}$.

Assumption 6

We assume that $\Delta \subseteq \mathcal{C}$, and Ξ is a set of relations:

- Each relation holds in $\mathcal{C}$.

Theorem B — Key assumptions

Assumption 5

We assume that $\mathcal{C}$ is a (strict) tensor category over $\mathbb{N}$.

Assumption 6

We assume that $\Delta \subseteq \mathcal{C}$, and Ξ is a set of relations:

- Each relation holds in $\mathcal{C}$.
- There is a morphism $\Gamma^* \to \Delta^{\circledast} : w \mapsto \widehat{w}$:
  - For all $x \in \Gamma$, we have $\widehat{x}\Phi = x\phi$.
  - For all $x \in \Delta$ and $m, n \in \mathbb{N}$, we have $\iota_m \oplus x \oplus \iota_n \approx \widehat{w}$ for some $w \in \Gamma^*$.
  - For all $(u, v) \in \Omega$, we have $\widehat{u} \approx \widehat{v}$.

Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

- The main work is establishing the properties of the terms $\widehat{w}$.

Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

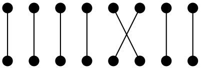
- The main work is establishing the properties of the terms $\widehat{w}$.
- Finding the definition is easy enough.

Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

- The main work is establishing the properties of the terms $\widehat{w}$.
- Finding the definition is easy enough.
- e.g., in the Brauer category $\mathcal{B}$:

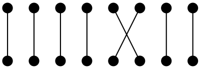
$\sigma_{5;8} \equiv$ 

Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

- The main work is establishing the properties of the terms $\widehat{w}$.
- Finding the definition is easy enough.
- e.g., in the Brauer category $\mathcal{B}$:
  - $\widehat{\sigma}_{5;8} = IIIIXII$

$\sigma_{5;8} \equiv$ 

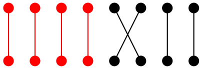
Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

- The main work is establishing the properties of the terms $\widehat{w}$.
- Finding the definition is easy enough.
- e.g., in the Brauer category $\mathcal{B}$:
  - $\widehat{\sigma}_{5;8} = IIIIXII = \iota_4$

$\sigma_{5;8} \equiv$

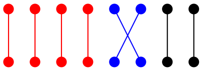


Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

- The main work is establishing the properties of the terms $\widehat{w}$.
- Finding the definition is easy enough.
- e.g., in the Brauer category $\mathcal{B}$:
  - $\widehat{\sigma}_{5;8} = IIIIXII = \iota_4 \oplus X$

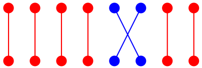
$\sigma_{5;8} \equiv$ 

Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

- The main work is establishing the properties of the terms $\widehat{w}$.
- Finding the definition is easy enough.
- e.g., in the Brauer category $\mathcal{B}$:
  - $\widehat{\sigma}_{5;8} = IIIIXII = \iota_4 \oplus X \oplus \iota_2$,

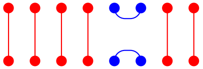
$\sigma_{5;8} \equiv$ 

Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

- The main work is establishing the properties of the terms $\widehat{w}$.
- Finding the definition is easy enough.
- e.g., in the Brauer category $\mathcal{B}$:
  - $\widehat{\sigma}_{5;8} = \iota_4 \oplus X \oplus \iota_2$,
  - $\widehat{\tau}_{5;8} = \iota_4 \oplus U \oplus \cap \oplus \iota_2$,

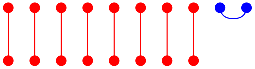
$\tau_{5;8} \equiv$ 

Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

- The main work is establishing the properties of the terms $\widehat{w}$.
- Finding the definition is easy enough.
- e.g., in the Brauer category $\mathcal{B}$:
  - $\widehat{\sigma}_{5;8} = \iota_4 \oplus X \oplus \iota_2$,
  - $\widehat{\tau}_{5;8} = \iota_4 \oplus U \oplus \cap \oplus \iota_2$,
  - $\widehat{\lambda}_8 = \iota_8 \oplus U$,

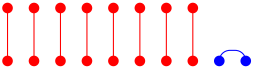
$\lambda_8 \equiv$ 

Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

- The main work is establishing the properties of the terms $\widehat{w}$.
- Finding the definition is easy enough.
- e.g., in the Brauer category $\mathcal{B}$:
  - $\widehat{\sigma}_{5;8} = \iota_4 \oplus X \oplus \iota_2$,
  - $\widehat{\tau}_{5;8} = \iota_4 \oplus U \oplus \cap \oplus \iota_2$,
  - $\widehat{\lambda}_8 = \iota_8 \oplus U$,
  - $\widehat{\rho}_8 = \iota_8 \oplus \cap$.

$\rho_8 \equiv$ 

Theorem B

Theorem B

If Assumptions 1–6 hold, then $\mathcal{C}$ has tensor presentation $\langle \Delta : \Xi \rangle$.

- The main work is establishing the properties of the terms $\widehat{w}$.
- Finding the definition is easy enough.
- e.g., in the Brauer category $\mathcal{B}$:
  - $\widehat{\sigma}_{5;8} = \iota_4 \oplus X \oplus \iota_2$,
  - $\widehat{\tau}_{5;8} = \iota_4 \oplus U \oplus \cap \oplus \iota_2$,
  - $\widehat{\lambda}_8 = \iota_8 \oplus U$,
  - $\widehat{\rho}_8 = \iota_8 \oplus \cap$.
- There is a Theorem C for categories like $\mathcal{T}$:
  - $\mathcal{C}_{m,n} = \varnothing \Leftrightarrow m > 0 = n$.

Theorem B — applications

Theorem

The Temperley-Lieb category $\mathcal{TL} \cong \langle U, \cap : \Xi \rangle$.

$$\cap \circ U = \iota_0,$$
$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U).$$

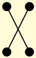
$U \equiv$  , $\cap \equiv$  , $I \equiv$  .

Theorem B — applications

Theorem

The Brauer category $\mathcal{B} \cong \langle X, U, \cap : \Xi \rangle$.

$$X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \quad X \circ U = U, \quad \cap \circ X = \cap,$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X),$$
$$(I \oplus \cap) \circ (U \oplus I) = I = (\cap \oplus I) \circ (I \oplus U),$$
$$(I \oplus X) \circ (U \oplus I) = (X \oplus I) \circ (I \oplus U),$$
$$(\cap \oplus I) \circ (I \oplus X) = (I \oplus \cap) \circ (X \oplus I).$$

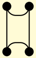
$X \equiv$ , $U \equiv$ , $\cap \equiv$ , $I \equiv$ .

Theorem B — applications

Theorem

The partition category $\mathcal{P} \cong \langle X, D, U, \cap : \Xi \rangle$.

$$
\begin{gathered}
X \circ X = I \oplus I, \quad \cap \circ U = \iota_0, \\
D \circ D = D = D \circ X = X \circ D, \\
(D \oplus I) \circ (I \oplus D) = (I \oplus D) \circ (D \oplus I), \\
(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \\
(X \oplus I) \circ (I \oplus D) \circ (X \oplus I) = (I \oplus X) \circ (D \oplus I) \circ (I \oplus X), \\
X \circ (I \oplus U) = U \oplus I, \quad (I \oplus \cap) \circ X = \cap \oplus I, \\
(I \oplus \cap) \circ D \circ (I \oplus U) = I, \quad D \circ (I \oplus U \oplus \cap) \circ D = D.
\end{gathered}
$$

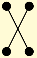
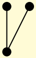
$X \equiv$ , $D \equiv$ , $U \equiv$ , $\cap \equiv$ , $I \equiv$ .

Theorem B — applications

Theorem

The transformation category $\mathcal{T} \cong \langle X, V, \cap : \Xi \rangle$.

$$X \circ X = \iota_2, \quad X \circ V = V,$$
$$(V \oplus I) \circ V = (I \oplus V) \circ V, \quad (I \oplus \cap) \circ V = I,$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X),$$
$$(\cap \oplus I) \circ X = I \oplus \cap, \quad (I \oplus V) \circ X = (X \oplus I) \circ (I \oplus X) \circ (V \oplus I).$$

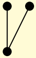
$X \equiv$  , $\quad V \equiv$  , $\quad \cap \equiv$  , $\quad I \equiv$  .

Theorem B — applications

Theorem

The partial transformation category $\mathcal{PT} \cong \langle X, V, U, \cap : \Xi \rangle$.

$$X \circ X = \iota_2, \quad \cap \circ U = \iota_0,$$
$$X \circ V = V, \quad V \circ U = U \oplus U,$$
$$(V \oplus I) \circ V = (I \oplus V) \circ V, \quad (I \oplus \cap) \circ V = I,$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X),$$
$$X \circ (U \oplus I) = I \oplus U, \quad (\cap \oplus I) \circ X = I \oplus \cap,$$
$$(I \oplus V) \circ X = (X \oplus I) \circ (I \oplus X) \circ (V \oplus I).$$

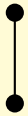
$X \equiv$  , $V \equiv$  , $U \equiv$  , $\cap \equiv$  , $I \equiv$  .

Theorem B — applications

Theorem

The symmetric inverse category $\mathcal{I} \cong \langle X, U, \cap : \Xi \rangle$.

$$X \circ X = \iota_2, \quad \cap \circ U = \iota_0,$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X),$$
$$X \circ (U \oplus I) = I \oplus U, \quad (\cap \oplus I) \circ X = I \oplus \cap.$$

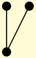
$X \equiv$  , $U \equiv$  , $\cap \equiv$  , $I \equiv$  .

Theorem B — applications

Theorem

Order-preserving transformations: $\mathcal{O} \cong \langle V, \cap : \Xi \rangle$.

$$(V \oplus I) \circ V = (I \oplus V) \circ V, \quad (I \oplus \cap) \circ V = I = (\cap \oplus I) \circ V.$$

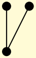
$V \equiv$  , $\quad \cap \equiv$  , $\quad I \equiv$  .

Theorem B — applications

Theorem

Order-preserving partial transformations: $\mathcal{PO} \cong \langle V, U, \cap : \Xi \rangle$.

$$\cap \circ U = \iota_0, \quad V \circ U = U \oplus U,$$
$$(V \oplus I) \circ V = (I \oplus V) \circ V, \quad (I \oplus \cap) \circ V = I = (\cap \oplus I) \circ V.$$

$V \equiv$  , $U \equiv$  , $\cap \equiv$  , $I \equiv$  .

Theorem B — applications

Theorem

Order-preserving partial bijections: $\mathcal{OI} \cong \langle U, \cap : \Xi \rangle$.

$$\cap \circ U = \iota_0.$$

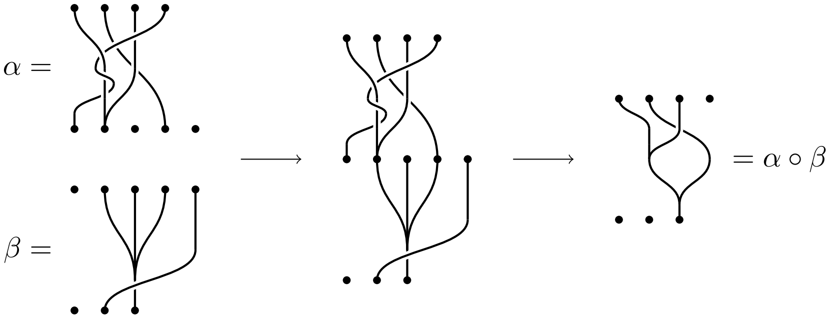
$U \equiv$  , $\cap \equiv$  , $I \equiv$  .

Theorem B — applications

- More applications come from (partial) braids/vines.

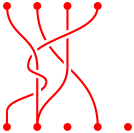
Theorem B — applications

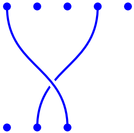
- More applications come from (partial) braids/vines.



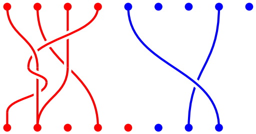
Theorem B — applications

- More applications come from (partial) braids/vines.

$\alpha =$ 

$\beta =$ 

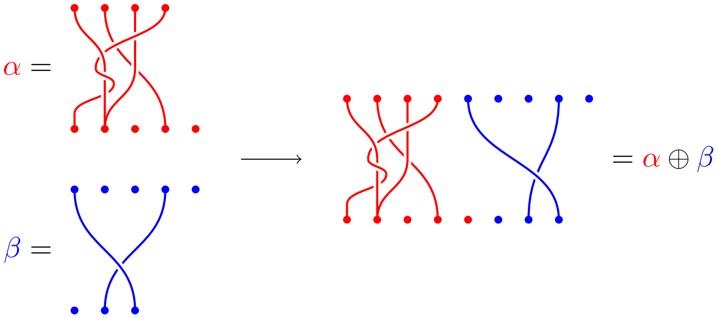
$\longrightarrow$



$= \alpha \oplus \beta$

Theorem B — applications

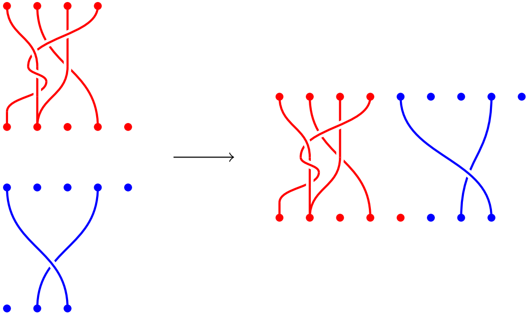
- More applications come from (partial) braids/vines.



- $\mathcal{PV}$ = the partial vine category.

Theorem B — applications

- More applications come from (partial) braids/vines.



$\alpha =$

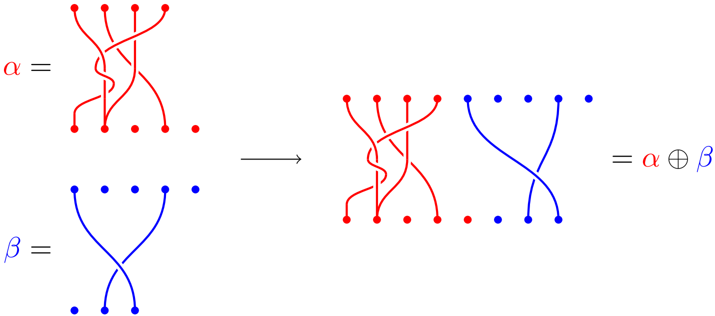
$\beta =$

$\longrightarrow$ $= \alpha \oplus \beta$

- $\mathcal{PV}$ = the partial vine category.
- $\mathcal{V}$ = the (full) vine category.

Theorem B — applications

- More applications come from (partial) braids/vines.



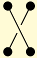
- $\mathcal{PV}$ = the partial vine category.
- $\mathcal{V}$ = the (full) vine category.
- $\mathcal{IB}$ = the partial braid category.

Theorem B — applications

Theorem

The partial vine category $\mathcal{PV} \cong \langle X, X^{-1}, V, U, \cap : \Xi \rangle$.

$$
\begin{gathered}
X \circ X^{-1} = X^{-1} \circ X = \iota_2, \quad \cap \circ U = \iota_0, \\
X \circ V = V, \quad V \circ U = U \oplus U, \\
(V \oplus I) \circ V = (I \oplus V) \circ V, \quad (I \oplus \cap) \circ V = I, \\
(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X), \\
X \circ (U \oplus I) = I \oplus U, \quad X \circ (I \oplus U) = U \oplus I, \\
(\cap \oplus I) \circ X = I \oplus \cap, \quad (I \oplus \cap) \circ X = \cap \oplus I, \\
(I \oplus V) \circ X = (X \oplus I) \circ (I \oplus X) \circ (V \oplus I), \\
(V \oplus I) \circ X = (I \oplus X) \circ (X \oplus I) \circ (I \oplus V).
\end{gathered}
$$

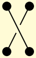
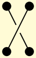
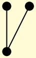
$X \equiv$ , $X^{-1} \equiv$ , $V \equiv$ , $U \equiv$ , $\cap \equiv$ , $I \equiv$ .

Theorem B — applications

Theorem

The (full) vine category $\mathcal{V} \cong \langle X, X^{-1}, V, \cap : \Xi \rangle$.

$$X \circ X^{-1} = X^{-1} \circ X = \iota_2, \quad X \circ V = V,$$
$$(V \oplus I) \circ V = (I \oplus V) \circ V, \quad (I \oplus \cap) \circ V = I,$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X),$$
$$(\cap \oplus I) \circ X = I \oplus \cap, \quad (I \oplus \cap) \circ X = \cap \oplus I,$$
$$(I \oplus V) \circ X = (X \oplus I) \circ (I \oplus X) \circ (V \oplus I),$$
$$(V \oplus I) \circ X = (I \oplus X) \circ (X \oplus I) \circ (I \oplus V).$$

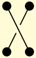
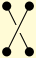
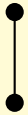
$X \equiv$ , $X^{-1} \equiv$ , $V \equiv$ , $\cap \equiv$ , $I \equiv$ .

Theorem B — applications

Theorem

The partial braid category $\mathcal{IB} \cong \langle X, X^{-1}, U, \cap : \Xi \rangle$.

$$X \circ X^{-1} = X^{-1} \circ X = \iota_2, \quad \cap \circ U = \iota_0,$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X),$$
$$X \circ (U \oplus I) = I \oplus U, \quad X \circ (I \oplus U) = U \oplus I,$$
$$(\cap \oplus I) \circ X = I \oplus \cap, \quad (I \oplus \cap) \circ X = \cap \oplus I.$$

$X \equiv$ , $X^{-1} \equiv$ , $U \equiv$ , $\cap \equiv$ , $I \equiv$ .

Theorem B — applications

Theorem

The partial braid category $\mathcal{IB} \cong \langle X, X^{-1}, U, \cap : \Xi \rangle$.

$$X \circ X^{-1} = X^{-1} \circ X = \iota_2, \quad \cap \circ U = \iota_0,$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X),$$
$$X \circ (U \oplus I) = I \oplus U, \quad X \circ (I \oplus U) = U \oplus I,$$
$$(\cap \oplus I) \circ X = I \oplus \cap, \quad (I \oplus \cap) \circ X = \cap \oplus I.$$

$X \equiv$ , $\quad X^{-1} \equiv$ , $\quad U \equiv$ , $\quad \cap \equiv$ , $\quad I \equiv$ .

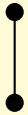
- $\mathcal{PV}$, $\mathcal{V}$ and $\mathcal{IB}$ are braided tensor categories (Joyal+Street).

Theorem B — applications

Theorem

The partial braid category $\mathcal{IB} \cong \langle X, X^{-1}, U, \cap : \Xi \rangle$.

$$X \circ X^{-1} = X^{-1} \circ X = \iota_2, \quad \cap \circ U = \iota_0,$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X),$$
$$X \circ (U \oplus I) = I \oplus U, \quad X \circ (I \oplus U) = U \oplus I,$$
$$(\cap \oplus I) \circ X = I \oplus \cap, \quad (I \oplus \cap) \circ X = \cap \oplus I.$$

$X \equiv$ , $X^{-1} \equiv$ , $U \equiv$ , $\cap \equiv$ , $I \equiv$ .

- $\mathcal{PV}$, $\mathcal{V}$ and $\mathcal{IB}$ are braided tensor categories (Joyal+Street).
- Can put the braids into the free data of the presentation.

Theorem B — applications

Theorem

The partial braid category $\mathcal{IB} \cong \langle X, X^{-1}, U, \cap : \Xi \rangle$.

$$X \circ X^{-1} = X^{-1} \circ X = \iota_2, \quad \cap \circ U = \iota_0,$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X),$$
$$X \circ (U \oplus I) = I \oplus U, \quad X \circ (I \oplus U) = U \oplus I,$$
$$(\cap \oplus I) \circ X = I \oplus \cap, \quad (I \oplus \cap) \circ X = \cap \oplus I.$$

$X \equiv$ , $X^{-1} \equiv$ , $U \equiv$ , $\cap \equiv$ , $I \equiv$ .

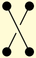
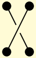
- $\mathcal{PV}$, $\mathcal{V}$ and $\mathcal{IB}$ are braided tensor categories (Joyal+Street).
- Can put the braids into the free data of the presentation.
- e.g., $\mathcal{IB} \cong \langle U, \cap : \cap \circ U = \iota_0 \rangle$.

Theorem B — applications

Theorem

The partial braid category $\mathcal{IB} \cong \langle X, X^{-1}, U, \cap : \Xi \rangle$.

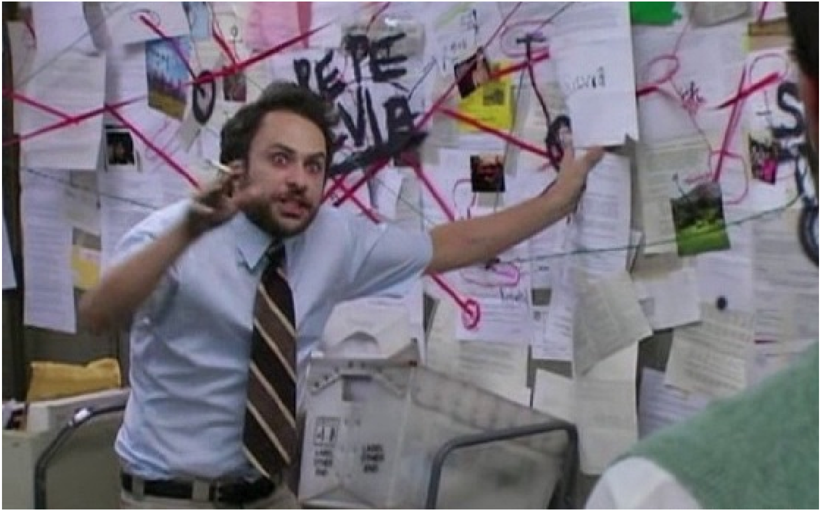
$$X \circ X^{-1} = X^{-1} \circ X = \iota_2, \quad \cap \circ U = \iota_0,$$
$$(X \oplus I) \circ (I \oplus X) \circ (X \oplus I) = (I \oplus X) \circ (X \oplus I) \circ (I \oplus X),$$
$$X \circ (U \oplus I) = I \oplus U, \quad X \circ (I \oplus U) = U \oplus I,$$
$$(\cap \oplus I) \circ X = I \oplus \cap, \quad (I \oplus \cap) \circ X = \cap \oplus I.$$

$X \equiv$ , $X^{-1} \equiv$ , $U \equiv$ , $\cap \equiv$ , $I \equiv$ .

- $\mathcal{PV}$, $\mathcal{V}$ and $\mathcal{IB}$ are braided tensor categories (Joyal+Street).
- Can put the braids into the free data of the presentation.
- e.g., $\mathcal{IB} \cong \langle U, \cap : \cap \circ U = \iota_0 \rangle$......

......the bicyclic braided tensor category?!

I could go on... and on...



CATEGORIEZ!!!!!1!!!

Thank you :-)


- Presentations for tensor categories
  - Coming soon to arXiv...