Congruences on ample semigroups

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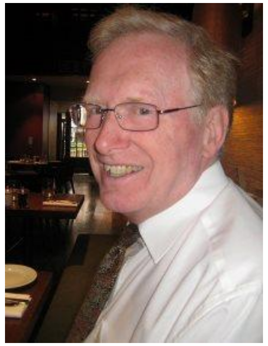

Ample semigroups

The ample semigroup is a special type (type A) of adequate semigroups.

The definition of adequate semigroups depends heavily on the generalized Green's relations; L^*, R^*, H^* and D^* where

$$a \; L^* \; b \text{ if and only if } (\forall s, t \in S^1) \; as = at \Leftrightarrow bs = bt,$$

$$a \; R^* \; b \text{ if and only if } (\forall s, t \in S^1) \; sa = ta \Leftrightarrow sb = tb.$$

and

$$H^* = L^* \cap R^*, \; D^* = L^* \vee R^*$$

The adequate semigroup S is the one in which every L^* - class and every R^* - class contains an idempotent, and the set E (= E (S)) of the idempotents of S is a semilattice.

In this case,
every L^* - class and every R^* - class contains a unique idempotent.

Denote the idempotent in L^*_a by a^* and the idempotent in R^*_a by $a^\dagger$.

Lemma [5] Let a, b be elements of an adequate semigroup S. Then:

- (i) a L^* b if and only if $a^* = b^*$ and a R^* b if and only if $a^\dagger = b^\dagger$.
- (ii) $(ab)^* = (a^* b)^*$ and $(ab)^\dagger = (ab^\dagger)^\dagger$;
- (iii) $(ab)^* b^* = (ab)^*$ and $a^\dagger (ab)^\dagger = (ab)^\dagger$.

An adequate semigroup S is called *ample* if for any $a \in S$ and $e \in E$:

$$ea = a\,(ea)^{*} \quad \text{and} \quad ae = (ae)^{\dagger}\,a.$$

From now on, unless otherwise stated, S is an ample semigroup and $E(S) = E$.

The aim is to extend results concerning congruences on inverse semigroups to ample semigroups.

We adopt trace - kernel approach.

If ρ is a congruence on S, trace of ρ $(\mathrm{tr}\,\rho) = \rho|_E$.

Kernel of ρ $(\ker \rho) = \{ a \in S : (a, e) \in \rho$ for some $e \in E \}$

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Admissible Congruences

A congruence ρ on an adequate semigroup S is admissible if :

$$as \ \rho \ at \Rightarrow a^* s \ \rho \ a^* t \text{ and } sa \ \rho \ ta \Rightarrow sa^\dagger \ \rho \ ta^\dagger \text{ for any } a \in S \text{ and } s,t \in S^1$$

Not every congruence on an adequate semigroup is admissible.

Lemma [1] If ρ is an admissible congruence on the adequate semigroup S and if a, b are elements of S such that $a \rho b$, then $a^* \rho b^*$ and $a^\dagger \rho b^\dagger$.

The converse is not true.

We remark that if ρ is an admissible congruence on S, then S/ρ is an ample semigroup when $*$ and $\dagger$ are defined on S/ρ by putting

$$(a\rho)^* = a^* \rho \textit{ and } (a \rho)^\dagger = a^\dagger \rho$$

Moreover, (see [3]), if $x\rho$ is an idempotent in S/ρ, then there exists an idempotent e in S such that $(x, e) \in \rho$.

$$E\,(S/\rho) = \{e\rho : e \in E\}$$

The natural homomorphism from S onto S/ρ is an admissible homomorphism in the following sense.

A homomorphism $\theta : S \to T$ of adequate semigroups is admissible if

$a\, L^*(S)\, b$ implies $a\theta\, L^*(T)\, b\theta$ and $a\, R^*(S) b$ implies $a\theta\, R^*(T)\, b\theta$.

The congruence μ

Let S be an adequate semigroup and E = E (S).

Define μ as follows:

$$(a, b) \in \mu \text{ if and only if } (ea)^* = (eb)^* \text{ and } (ae)^\dagger = (be)^\dagger \text{ for all } e \in E.$$

In [5] it is shown that μ is the maximum congruence contained in H^*. From [3] we conclude that μ is an admissible congruence on any ample semigroup. It follows from [1] that μ is the maximum idempotent-separating admissible congruence on *S*.

In fact we have

Proposition [2] *If* ρ *is an admissible congruence on S, then* ρ *is idempotent - separating if and only if* $\rho \subseteq H^*$.

Corollary: tr $\mu = i_E$.

The congruence σ

The relation σ on S is defined *for any* $a, b \in S$ by the following rule:

$$(a, b) \in \sigma \text{ if and only if } ae = be \text{ for some } e \in E.$$

In [8] it is shown that σ is the minimum cancellative congruence on S.

As S is an ample semigroup, $ea = a(ea)^*$ and $ae = (ae)^\dagger a$ for any $a \in S$, $e \in E$, then – alternatively - σ can be given as:

$$(a,b) \in \sigma \text{ if and only if } fa = fb \text{ for some } f \in E.$$

tr $\sigma = E \times E$.

Congruences with the same trace

Definition

A congruence π on E is said to be *normal* if for any $e, f \in E$ and $a \in S$;

$$e \,\pi\, f \text{ implies } (e\,a)^{*} \,\pi\, (f\,a)^{*} \text{ and } (a\,e)^{\dagger} \,\pi\, (a\,f)^{\dagger}.$$

Lemma *If π is a normal congruence on E, then for any elements a, b in S, the following two statements are equivalent:*

(1) $a^{*} \pi b^{*}$, $ae = be$ for some $e \in E$, $e \,\pi\, a^{*}$;

(2) $a^{\dagger} \pi\, b^{\dagger}$, $fa = f\,b$ for some $f \in E$, $f \,\pi\, a^{\dagger}$.

Theorem [1,7] *For any normal congruence π on E, the relation:*

$$\sigma_{\pi} = \{ (a,b) \in S \times S : a^{*} \,\pi\, b^{*},\ a\,e = b\,e \ \text{ for some } e \in E,\ e \,\pi\, a^{*} \}$$

is the minimum congruence on S whose restriction to E is π. Further, σ_{π} is an admissible congruence.

Let π be a normal congruence on E.

Define μ_π on S by the following rule:
$(a, b) \in \mu_\pi$ if and only if $(ea)^* \,\pi\, (eb)^*$ and $(ae)^\dagger \,\pi\, (be)^\dagger$ for any $e \in E$.

The following Lemma gives an alternative description of μ_π .

Lemma *Let* π *be a normal congruence on* E*. Then for any elements* a, b *of* S*, the following statements are equivalent:*

(i) $(a, b) \in \mu_\pi$.

(ii) $(ea)^* \,\pi\, (fb)^*$ *and* $(ae)^\dagger \,\pi\, (bf)^\dagger$ *for any* $e, f \in E$ *with* $e \,\pi\, f$.

(iii) $(a\sigma_\pi, b\sigma_\pi) \in \mu(S/\sigma_\pi)$.

Theorem [1] *The relation* μ_π *is the maximum admissible congruence on* S *whose restriction to* E *is* π.

Let ρ be an admissible congruence on S.

tr ρ is a normal congruence on E.

$\sigma_{tr}\,\rho$, $\mu_{tr}\,\rho$ are respectively the minimum and the maximum admissible congruence on S such that:

$$\text{tr}\ \sigma_{tr\rho} = \text{tr}\ \rho = \text{tr}\ \mu_{tr\rho}$$

Note: $\sigma_{tr\rho} \subseteq \rho \subseteq \mu_{tr\rho}$.

We may put $\sigma_{tr\rho} = \sigma_\rho$ and $\mu_{tr\rho} = \mu_\rho$

Where

$$\sigma_\rho = \{(\,a\,,b\,) \in S \times S;\ a^{*}\,\rho b^{*}\,,a\,e = b\,e \text{ for some } e \in a^{*}\,\rho \cap E\}.$$

$$= \{(a,\,b) \in S \times S;\ a^{\dagger}\rho\ b^{\dagger},\ f\,a = f\,b \text{ for some } f \in a^{\dagger}\rho \cap E\}.$$

and

$$\mu_\rho = \{\ (\,a,\,b\,) \in S \times S\ ;\ (e\,a)^{*}\ \rho\ (e\,b)^{*}, \text{ and } (a\,e)^{\dagger}\ \rho\ (\,b\,e\,)^{\dagger} \text{ for all } e \in E\ \}.$$

Theorem [1] *The following statements concerning admissible congruences* ρ *and* τ *on the ample semigroup* S *are equivalent:*

(i) $\operatorname{tr}\rho = \operatorname{tr}\tau$.

(ii) $\rho \subseteq \mu_\tau$; $\mu_\tau/\rho = \mu(S/\rho)$.

(iii) $a\rho\, \mu(S/\rho)\, b\rho \Leftrightarrow a\tau\, \mu(S/\tau)\, b\tau$, $(a, b \in S)$.

(iv) $a\rho\, H^*(S/\rho)\, b\rho \Leftrightarrow a\tau\, H^*(S/\tau)\, b\tau$, $(a, b \in S)$.

(v) $\rho \cap \tau|_{e\rho}$ *and* $\rho \cap \tau|_{e\tau}$ *are cancellative congruences,* $(e \in E)$.

(vi) $\rho/\rho \cap \tau$ *and* $\tau/\rho \cap \tau$ *are congruences contained in* $H^*(S/\rho \cap \tau)$.

The Kernel of σ

Proposition [1] *If* ρ *is admissible congruence on* S *and* $a \in \ker \rho$, *then* $(a^\dagger, a^*) \in \mathrm{tr}\, \rho$.

Proposition [1] *For any admissible congruence* ρ *on* S:

$\ker \sigma_{\rho^*} = \{ a \in S: ae = e \text{ for some } e \in a^* \rho \cap E \}$.

$= \{ a \in S: ea = e \text{ for some } e \in a^\dagger \rho \cap E \}$.

Corollary *For any normal congruence* π *on* E.

$\ker \sigma_\pi = \{ a \in S: ae = e \text{ for some } e \in E, e \,\pi\, a^* \}$.

$= \{ a \in S: ea = e \text{ for some } e \in E, e \,\pi\, a^\dagger \}$.

Corollary

$\ker \sigma = \{ a \in S: ae = e \text{ for some } e \in E \}$.

Note: $\sigma = \sigma_\omega$

The Kernel of μ

Proposition [1] *Let* ρ *be an admissible congruence on S. Then:*

$\ker \mu_\rho = \{ a \in S : ea \ \rho \ a e, \text{ for all } e \in E \}.$

$= \{a \in S : (e a)^* \ \rho \ e a^*, \text{ for all } e \in E \}.$

$= \{a \in S : (a e)^\dagger \ \rho \ a^\dagger e, \text{ for all } e \in E \}.$

It follows that

Corollary *For any normal congruence* π *on E.*

$\ker \mu_\pi = \{a \in S : (e a)^* \ \pi \ e a^*, \text{ for any } e \in E \}.$

$= \{a \in S : (a e)^\dagger \ \pi \ a^\dagger e, \text{ for any } e \in E \}.$

Corollary

$\ker \mu = \{a \in S : a e = e a, \text{ for any } e \in E\}.$

$= \{a \in S : (e a)^* = e a^*, \text{ for any } e \in E \}.$

Note: $\mu = \mu_i$.

Congruences with the same kernel

Definition. A normal subsemigroup of S is a full subsemigroup N with the following conditions

(1) for any $x, y \in S, n \in N, x\,y \in N$ together imply $x\,n\,y \in N$, and

(2) for any $x, y \in S, n \in N, x\,n\,y \in N$ together imply; $x\,n^{\dagger}\,y \in N, x\,n^{*}\,y \in N$.

Examples:

(1) E is a normal subsemigroup of S.

(2) (Q,.) is a normal subsemigroup of (R,.).

(3) (Z,.) is not a normal subsemigroup of (Q,.).

(4) If S is an inverse semigroup and N is a full subsemigroup of S which satisfies condition (1), then N is a normal subsemigroup.

(5) If G is a group, N is a subgroup of G, then N is a normal subgroup if and only if for any x, y ∈ S, n ∈ N,

$$x\,y \in N \text{ implies } x\,n\,y \in N$$

Every non-normal subgroup does not satisfy condition (1).

There exists a non-normal subgroup which satisfies condition (2).

Example, let:

$$S = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x,\ y \in \mathbb{R},\ xy \neq 0 \right\} \text{ and } T = \left\{ \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} : x,\ y \in \mathbb{R},\ xy \neq 0 \right\}.$$

Put $G = S \cup T$. G is a group (so it is an ample monoid) under the matrix multiplication where $H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} : 0 \neq x \in \mathbb{R} \right\}$ is a subgroup. Since for any $\alpha = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ in T, $\alpha^{-1} = \begin{bmatrix} 0 & b^{-1} \\ a^{-1} & 0 \end{bmatrix}$ and if we take $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$ in H provided that $x \neq 1$ we find $\alpha\,\gamma\,\alpha^{-1} \notin H$. Then H is not normal. Let $\alpha,\ \beta \in G,\ \gamma \in H \left(\gamma^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \gamma^{*} \right)$.

Notice that:

when $\alpha,\ \beta \in S$ and $\alpha\,\gamma\,\beta \in H$, then $\alpha\,\beta \in H$,

when $\alpha \in S$ and $\beta \in T$, then $\alpha\,\gamma\,\beta \notin H$,

when $\alpha \in T$ and $\beta \in S$, then $\alpha\,\gamma\,\beta \notin H$,

when $\alpha,\ \beta \in T$, then $\alpha\,\gamma\,\beta \in H$ if $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $a\,d = 1$, where $\alpha = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}$, and in this case $\alpha\,\beta \in H$. Therefore, whenever $\alpha,\ \beta \in G$ and $\gamma \in H$ such that $\alpha\,\gamma\,\beta \in H$, we get $\alpha\,\beta \in H$. Hence H satisfies condition (2).

In conclusion, conditions (1) and (2) are independent.

Proposition *The kernel of any admissible congruence γ on S is a normal subsemigroup of S.*

For any subset N of a semigroup S, the well known *syntactic congruence* η_N of N on S is defined as follows:

$$\eta_N = \{ (a, b) \in S \times S; \text{ for any } x, y \in S^1; x a y \in N \Longleftrightarrow x b y \in N \}.$$

Proposition *Let N be a normal subsemigroup of S. Then the relation η_N is the maximum congruence on S whose kernel is N.*

Example Consider the ample monoid of integers Z with its normal subsemigroup N = {-1, 0, 1}.

The syntactic congruence η_N of N in Z is not admissible.

Definition. A congruence ρ on S is said to be an *idempotent-pure congruence* if $\ker \rho = E$.

Corollary *The relation*

$$\eta_E = \{(a, b) \in S \times S: \textit{for any } x, y \in S^1, x\, a\, y \in E \Leftrightarrow x\, b\, y \in E\}$$

is the maximum idempotent-pure congruence on S.

The congruence η_E *is not necessarily admissible.*

Example. Consider Z to be the ample monoid of integers with the semilattice of idempotents $E = \{0, 1\}$.

The maximum idempotent-pure congruence η_E on Z is not admissible.

Lemma

Let N be a normal subsemigroup of S and τ_N *be the relation on S defined by:*

$$\tau_N = \{ (x n_1 y, x n_2 y) : x, y \in S^1; n_1, n_2 \in N; n^\dagger_1 = n^\dagger_2 \}.$$

Then

(1) τ_N *is reflexive, symmetric and compatible relation on S.*

(2) $N = \{ a \in S : (a, e) \in \tau_N$ *for some* $e \in E \}$.

(3) τ_N *is contained in any admissible congruence on S whose kernel is N.*

Proposition [1]

Let λ_N *be the transitive closure of* τ_N ($\lambda_N = \tau^t_N$). *Then* λ_N *is a congruence on S whose kernel is N and it is contained in any admissible congruence on S whose kernel is N.*

Combine the previous two propositions in.

Corollary

If ρ *is an admissible congruence on S and N* = ker ρ, *then*

$$\lambda_N \subseteq \rho \subseteq \eta_N \quad \text{and} \quad \ker \lambda_N = \ker \rho = ker\ \eta_N .$$

Corollary.

If ρ is an admissible consgurence on S and N = ker ρ, then the following congruences:

(1) The minimum admissible congruence on S containing λ_N *,*

(2) The minimum admissible congruence on S whose kernel is N,

exist and they are equal.

Denote this congruence by ρ_k *.*

Congruence pairs

Definition. Let N be a normal subsemigroup of the ample semigroup S and π be a normal congruence on E; (π, N) is a congruence pair for S if:

(1) for any $n \in N$; $n^{\dagger} \, \pi \, n^{*}$.

(2) *for any* $x, y \in S$, *and any* $e, f \in E$; $x\,e\,y \in N$ *and* $e \,\pi\, f$ *together imply* $x\,f\,y \in N$.

Lemma

If ρ is an admissible congruence on S, then (tr ρ, ker ρ) is a congruence pair for S.

A congruence ρ on S is said to be associated with a congruence pair (π, N) if tr ρ = π and ker ρ = N.

Theorem [1],

Let ρ be an admissible congruence on S where ker $\rho = N$ and tr $\rho = \pi$. Then

(1) The relation $\mu_\pi \cap \eta_N$ is a congruence on S associated with the congruence pair (π, N).

(2) The relation $\sigma_\pi \vee \lambda_N$ is a congruence on S associated with the congruence pair (π, N).

(3) $\sigma_\pi \vee \lambda_N \subseteq \rho \subseteq \mu_\pi \cap \eta_N$.

Example (contributed by J.B Fountain).

Let $M = \{a, b\}^*$ be the free monoid on the elements a and b. Let π be the universal relation and $N = \{1\}$. Consider the congruence pair (π, N) for M. Let $T = \{c\}^*$ be the free monoid on the element c. Let $\varphi : M \to T$ be the admissible homomorphism determined by; $a\varphi = c = b\varphi$ and put $\rho_1 = \varphi \circ \varphi^{-1}$. Then ρ_1 is admissible congruence, tr $\rho_1 = \pi$ and ker $\rho_1 = N$. Let $\psi : M \to T$ be the admissible homomorphism determined by; $a\,\psi = c$, $b\,\psi = c^2$ and put $\rho_2 = \psi \circ \psi^{-1}$. Then ρ_2 is admissible congruence, tr $\rho_2 = \pi$ and ker $\rho_2 = N$. So we can have two different admissible congruences on M associated with the same congruence pair (π, N).

Certain minimum admissible congruences

Notations:

For any admissible congruence ρ on S,

The minimum admissible congruence on S whose restriction to E is tr ρ is denoted by ρ_t .

The minimum admissible congruence on S whose kernel is ker ρ *is denoted by* ρ_k .

(1) $\omega_t = \sigma$ (ω is the universal congruence on S)

(2) The relation $\eta = \{(x\, a\, y,\ x\, b\, y) : a, b \in S,\ x, y \in S^1,\ a\, D^* b\}^t$ *is the minimum semilattice admissible congruence on S.*

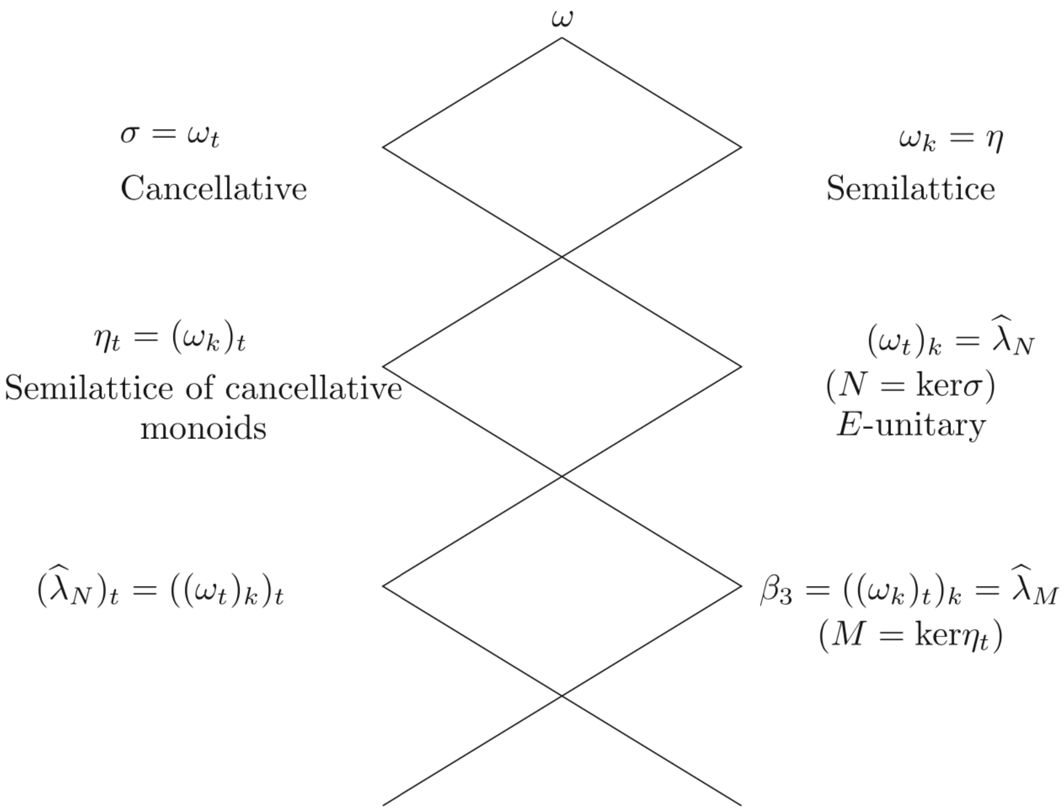
Proposition. $\eta = \omega_k$.

(3) The relation $\eta_t = \{ (a, b) \in S \times S : (a, b) \in \eta,\ ae = be \text{ for some } e \in a\eta \cap E \}$.

is the minimum semilattice of cancellative monoids admissible congruence on S.

($\eta_t = (\omega_k)_t$.

(4) The minimum E – unitary admissible congruence on S is $(\omega_t)_k$ [$= \sigma_k$].



A network of admissible congruences

The minimum admissible congruence of a kernel-class

Let γ be an admissible congruence on S. The minimum admissible congruence on S whose trace is tr γ is denoted by γ_t .

We shall characterize the minimum admissible congruence on S whose kernel is ker γ_t provided that γ satisfies the following condition:

For some positive integer n, $y^{n+1} \rho y^n \Rightarrow y^2 \rho y$. (for any $y \in S$).

In this case γ is called a *P-congruence.*

Notations. $\gamma_1 = \gamma_t$, $\gamma_2 = (\gamma_t)_k$, $\gamma_3 = ((\gamma_t)_k)_t$.

γ_2 and γ_3 will be investigated under the assumption that γ is a *P-congruence.*

Lemma.

If γ is a P-congruence on S, then:

(1) γ_k *is a P-congruence.*

(2) γ_t *is a P-congruence.*

let γ be a fixed but arbitrary P-congruence on S.

For any positive integer n, a congruence ρ on S is a Q_n-congruence associated with γ if ρ satisfies the following condition:

(Q_n) For any $x, y \in S$, $x\, y^{n+1}\, \rho\, x\, y^n$, $x\, \gamma\, y \Rightarrow y^2\, \rho\, y$.

We may write " Q -congruence associated with γ " to indicate that:

" Q_n -congruence associated with γ for any positive integer n".

Remark. γ_1 and γ_2 are Q –congruences associated with γ.

γ_1 is admissible, it is a Q-congruence associated with γ . So then the minimum admissible Q-congruence associated with γ exists. Let ψ be such a congruence. Since γ_2 exists as an admissible congruence and γ_2 is a Q-congruence associated with γ, then clearly $\psi \subseteq \gamma_2$.

Therefore, ker $\psi \subseteq$ ker γ_2. By definition, any admissible congruence on S whose kernel is ker γ_2 contains γ_2. It suffices for $\psi = \gamma_2$ is to show that: ker $\gamma_2 \subseteq$ ker ψ.

Recall that ker γ_2 = ker γ_1. Let $a \in$ ker γ_2 and $e \in E$ such that $ae = e$, $e \, \gamma \, a^*$ $(\gamma_1 \subseteq \gamma)$.

Notice that

$$e \, \gamma \, a^* \Rightarrow a \, e \, \gamma \, a \Rightarrow e \, \gamma \, a.$$

It is clear that:

$$(ae)^\dagger = e,\ ae = (a\,e)^\dagger a = ea \text{ (S is ample)},\ ea = e \text{ and } ea^{n+1} = ea^n.$$

In particular $ea^{n+1} \, \psi \, ea^n$, $e \, \gamma \, a$. Since ψ is a Q-congruence associated with γ, then $a^2 \, \psi \, a$ and as ψ is admissible, $a \in$ ker ψ. Hence $\psi = \gamma_2$

Theorem [2]

Let n be a positive integer. Then the relation γ_2 *is the minimum admissible Q-congruence associated with* γ.

Corollary

Any admissible congruence ρ *on S such that* $\gamma_2 \subseteq \rho \subseteq \gamma$ *is a Q- congruence associated with* γ.

Corollary

If ρ *is an admissible congruence on S and for some positive integer n,* ρ *is a* Q_n*-congruence associated with* γ*, then* ρ *is a P-congruence.*

An assistant minimum congruence

Let T be an ample semigroup and F be its semilattice of idempotents.

Let δ be a P-congruence on T satisfying *the condition* (K) where:

(K) For any $x, y \in T$ and any positive integer n :

$$x\,y^{n+1} = x\,y^{n},\ x\ \delta\ y,\ e \in F \Rightarrow e\,y = y\,e.$$

The aim is to find an admissible congruence on T which is a Q-congruence associated with δ and could be used to characterise γ_3.

For any $e \in F$, let:

$$N_e = \{a \in \mathrm{H}_e^* : fa = f \text{ for some } f \in F,\ f\delta e\}.$$

Put $N = \cup_{e \in F} N_e$. Also let $\delta_1 = \delta_t$ and $\delta_2 = (\delta_t)_k$.

Proposition. The relation

$$\lambda = \{(x\,n\,y, x\,m\,y) : x, y \in T, n, m \in N_e \textit{ for some } e \in F\}^t,$$

is a congruence on S whose kernel is N and is contained in any admissible congruence whose kernel is N.

Corollary. The congruence λ is contained in both H^* and δ_2.

Since δ_2 is an admissible congruence and $\lambda \subseteq \delta_2$, then the minimum admissible congruence $\widehat{\lambda}$ on T containing λ exists. As $\lambda \subseteq \widehat{\lambda} \subseteq \delta_2$ and $\ker \lambda = N = \ker \delta_2$, then $\ker \widehat{\lambda} = N$. Also λ is a congruence included in H^* and the relation μ is the maximum congruence in H^* [6]. Therefore $\lambda \subseteq \mu$. But μ itself is admissible on T (see [3]or [4]). Hence $\widehat{\lambda} \subseteq \mu$ and $\widehat{\lambda}$ is an idempotent-separating congruence.

Lemma [2] *For the admissible congruence* $\widehat{\lambda}$ *on T the following statements hold.*

(1) $\ker \widehat{\lambda} = N$ *and* $\widehat{\lambda} \subseteq \delta_2$.

(2) The relation $\widehat{\lambda}$ *is an idempotent-separating congruence.*

Theorem [2] *The relation* $\widehat{\lambda}$ *is an admissible Q-congruence associated with* δ.

Corollary [2] *The admissible congruence* $\widehat{\lambda}$ *is equal to* δ_2 *on T* .

The minimum admissible congruence of a trace-class

A congruence ρ on S is called an R_n- *congruence associated with* γ for some positive integer n if ρ satisfies for such positive integer n, the following condition.

(R_n) For any $x, y \in S$,

$$x\, y^{n+1} \rho\, x\, y^n,\; x\, \gamma\, y,\; e \in E \Rightarrow\; e\, y\, \rho\, y\, e.$$

It seems the R_n-congruence condition weakens the Q_n-congruence condition.

We use the statement "R-congruence associated with γ" to indicate that "R_n-congruence associated with γ for any positive integer n".

Proposition.

If ρ *is an admissible* R_n*-congruence associated with* γ *for some positive integer* n*, then* ρ *is a* P*-congruence.*

Lemma

The relations: γ, γ_1, γ_2 *and* γ_3 *are admissible* R*-congruences associated with* γ .

The minimum admissible R-congruence θ associated with γ exists.

Corollary

The minimum admissible R*-congruence* θ *associated with* γ *is included in* γ_3.

Lemma

If there exists an admissible *Q-congruence* β associated with γ such that $tr\,\theta = tr\,\beta$, then $tr\,\theta = tr\,\gamma_3$.

In this case:

β is an admissible *R-congruence* associated with γ and $\theta \subseteq \beta$.

As θ is an admissible congruence on S, S/θ is an ample semigroup, $\theta \subseteq \gamma_3$.

Since $\gamma_3 \subseteq \gamma$, then we have a well-defined congruence γ/θ on S/θ.

Let $T = S/\theta$ and $\delta = \gamma/\theta$.

The transitional objective is to determine a candidate for β.

Lemma

The congruence relation δ is a P-congruence on T satisfying condition (K).

Consider - as before - for any $e \in E(T)$, the set:

$$N_e = \{a \in \mathrm{H}_e^*(T) : fa = f \text{ for some } f \in E(T),\ f\delta e\},$$

and form

$$N = \bigcup_{e \in E(T)} N_e.$$

N is the kernel of δ_1, and the relation:

$$\lambda = \{(xny, xmy) : x, y \in T, n, m \in N_e \text{ for some } e \in E(T)\}^t$$

is a congruence whose kernel is N. The congruence λ on T is contained in any admissible congruence on T whose kernel is N.

Let $\widehat{\lambda}$ be the minimum admissible congruence on T containing λ.

Define a relation ρ on S by the rule that for any $x, y \in S$,

$$(x, y) \in \rho \text{ if and only if } (x\theta, y\theta) \in \widehat{\lambda}.$$

It is readily:

(1) ρ is a congruence,

(2) $\theta \subseteq \rho$, and

(3) $\widehat{\lambda} = \rho/\theta$.

Moreover, we have:

(a) The relation ρ *is an admissible Q-congruence associated with* γ .

(b) tr ρ = tr θ.

Now, ρ satisfies the condition of β

Therefore,

tr θ = tr γ_3 and $\gamma_3 \subseteq \theta$ but $\theta \subseteq \gamma_3$. Hence,

Theorem [2]

The relation γ_3 is the minimum admissible R-congruence associated with γ .

Proposition

If ρ is an admissible congruence on S such that $\rho \subseteq \gamma_1$, then the following statements are equivalent.

(1) For any positive integer n, ρ satisfies (R_n).

(2) There exists a positive integer n such that ρ satisfies (R_n).

(3) $\gamma_3 \subseteq \rho$.

A min-network

Recall

(1) $\alpha_1 = \omega_t = \sigma$

where σ is the minimum cancellative congruence on S ;

(2) $\beta_1 = \omega_k = \eta$

where η is the minimum semilattice congruence on S ;

(3) $\alpha_2 = (\beta_1)_t = \eta_t$

where η_t is the minimum semilattice cancellative monoids admissible congruence on S;

(4) $\beta_2 = (\alpha_1)_k = \widehat{\lambda_N}$

where $\widehat{\lambda_N}$ is the minimum admissible congruence on S containing λ_N ,

$N = \ker \sigma$ and λ_N is the congruence on S whose kernel is N and it is contained in any admissible congruence on S whose kernel is N.

(5) $\alpha_3 = (\beta_2)t = (\widehat{\lambda_N})_t = ((\omega_t)_k)_t$,

which is the minimum *E*- unitary admissible congruence on *S*.

(6) $\beta_3 = (\alpha_2)_k = (\eta_t)_k = \widehat{\lambda_M}$

Which is the minimum admissible congruence on S containing λ_M, (where $M = \ker \eta_t$).

(7) $\alpha_4 = (\beta_3)_t = ((\eta_t)_k)_t$

where its existence as an admissible congruence is based on the admissible congruence $(\eta_t)_k$ mentioned in (6).

Since $(\widehat{\lambda_N})_t$ as written in (5)is also an existed admissible congruence,

put $U = \ker (\widehat{\lambda_N})_t$ and define λ_U as λ_N in (4) whose kernel is *U*.

(8) β_4 will be the minimum admissible congruence containing λ_U .

Proposition

The congruences in the following two sequences

I $\alpha_1 \supseteq \beta_2 \supseteq \alpha_3 \supseteq \cdots \supseteq \alpha_{2n+1} \supseteq \beta_{2n+2} \ldots,$

II $\beta_1 \supseteq \alpha_2 \supseteq \beta_3 \supseteq \cdots \supseteq \beta_{2n+1} \supseteq \alpha_{2n+2} \ldots,$

are existed admissible congruences where for any positive integer n

$\alpha_n = (\beta_{n-1})_t$, $\beta_n = (\alpha_{n-1})_k$ *and* $\alpha_0 = \omega = \beta_0$.

Since the universal relation ω on *S* is *P*-congruence, then β_1 is *P*- congruence, the relations α_1, β_2 are *P*-congruences. The relation α_2 is *P*-congruence and α_3 is *P*-congruence, so then inductively we have:

Corollary

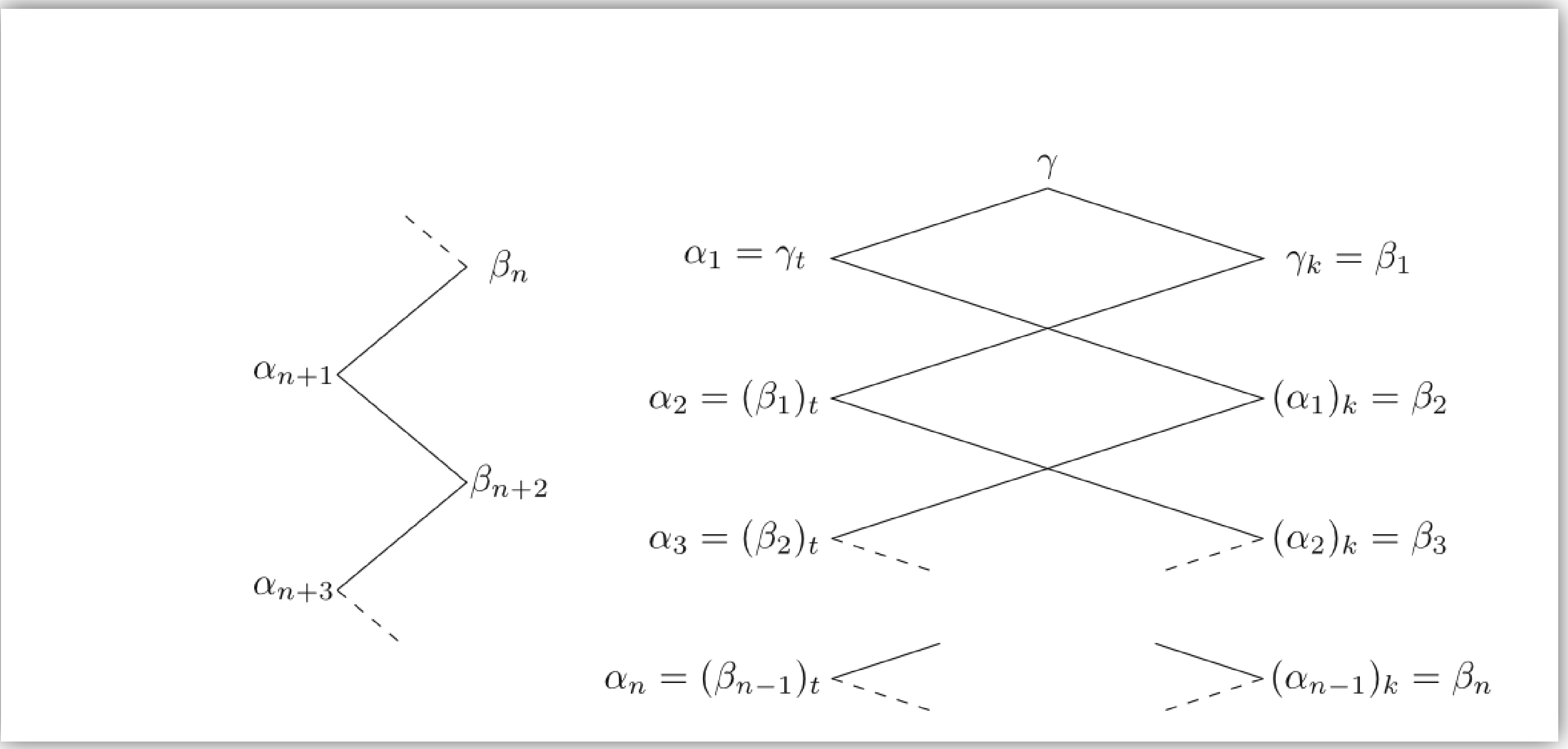
All congruences of the two sequences I and II are P –congruences.

Let γ be an admissible congruence on S. Recall that γ_t (= α_1, say) is admissible congruence on S. γ_k (= β_1, say) and $\beta_2 = (\alpha_1)_k$ are admissible congruences. The process can be continued to have the following two sequences of admissible congruences:

(1) $\alpha_1 \supseteq \beta_2 \supseteq \cdots \supseteq \beta_{2n} \supseteq \alpha_{2n+1} \supseteq \cdots$

(2) $\beta_1 \supseteq \alpha_2 \supseteq \cdots \supseteq \alpha_{2n} \supseteq \beta_{2n+1} \supseteq \cdots$

where ker β_n = ker α_{n-1}, tr α_n = tr β_{n-1} for any positive integer n; $\alpha_0 = \gamma = \beta_0$.



If γ is P-congruence, then the relation β_1 is P-congruence and so are α_1 , β_2 .

The relation α_2 is P-congruence and the relation α_3 is P -congruence, so then inductively all the congruences in the sequences (1) and (2) are P-congruences.

For any positive integer n, $((\beta_n)_t)_k$ is the minimum Q-congruence associated with β_n where:

$$\beta_{n+2} = (\alpha_{n+1})_k = ((\beta_n)_t)_k .$$

Hence we have

Proposition

For any positive integer n, the congruence β_{n+2} is the minimum admissible Q-congruence associated with β_n.

Moreover,

β_n is an admissible R-congruence associated with β_n (for any positive integer n).

Let θ_n be the minimum admissible R-congruence associated with β_n. Consider the ample semigroup $T_n = S/\theta_n$.

Let $\delta_n = \beta_n/\theta_n$. For any $e \in E(T_n)$,

let N_e be as before. Construct $N_n = \bigcup_{e \in E(T_n)} N_e$. Let:

$$\lambda_{N_n} = \{(xay, xby) : x, y \in T_n, a, b \in N_e \text{ for some } e \in E(T_n)\}^t.$$

λ_{N_n} is a congruence on T_n whose kernel is N_n and it is contained in any admissible congruence whose kernel is N_n. Let $\widehat{\lambda}_{N_n}$ be the minimum admissible congruence on S containing λ_{N_n}.

$\widehat{\lambda}_{N_n}$ is an admissible Q-congruence associated with δ_n. In this case:

$$\theta_n = (((\beta_n)_t)_k)_t.$$

Since $\alpha_{n+3} = (\beta_{n+2})_t = ((\alpha_{n+1})_k)_t = (((\beta_n)_t)_k)_t$, hence we have

Proposition

For any positive integer n, the congruence α_{n+3} is the minimum admissible R-congruence associated with β_n.

Closing statements

(1) Enriching the topic of congruences on ample semigroups.

The approach presented in this seminar based on [9, 10]. Another approach might emerge based on the new approach adopted in

Feng, Y. Y., Wang, L. M., Zhang, L., & Huang, H. Y. (2019). A new approach to a network of congruences on an inverse semigroup. In *Semigroup Forum* (Vol. 99, pp. 465 - 480).

As some results related to congruences on ample semigroups are also appear in [7]. The research may continue from the same (or different) viewpoint to enrich the present results.

Do we have to stick with the conditions: P-congruence, Q-congruence and R-congruence?

(2) Extending toward the congruences on restriction semigroups.

The properties of congruences on ample semigroups presented in this seminar might be extended to classes of semigroups related to restrictions semigroups (weak type-A semigroups).

In fact, the process has already been started by

Gould, V. (2012). Restriction and Ehresmann semigroups. In *Proceedings of the International Conference on Algebra 2010: Advances in Algebraic Structures* (pp. 265-288).

and continued by extending some results from [1] to restriction semigroups in

Zhang, Z., Guo, J., & Guo, X. Congruence-free restriction semigroups. *Italian Journal of Pure and Applied Mathematics*, 634.

We may also look at the congruences on Fountains semigroups. This has been started by,

El-Qallali, A, Congruences on Fountain semigroups. Preprint.

Thank you.

The floor is open for questions and comments.