Endomorphisms of the random graph

Martyn Quick

University of
St Andrews

York Semigroup Seminar
7th March 2018

This is joint work with:

- Bob Gray (UEA)
- Jay McPhee (formerly St Andrews)
- James Mitchell (St Andrews)
- Igor Dolinka (Novi Sad)

This work appears in: **DGMMQ** "Automorphism groups of countable algebraically closed graphs and endomorphisms of the random graph," *Math. Proc. Camb. Phil. Soc.* **160** (2016).

All graphs considered are countable simple graphs:
No multiple edges and no loops.

The random graph R

[Arises in model theory]

Start with vertices: $v_1, v_2, \ldots$.
For each pair of vertices, toss a coin:
If **H** the vertices are joined; if **T** the vertices are not joined by an edge.

With probability 1, the resulting graph, the random graph R, is existentially closed:

> If A and B are disjoint finite sets of vertices, there exists some vertex v that is joined to all the vertices in A and to none of the vertices in B.

This property uniquely characterises R.
A back-and-forth argument shows that any two countable graphs satisfying the condition are isomorphic.

More properties of the random graph

R is homogeneous:

Every isomorphism $\phi\colon \Gamma_1 \to \Gamma_2$ between finite subgraphs of R can be extended to an automorphism $\hat{\phi}$ of$R$.

R is the Fraïssé limit of the finite graphs

The class $\mathcal{C}$ of finite graphs satisfy the hereditary property, joint embedding property and amalgamation property.
Fraïssé's Theorem says $\mathcal{C}$ has a Fraïssé limit.
This is the random graph R: $\mathrm{age}(R) = \mathcal{C}$.

Theorem (Truss, 1985)

The automorphism group of R is simple.

Construction of the random graph

If $\Gamma = (V, E)$ is any countable graph, enumerate the finite subsets of V as $(A_i)_{i \in \mathbb{N}}$. Define $\mathcal{G}(\Gamma)$ to be the graph with vertices

$$V \cup \{ v_i \mid i \in \mathbb{N} \},$$

edges E plus new edges joining each v_i to each vertex in A_i for all $i \in \mathbb{N}$. Then

- Γ is a subgraph of $\mathcal{G}(\Gamma)$,
- given two disjoint finite subsets A and B of V, there exists some v joined to every vertex of A and to none of the vertices in B (namely v_i when $A = A_i$).

Now define $\Gamma_0 = \Gamma$ and $\Gamma_{n+1} = \mathcal{G}(\Gamma_n)$ for each n.

Observation

$\Gamma_\infty = \mathcal{G}^\infty(\Gamma) = \varinjlim \Gamma_n = \bigcup_{n=0}^{\infty} \Gamma_n$ is isomorphic to the random graph R.

Green's relations on $M = \operatorname{End} R$

$$f \mathrel{\mathscr{L}} g \quad \text{when } Mf = Mg \qquad (\mathscr{R} \text{ sim.})$$
$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}$$
$$\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$$

- Maximal subgroups of $\operatorname{End} R$ are the $\mathscr{H}$-classes of idempotents ($f^2 = f$).
- Regular $\mathscr{D}$-classes are those that contain group $\mathscr{H}$-classes.
- If f is an idempotent, then $f|_{\operatorname{im} f} = \operatorname{id}$ and

$$H_f \cong \operatorname{Aut}(\operatorname{im} f).$$

Indeed, if $g \in \operatorname{Aut}(\operatorname{im} f)$, then $fg \in \operatorname{End} R$ satisfies

$$(fg^{-1})(fg) = f, \quad (fg)f = fg, \qquad (fg)(fg^{-1}) = f, \quad f(fg) = fg$$

so $fg \mathrel{\mathscr{H}} f$. The isomorphism is $fg \leftrightarrow g$.

Idempotents in $\operatorname{End} R$

Upshot: Need to understand the idempotent endomorphisms f of R.

Note that since R is existentially closed, it is also algebraically closed:

a.c.: If A is any finite set of vertices, there exists some vertex v joined to all vertices in A.

This is inherited by images: $\operatorname{im} f$ is algebraically closed.
Conversely, if Γ is a.c., we can extend the identity map to a homomorphism $\mathcal{G}^{\infty}(\Gamma) \to \Gamma$.

Theorem (Bonato–Delić, 2000)

There is an idempotent endomorphism f of R with $\operatorname{im} f \cong \Gamma$ if and only if Γ is a.c.

Uncountably many idempotent endomorphisms with given image

Suppose $\Gamma_0 = \Gamma$ is a.c. $\qquad \Gamma_{n+1} = \mathcal{G}(\Gamma_n)$.
Assume we've constructed $f_n \colon \Gamma_n \to \Gamma$ with $f_n|_\Gamma = \mathrm{id}$.

In Γ_{n+1} have vertices v_i corresponding to finite $A_i \subseteq V(\Gamma_n)$.

Extend f_n as follows:

- Assume images of v_1, v_2, ..., v_k have already been specified; i.e., have defined f_{n+1} on the subgraph induced by $V \cup \{v_1, v_2, \dots, v_k\}$.
- Γ is a.c. $\Rightarrow$ $\exists w$ adjacent to every vertex of $(A_{k+1} \cup \{v_1, \dots, v_k\}) f_{n+1}$.
- Extend: Define $v_{k+1} \mapsto w$.

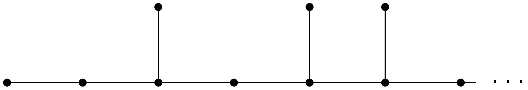
There are infinitely many choices for w. (Need only more than one!)

Conclusion: $2^{\aleph_0}$ extensions to $\Gamma_\infty \cong R$.

Constructing a.c. graphs

Let Γ be *any* countable graph and $S \subseteq \{2, 3, 4, \dots\}$.

Construct L_S:



Write † to denote the complement. Then

$$(\Gamma \mathbin{\dot\cup} L_S)^\dagger \text{ is a.c.}$$

and, provided $L_S \ncong \Gamma$,

$$\operatorname{Aut}(\Gamma \mathbin{\dot\cup} L_S)^\dagger \cong \operatorname{Aut}(\Gamma \mathbin{\dot\cup} L_S) \cong \operatorname{Aut}\Gamma \times \operatorname{Aut} L_S \cong \operatorname{Aut}\Gamma.$$

Conclusion: $2^{\aleph_0}$ a.c. graphs with specified automorphism group.

The maximal subgroups of $\operatorname{End} R$

Theorem (DGMMQ)

(i) *Let Γ be a countable graph.*
There are $2^{\aleph_0}$ regular $\mathscr{D}$-classes of $\operatorname{End} R$ whose group $\mathscr{H}$-classes are isomorphic to $\operatorname{Aut} \Gamma$.

(ii) *Every regular $\mathscr{D}$-class of $\operatorname{End} R$ contains $2^{\aleph_0}$ group $\mathscr{H}$-classes.*

Every group that could appear as a maximal subgroup of $\operatorname{End} R$ occurs and does so as many times as it possibly could.

The maximal subgroups of $\operatorname{End} R$

Theorem (DGMMQ)

(i) *Let Γ be a countable graph.*
There are $2^{\aleph_0}$ regular $\mathscr{D}$-classes of $\operatorname{End} R$ whose group $\mathscr{H}$-classes are isomorphic to $\operatorname{Aut} \Gamma$.

(ii) *Every regular $\mathscr{D}$-class of $\operatorname{End} R$ contains $2^{\aleph_0}$ group $\mathscr{H}$-classes.*

Proof.

(i) Take $S \subseteq \{2, 3, \dots\}$ with $L_S \ncong \Gamma$. There is an idempotent f_S with image $\cong (\Gamma \dot{\cup} L_S)^\dagger$. Then

$$H_{f_S} \cong \operatorname{Aut}(\operatorname{im} f_S) \cong \operatorname{Aut} \Gamma.$$

For $S \neq T$, these lie in different $\mathscr{D}$-classes because $L_S \ncong L_T$, so $\operatorname{im} f_S \ncong \operatorname{im} f_T$.

(ii) For each a.c. graph Γ, there are $2^{\aleph_0}$ idempotents with image $\cong \Gamma$. □

$\mathscr{L}$- and $\mathscr{R}$-classes in **regular** $\mathscr{D}$-classes

Theorem (DGMMQ)

Every regular $\mathscr{D}$-class in $\operatorname{End} R$ contains $2^{\aleph_0}$ many $\mathscr{L}$- and $\mathscr{R}$-classes.

For f, g regular:

$$\begin{aligned} f \mathrel{\mathscr{L}} g \quad &\text{iff} \quad Vf = Vg \\ f \mathrel{\mathscr{R}} g \quad &\text{iff} \quad \ker f = \ker g \\ f \mathrel{\mathscr{D}} g \quad &\text{iff} \quad \operatorname{im} f \cong \operatorname{im} g \end{aligned}$$

[$\Rightarrow$ holds without the regularity assumption.]

$2^{\aleph_0}$ $\mathscr{R}$-classes: Given an a.c. graph Γ, there are $2^{\aleph_0}$ idempotents with image $\cong \Gamma$ (extend the identity map on Γ).

All such f are $\mathscr{L}$-related, but not $\mathscr{R}$-related.

Uncountably many regular $\mathscr{L}$-classes

Start with an a.c. graph Γ (having vertices v_i).
Construct $\Gamma^\sharp$ with vertices

$$V^\sharp = \{ v_{i,0}, v_{i,1} \mid i \in \mathbb{N} \}$$

and edges

$$(v_{i,0}, v_{j,0}),\ (v_{i,0}, v_{j,1}),\ (v_{i,1}, v_{j,0}),\ (v_{i,1}, v_{j,1})$$

whenever (v_i, v_j) is an edge in Γ.
Note

- $\Gamma^\sharp$ is also algebraically closed.
- For any sequence $\mathbf{b} = (b_i)$ with $b_i \in \{0,1\}$, the subgraph $\Lambda_{\mathbf{b}}$ induced by $\{ v_{i,b_i} \mid i \in \mathbb{N} \}$ is isomorphic to Γ.

Build a copy of R (as $\mathcal{G}^\infty(\Gamma^\sharp)$) around $\Gamma^\sharp$.
Hence construct idempotent f in $\operatorname{End} R$ with $\operatorname{im} f = \Gamma^\sharp$.
Given $\mathbf{b}$, apply the map $\phi_{\mathbf{b}}$ that maps $v_{i,0}, v_{i,1} \mapsto v_{i,b_i}$.
Note the $f\phi_b$ are $\mathscr{D}$-related but not $\mathscr{L}$-related.

What about **non-regular** $\mathscr{D}$-classes?

Our conclusions are less complete.
Write $R = (V, E)$.
If $f \in \operatorname{End} R$, the key is understanding the difference between

$$\operatorname{im} f = (Vf, Ef) \qquad \text{vs.} \qquad \langle Vf \rangle = \big(Vf, E \cap (Vf \times Vf)\big).$$

$f \in \operatorname{End} R$ is regular if $\exists g$ with $fgf = f$.

$$f \text{ regular} \quad \Rightarrow \quad \operatorname{im} f = (Vf, Ef) = \langle Vf \rangle$$

Proposition (Cameron–Nešetřil, 2006)

Let $\Gamma = (V', E')$ be a countable graph. Then Γ is algebraically closed if and only if $(V', F) \cong R$ for some $F \subseteq E'$.

We use this to construct a injective homomorphism $f \colon R \to \Gamma$ such that $\operatorname{im} f = (V', F) \neq \langle Vf \rangle = (V', E')$.

Uncountably many non-regular $\mathscr{D}$-classes

Let Γ be an a.c. graph with $\Gamma \ncong R$.
Create $\Gamma^\sharp$ with vertices $\{ v_{i,0}, v_{i,1} \mid i \in \mathbb{N} \}$. Set $\Lambda_0 = \langle v_{i,0} \mid i \in \mathbb{N} \rangle \cong \Gamma$.
Build $R = \mathcal{G}^\infty(\Gamma^\sharp) = (V, E)$.

Use Cameron–Nešetřil: there is an injective endomorphism $f \colon R \to R$ with $Vf = \{ v_{i,0} \mid i \in \mathbb{N} \}$. So $\operatorname{im} f \cong R$ and $\langle Vf \rangle = \Lambda_0 \cong \Gamma$.
In particular, f is not regular.

If $\mathbf{b} = (b_i) \in \{0,1\}^{\mathbb{N}}$, the map $v_{i,j} \mapsto v_{i,j+b_i}$ is an automorphism of $\Gamma^\sharp$.
It extends to an automorphism $\psi_{\mathbf{b}}$ of R.
Then $f\psi_{\mathbf{b}}$ is $\mathscr{R}$-related to f.
No pair of these are $\mathscr{L}$-related.

Can also create $2^{\aleph_0}$ many $\mathscr{R}$-classes in D_f.
Varying Γ yields $2^{\aleph_0}$ many $\mathscr{D}$-classes.

Summary for non-regular $\mathscr{D}$-classes

Theorem (DGMMQ)

(i) *There exists a non-regular injective endomorphism f of R such that the $\mathscr{D}$-class of f contains $2^{\aleph_0}$ many $\mathscr{L}$- and $\mathscr{R}$-classes.*

(ii) *There are $2^{\aleph_0}$ many non-regular $\mathscr{D}$-classes in $\operatorname{End} R$.*

Questions

1. Can the injectivity condition in (i) be removed?
2. Does (i) hold for all non-regular $\mathscr{D}$-classes?

Schützenberger Groups

If the $\mathscr{H}$-class of $f \in \operatorname{End} R$ is not a group, can create the Schützenberger group $\mathcal{S}_H$.

This highlights the distinction between $\operatorname{im} f = (Vf, Ef)$ and $\langle Vf \rangle$ for certain f arising via Cameron–Nešetřil:

Let $\Gamma_0 = (V_0, E_0)$ be a.c. and construct R as $R = \mathcal{G}^{\infty}(\Gamma_0)$. There is an injective endomorphism f with $Vf = V_0$.

Proposition

Let $H = H_f$ for such f. Then

$$\mathcal{S}_H \cong \operatorname{Aut}(\operatorname{im} f) \cap \operatorname{Aut}\langle Vf \rangle.$$

By a suitable construction of Γ_0 around a particular graph Γ obtain:

Theorem (DGMMQ)

Let Γ be a countable graph. There are $2^{\aleph_0}$ many non-regular $\mathscr{D}$-classes in $\operatorname{End} R$ that have Schützenberger groups isomorphic to $\operatorname{Aut} \Gamma$.

Directed graphs & bipartite graphs

Also have analogous results for the endomorphism of the countable universal homogeneous directed graph D and the countable universal homogeneous bipartite graph B.

Definition of bipartite graphs?
The partition is preserved by a homomorphism, but the parts may be interchanged.

Some unusual observations for bipartite graphs: e.g., the finite complete bipartite graphs are a.c.

Some results for bipartite graphs, I

Maximal subgroups / group $\mathscr{H}$-classes:

Theorem (DGMMQ)

1. *Let* Γ *be a countable graph.*
   There are $2^{\aleph_0}$ *regular* $\mathscr{D}$*-classes of* $\operatorname{End} B$ *whose group* $\mathscr{H}$*-classes are isomorphic to* $\operatorname{Aut} \Gamma$*.*
2. *Let* f *be an idempotent.*
   If $\operatorname{im} f \ncong K_{1,1}$*, then* D_f *contains* $2^{\aleph_0}$ *many group* $\mathscr{H}$*-classes.*
   If $\operatorname{im} f \cong K_{1,1}$*, then* D_f *contains* $\aleph_0$ *many group* $\mathscr{H}$*-classes (each* $\cong C_2$*).*

Some results for bipartite graphs, II

$\mathscr{L}$- and $\mathscr{R}$-classes in regular $\mathscr{D}$-classes:

Theorem (DGMMQ)

Let f be a regular endomorphism of B.

1. *If $\operatorname{im} f$ is infinite, D_f contains $2^{\aleph_0}$ many $\mathscr{L}$- and $\mathscr{R}$-classes.*
2. *If $\operatorname{im} f$ is finite but not $K_{1,1}$, then D_f contains $\aleph_0$ many $\mathscr{L}$-classes and $2^{\aleph_0}$ many $\mathscr{R}$-classes.*
3. *If $\operatorname{im} f \cong K_{1,1}$, then D_f contains $\aleph_0$ many $\mathscr{L}$-classes and one $\mathscr{R}$-class.*

The end!

Thank you for your attention!