Faithful representations of diagram monoids

Marianne Johnson
University of Manchester

Joint work with

James East (right) and Mark Kambites (left)
(with many thanks to James for sharing a version of these slides)

Faithful linear and relational representations of diagram monoids

- JE, MJ, MK
- `arXiv:2512.?????` (?)

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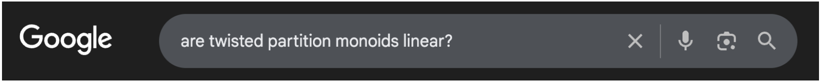

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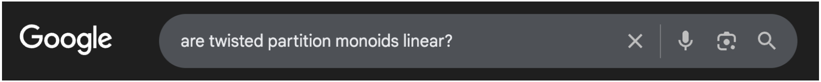

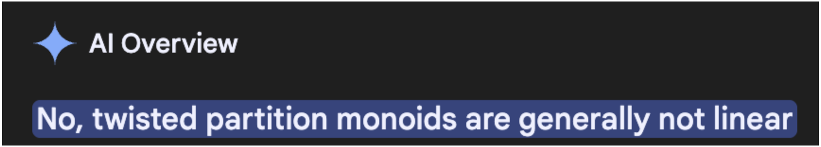

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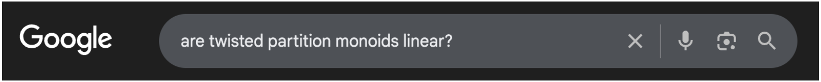

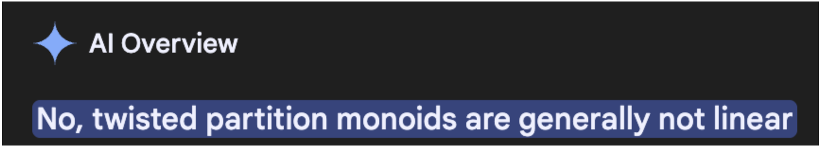

Theorem

Twisted partition monoids are linear.

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Abstract groups:

- permutation groups,
- matrix groups (over fields).

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Say that a representation is **faithful** if the morphism is injective.

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Today: matrix representations of:

- partition monoids (finite),
- twisted partition monoids (finite or infinite).

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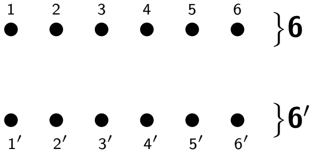

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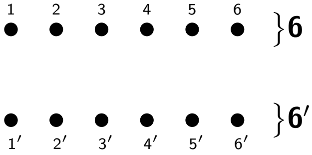

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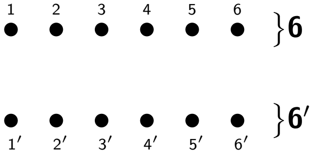

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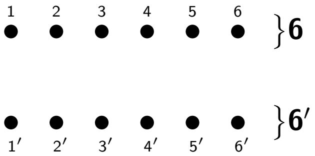

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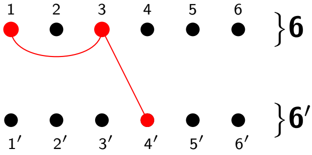

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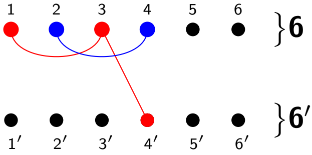

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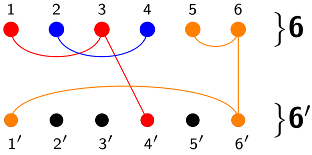

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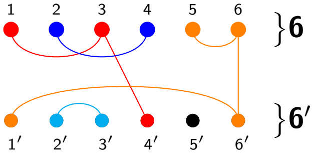

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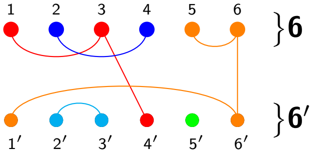

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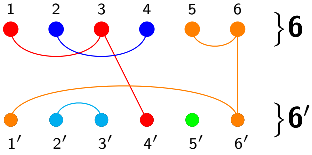

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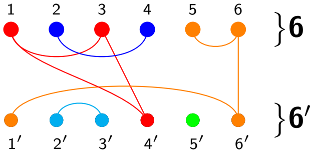

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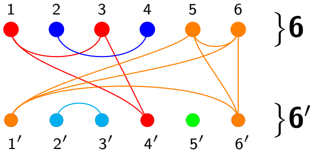

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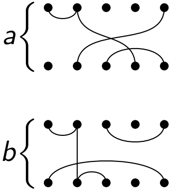

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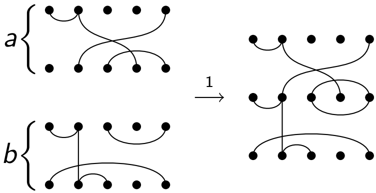

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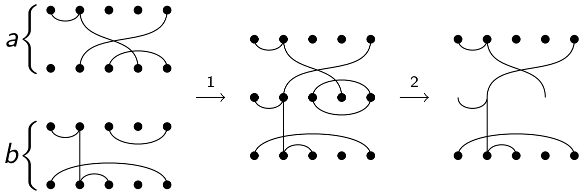

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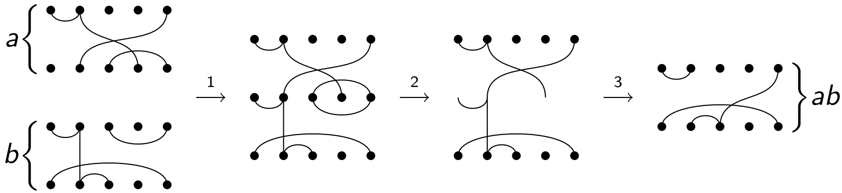
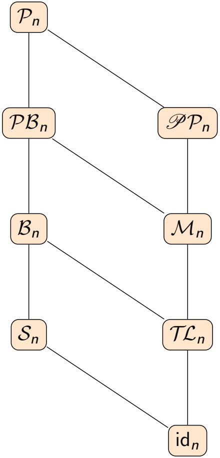
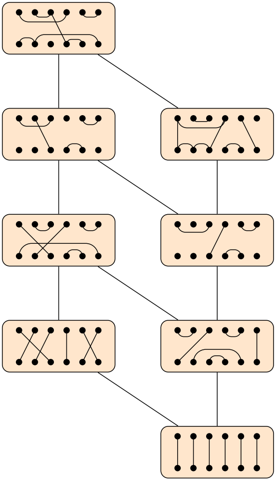


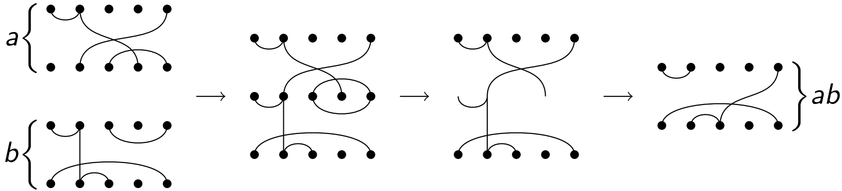
Diagram monoids — submonoids of $\mathcal{P}_n$





- Brauer, Temperley–Lieb (a.k.a. Jones), Motzkin, and more......

Twisted partition monoids — $\mathcal{P}_n^\Phi$



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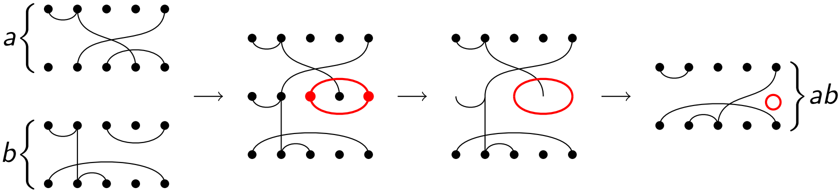

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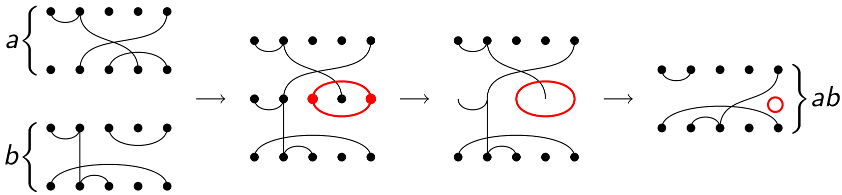

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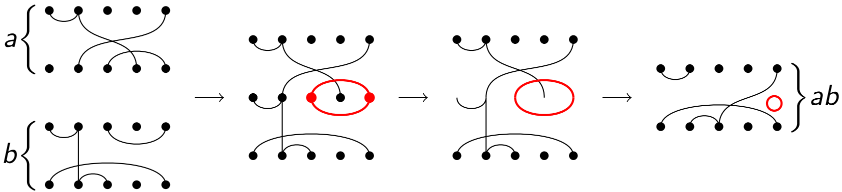

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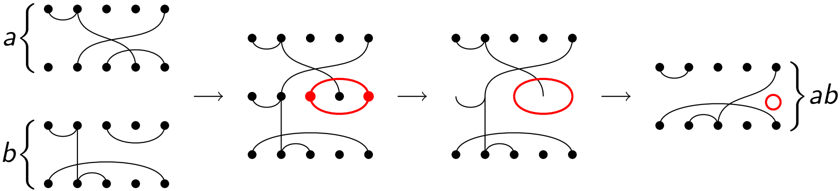

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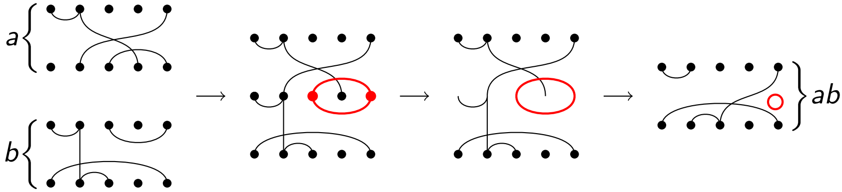

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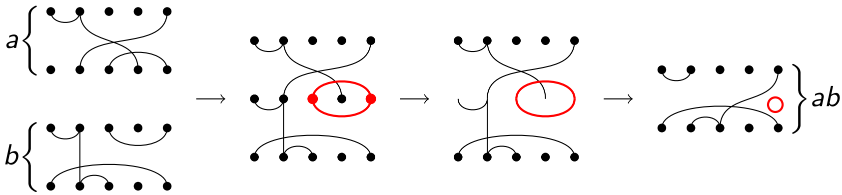

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  - So above, $(0, a) \cdot (0, b) = (1, ab)$.

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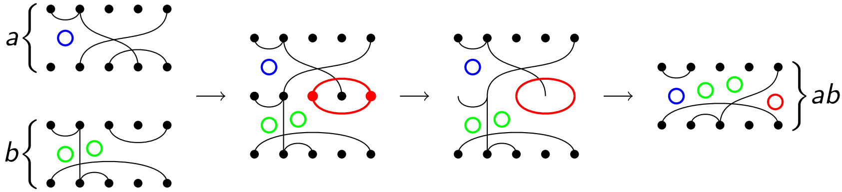

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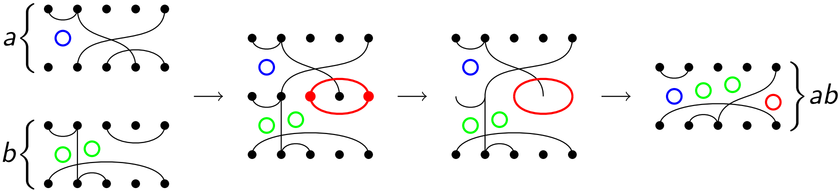

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Theorem (Reinis Cirpons, James East, James Mitchell, 2024)

The least k such that $\mathcal{P}_n$ embeds in $\mathcal{T}_k$ is

$$k = 1 + \frac{b_{n+2} - b_{n+1} + b_n}{2},$$

where b_n is the nth Bell number.

Can we do better? (Note: these numbers are BIG!)

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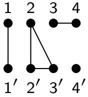
Example: $a =$ 

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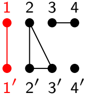
Example: $a =$ 

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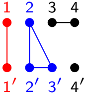
Example: $a =$ 

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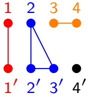
Example: $a =$ 

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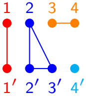
Example: $a =$ 

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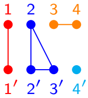
Example: $a =$ 

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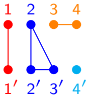
Example: $a =$ 

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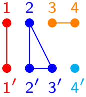
Example: $a =$ 

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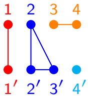
Example: $a =$  has blocks $\{1, 1'\}$, $\{2, 2', 3'\}$, $\{3, 4\}$, $\{4'\}$.

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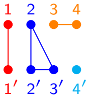
Example: $a =$  has blocks $\{1, 1'\}$, $\{2, 2', 3'\}$, $\{3, 4\}$, $\{4'\}$.

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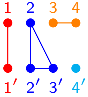
Example: $a =$ 

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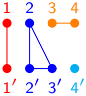
Example: $a =$ 

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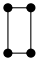
- $a =$ 

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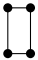
- $a =$  $\mapsto \overline{a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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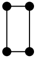
- $a =$  $\mapsto \overline{a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
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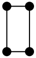
- $a =$  $\mapsto \overline{a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
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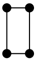
- $a =$  $\mapsto \overline{a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
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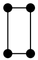
$$\overline{a}_{X,Y} = \begin{cases} 1 & \text{if } X \cup Y' \text{ is a union of blocks of } a, \\ 0 & \text{otherwise.} \end{cases}$$

- $a =$  $\mapsto \overline{a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
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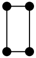
- $a =$  $\mapsto \overline{a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
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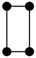
- $a =$  $\mapsto \overline{a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
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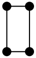
- $a =$  $\mapsto \overline{a} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
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- $\overline{b}^2 = 4\overline{b} = \overline{b}$ if $1 + 1 = 1$!

Main result

For $a \in \mathcal{P}_n$ define $\overline{a} \in M_{2^n}(R)$ by

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1. If R is AI, then $\overline{a}\overline{b} = \overline{ab}$ for $a, b \in \mathcal{P}_n$.

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Main corollaries — Brauer and Temperley–Lieb

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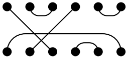
  -  $\in \mathcal{B}_6$.

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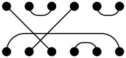
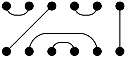
  -  $\in \mathcal{B}_6$.
  -  $\in \mathcal{TL}_6$.

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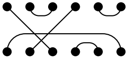
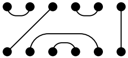
  -  $\in \mathcal{B}_6$.
  -  $\in \mathcal{TL}_6$.

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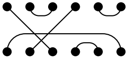
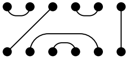
  -  $\in \mathcal{B}_6$.
  -  $\in \mathcal{TL}_6$.

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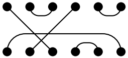
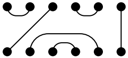
-  $\in \mathcal{B}_6$.
-  $\in \mathcal{TL}_6$.

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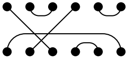
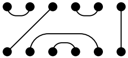
  -  $\in \mathcal{B}_6$.
  -  $\in \mathcal{TL}_6$.

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- Both are faithful for $\mathcal{B}_n$ when n is odd

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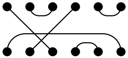
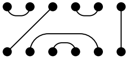
  -  $\in \mathcal{B}_6$.
  -  $\in \mathcal{TL}_6$.

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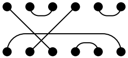
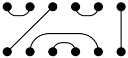
-  $\in \mathcal{B}_6$.
-  $\in \mathcal{TL}_6$.

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- Both are faithful for $\mathcal{B}_n$ when n is odd (but not for n even).
- Both are faithful for $\mathcal{TL}_n$ (for any n).

But we can do better for $\mathcal{TL}_n$!

Temperley–Lieb

Temperley–Lieb

Theorem

There is a faithful rep $\mathcal{TL}_n \to M_{f_n}(\mathbb{B}) : a \mapsto \underline{a}$, where f_n is the nth Fibonacci number.

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Matrix $\underline{a}$ is the restriction of $\overline{a}$ to 'even-gap' subsets of $\mathbf{n}$:

Temperley–Lieb

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- An even-gap subset $X = \{x_1 < x_2 < \cdots < x_k\} \subseteq \mathbf{n}$ satisfies:
  - $x_i \equiv i \bmod 2$,
  - $x_k \equiv n \bmod 2$.

For example $\{3, 8, 9, 10, 17, 18\}$ is an even gap susbet of $\mathbf{20}$:
$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$.

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The proof is pretty involved!

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The proof is pretty involved! There is no decomp $\overline{a} = \begin{bmatrix} \underline{a} & 0 \\ 0 & ? \end{bmatrix}$.

Summary

Monoid	Dimension of our faithful rep
$\mathcal{P}_n$, $\mathcal{P}_n^{\Phi}$, $\mathcal{P}_{n,d}^{\Phi}$	2^n
$\mathcal{B}_n$, $\mathcal{B}_n^{\Phi}$, $\mathcal{B}_{n,d}^{\Phi}$	2^n or 2^{n-1}
$\mathcal{TL}_n$, $\mathcal{TL}_n^{\Phi}$, $\mathcal{TL}_{n,d}^{\Phi}$	f_n, 2^{n-1}, 2^{n-1}

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Obvious Open Question

Are the above dimensions minimal?

Summary

Monoid	Dimension of our faithful rep
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- Thanks for listening! :-)