The Heights of Green's Posets of Semigroups

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Green's relations and associated posets

Let S be a semigroup.

Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ on S are defined by

$$a\,\mathcal{L}\,b \Leftrightarrow S^1a = S^1b, \quad a\,\mathcal{R}\,b \Leftrightarrow aS^1 = bS^1, \quad \mathcal{H} = \mathcal{L} \cap \mathcal{R},$$
$$\mathcal{D} = \mathcal{L} \vee \mathcal{R}\,(= \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}), \quad a\,\mathcal{J}\,b \Leftrightarrow S^1aS^1 = S^1aS^1.$$

For each Green's relation $\mathcal{K}$, we denote the $\mathcal{K}$-class of $a \in S$ by K_a.

For $\mathcal{K} \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}\}$, there is a natural partial order on the set of $\mathcal{K}$-classes of S; e.g. for $\mathcal{K} = \mathcal{L}$,

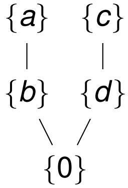
$$L_a \leq L_b \Leftrightarrow S^1a \subseteq S^1b.$$

The $\mathcal{K}$**-height** of S, denoted by $H_{\mathcal{K}}(S)$, is the size of a maximum chain of $\mathcal{K}$-classes of S, if such a chain exists, or is infinite.

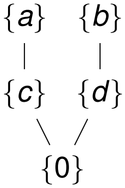
Example

Let S be the semigroup with multiplication table

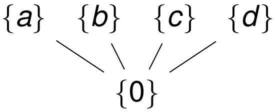
	a	b	c	d	0
a	a	0	c	0	0
b	b	0	d	0	0
c	0	0	0	0	0
d	0	0	0	0	0
0	0	0	0	0	0



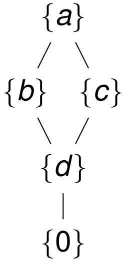
$\mathcal{L}$-classes, $H_{\mathcal{L}}(S) = 3$



$\mathcal{R}$-classes, $H_{\mathcal{R}}(S) = 3$



$\mathcal{H}$-classes, $H_{\mathcal{H}}(S) = 2$



$\mathcal{J}$-classes, $H_{\mathcal{J}}(S) = 4$

Stability

S is **left stable** (resp. **right stable**) if $\mathcal{L}$-classes (resp. $\mathcal{R}$-classes) within the same $\mathcal{J}$-class are incomparable.
S is **stable** if it is both left stable and right stable.
$\mathcal{D} = \mathcal{J}$ in stable semigroups.

Lemma. If $H_{\mathcal{L}}(S) < \infty$ (resp. $H_{\mathcal{R}}(S) < \infty$) then S is left stable (resp. right stable).

S is *uniformly group-bound* if there exists some $n \in N$ such that for every $a \in S$ we have a^n belongs to a subgroup of S.
(Uniformly) group-bound semigroups are stable.

Lemma. If $H_{\mathcal{H}}(S) < \infty$, then S is uniformly group-bound.

Proof. $H_a \geq H_{a^2} \geq H_{a^3} \geq \cdots$. Then $H_{a^n} = H_{a^{2n}}$ for some $n \leq H_{\mathcal{H}}(S)$, and hence H_{a^n} is a group.

If S is left stable, then $H_{\mathcal{H}}(S) \leq H_{\mathcal{R}}(S)$ and $H_{\mathcal{L}}(S) \leq H_{\mathcal{J}}(S)$.
If S is right stable, then $H_{\mathcal{H}}(S) \leq H_{\mathcal{L}}(S)$ and $H_{\mathcal{R}}(S) \leq H_{\mathcal{J}}(S)$.
Consequently, if S is stable, then

$$H_{\mathcal{H}}(S) \leq \min\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big) \quad \text{and} \quad \max\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big) \leq H_{\mathcal{J}}(S).$$

There exist semigroups with $\mathcal{L}$-height 1 and infinite $\mathcal{R}$-height, and vice versa (e.g. left/right simple semigroups that are not completely simple).

If $H_{\mathcal{L}}(S) = 1$ (or $H_{\mathcal{R}}(S) = 1$), then $H_{\mathcal{J}}(S) = 1$.
If $H_{\mathcal{L}}(S) = 2$ (or $H_{\mathcal{R}}(S) = 2$) then $H_{\mathcal{J}}(S) \in \{2, 3\}$.

There exist semigroups with $\mathcal{L}$-height 3 and infinite $\mathcal{J}$-height.

Bounds on the $\mathcal{R}$-height

Theorem. If $H_{\mathcal{L}}(S) = n < \infty$ and S is (right) stable, then

$$\left\lceil \frac{\ln(n+1)}{\ln 2} \right\rceil \leq H_{\mathcal{R}}(S) \leq 2^n - 1.$$

Construction. Let S be a semigroup with zero z. Let $\mathcal{U}(S) = S \cup \{x_s : s \in S^1\}$ (where $1 \notin S$). Define a multiplication on $\mathcal{U}(S)$, extending that on S, by

$$ax_s = x_s, \quad x_s a = x_{sa} \quad \text{and} \quad x_s x_t = x_z$$

for all $a \in S$ and $s, t \in S^1$. Then $\mathcal{U}(S)$ is a semigroup with zero x_z.

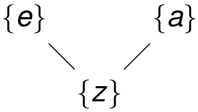
Proposition. Letting $U = \mathcal{U}(S)$, we have $H_{\mathcal{L}}(U) = H_{\mathcal{L}}(S) + 1$ and $H_{\mathcal{R}}(U) = 2H_{\mathcal{R}}(S) + 1$.

Theorem. For every $n \in \mathbb{N}$, there exists a $\mathcal{J}$-trivial semigroup S of order $2^n - 1$ such that $H_{\mathcal{L}}(S) = n$ and $H_{\mathcal{R}}(S) = 2^n - 1$.

Example: $S = \mathcal{U}(\{e\})$

$S = \mathcal{U}(\{e\})$ is the semigroup $\{e, a\,(= x_1), z\,(= x_e)\}$ with multiplication

$$e^2 = e, \quad ea = a, \quad ae = a^2 = zs = sz = z \quad (s \in S).$$

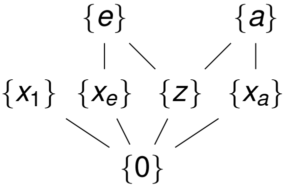


$\mathcal{L}$-classes of S

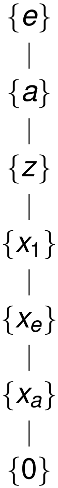
$\mathcal{R}$-classes of S

Example: $\mathcal{U}(S)$

	e	a	z	x_1	x_e	x_a	0
e	e	a	z	x_1	x_e	x_a	0
a	z	z	z	x_1	x_e	x_e	0
z	z	z	z	x_1	x_e	x_a	0
x_1	x_e	x_a	0	0	0	0	0
x_e	x_e	x_a	0	0	0	0	0
x_a	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0



$\mathcal{L}$-classes of $\mathcal{U}(S)$



$\mathcal{R}$-classes of $\mathcal{U}(S)$

$H_{\mathcal{L}}(S)$	Set of possible values for $H_{\mathcal{R}}(S)$
1	$\{1\}$
2	$\{2, 3\}$
3	$\{2, \ldots, 7\}$
4	$\{3, \ldots, 15\}$
5	$\{3, \ldots, 31\}$
6	$\{3, \ldots, 63\}$
7	$\{3, \ldots, 127\}$
8	$\{4, \ldots, 255\}$

Table: For some small natural numbers n, the range of possible values of $H_{\mathcal{R}}(S)$ for a stable semigroup S with $H_{\mathcal{L}}(S) = n$.

Bound on the $\mathcal{J}$-height

Theorem. If $2 \leq H_{\mathcal{L}}(S) < \infty$ and $2 \leq H_{\mathcal{R}}(S) < \infty$, then

$$H_{\mathcal{J}}(S) \leq H_{\mathcal{L}}(S) + H_{\mathcal{R}}(S) - 2.$$

Consequently, letting $\min\big(H_{\mathcal{L}}(S), H_{\mathcal{R}}(S)\big) = n$, we have

$$H_{\mathcal{J}}(S) \leq 2^n + n - 3.$$

Example

Let S be the semigroup with multiplication table

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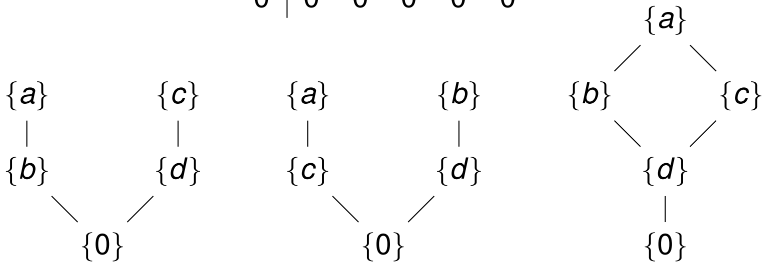
c	0	0	0	0	0
d	0	0	0	0	0
0	0	0	0	0	0



$\mathcal{L}$-classes $\mathcal{R}$-classes $\mathcal{J}$-classes

Corollary of previous results

For any semigroup S, the following are equivalent:

1. $H_{\mathcal{L}}(S)$ and $H_{\mathcal{H}}(S)$ are finite;
2. $H_{\mathcal{L}}(S)$ is finite and S is uniformly group-bound;
3. $H_{\mathcal{L}}(S)$ is finite and S is stable;
4. $H_{\mathcal{L}}(S)$ is finite and S is right stable;
5. $H_{\mathcal{L}}(S)$ and $H_{\mathcal{R}}(S)$ are finite;
6. $H_{\mathcal{L}}(S)$, $H_{\mathcal{R}}(S)$ and $H_{\mathcal{H}}(S)$ are finite;
7. $H_{\mathcal{R}}(S)$ and $H_{\mathcal{H}}(S)$ are finite;
8. $H_{\mathcal{R}}(S)$ is finite and S is uniformly group-bound;
9. $H_{\mathcal{R}}(S)$ is finite and S is stable;
10. $H_{\mathcal{R}}(S)$ is finite and S is left stable.

Moreover, if any (and hence all) of the conditions (1)-(10) hold, then $H_{\mathcal{J}}(S)$ is finite.

Poset of idempotents

There is a partial order on the set E of idempotents of S given by

$$e \leq f \Leftrightarrow e = ef = fe.$$

We denote the height of the resulting poset by $H_E(S)$.

Lemma. $e \leq f \Leftrightarrow H_e \leq H_f$. Consequently, $H_E(S) \leq H_{\mathcal{H}}(S)$.

S is *regular* if $a \in aSa$ for all $a \in S$.

Proposition. If S is regular, then

$$H_{\mathcal{L}}(S) = H_{\mathcal{R}}(S) = H_{\mathcal{H}}(S) = H_E(S).$$

Example. The bicyclic monoid $B = \langle a, b \mid ab = 1 \rangle$ has an infinite chain of idempotents $E = \{ba > b^2a^2 > b^3a^3 > \cdots\}$, so $H_E(B) = \infty$, but $H_{\mathcal{J}}(B) = 1$ (since B has a single $\mathcal{J}$-class).

Proposition. If S is regular and stable, then

$$H_{\mathcal{L}}(S) = H_{\mathcal{R}}(S) = H_{\mathcal{H}}(S) = H_E(S) = H_{\mathcal{J}}(S).$$

Thanks for listening